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Bayes Estimation of Two Ordered Exponential Means

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Abstract

Bayes estimation of parameters is considered for two independent exponential distributions with ordered means. Order restricted Bayes estimators for means are obtained with respect to inverted gamma, noninformative prior and uniform prior distributions, and their asymptotic properties are established. It is shown that the maximum likelihood estimator, restricted maximum likelihood estimator, unrestricted Bayes estimator, and restricted Bayes estimator of the mean are all consistent and have the same limiting distribution. These estimators are compared with the corresponding unrestricted Bayes estimators by Monte Carlo simulation.

Keywords : Asymptotic normality, Bayes estimator, Consistency, Order restricted Bayes estimator

1. Introduction

The problem of estimating ordered means when the ordering is known arises in various industrial experiments. For example, there are situations where one is interested in the estimation of the mean lifetimes of two components having exponential life distributions; one is an improvement of the other, and naturally the improved component should have a greater mean lifetime than that of the original component.

Order restricted inference has been extensively studied. Barlow et al.(1972) and Robertson et al.(1988) assembled much of the early works and served as a basis for subsequent researches in this area. For the exponential distributions, Jin and Pal(1991) considered the problem of simultaneous estimation of location parameters

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of two independent exponential distributions when location and/or scale parameters are ordered. Kaur and Singh(1991) obtained order restricted maximum likelihood estimators(MLEs) for two exponential means. Vijayasree and Singh(1991) found a class of admissible estimators for two ordered exponential means. Singh et al.(1993) considered the problem of estimating the location and scale parameters when the two location parameters are ordered. Parnami and Singh(1993) considered a sequential approach for estimation of ordered means. Broffitt(1984) considered, from Bayesian viewpoint, the problem of estimating the ordered parameters of an exponential family of distributions.

In this paper, we consider the problem of Bayes estimation of ordered means for two independent exponential distributions when the mean of Y is greater than that of X. Three restricted Bayes estimators(RBEs) of the means for each of the inverted gamma, noninformative, and uniform priors are obtained in Section 2. In Section 3, the asymptotic properties of the estimators are established. Results of comparative studies based on Monte Carlo simulation are summarized in Section 4.

2. Bayes Estimation of Ordered Means

Let $X_1, X_2, \cdots, X_{n_1}$ and $Y_1, Y_2, \cdots, Y_{n_2}$ be the two independent samples from exponential distributions with means $\theta_1 = 1/\lambda_1$ and $\theta_2 = 1/\lambda_2$, respectively, satisfying the restriction $\theta_1 < \theta_2$. The likelihood function can be written as

$$l(\theta_1, \theta_2, | s_1, s_2) \propto \Pi^2_{i=1} \frac{1}{\theta_i^{n_i}} \exp\left(-\frac{s_i}{\theta_i}\right), \quad \theta_1 < \theta_2 \tag{1}$$

where $s_1 = \sum_{j=1}^{n_1} x_i$ and $s_2 = \sum_{j=1}^{n_2} y_i$. Note that $\widehat{\theta}_i = S_i / n_i$ is the MLE and uniformly minimum variance unbiased estimator of θ_i, $i = 1, 2$.

We know, from Robertson *et al.*(1988), that the order restricted maximum likelihood estimators(RMLEs) of θ_1 and θ_2 are

$$\widehat{\theta_{1R}} = \min\left(\widehat{\theta}_1, \frac{\widehat{\theta}_1 + \widehat{\theta}_2}{2}\right), \tag{2a}$$

$$\widehat{\theta_{2R}} = \max\left(\widehat{\theta}_2, \frac{\widehat{\theta}_1 + \widehat{\theta}_2}{2}\right). \tag{2b}$$

In what follows, $IG(\alpha_0, \beta_0)$ will denote the inverted gamma distribution whose probability density function(pdf) is given by

$$g(\theta; c, d) = \frac{d^c}{\Gamma(c)} \theta^{-(c+1)} e^{-d/\theta}, \quad \theta > 0.$$

The following result will be used subsequently. If θ_1 and θ_2 are independent

$IG(\alpha_1, \beta_1)$ and $IG(\alpha_2, \beta_2)$ random variables, respectively, then $\beta_2\theta_1/(\beta_2\theta_1+\beta_1\theta_2)$ has a beta pdf with parameters α_2 and α_1. Moreover,

$$P[\theta_1 < \theta_2] = P\left[\frac{\beta_2\theta_1}{\beta_2\theta_1+\beta_1\theta_2} < \frac{\beta_2}{\beta_1+\beta_2}\right] = I\left(\frac{\beta_2}{\beta_1+\beta_2}; \alpha_2, \alpha_1\right), \tag{3}$$

where $I(x; r, s)$ is the beta distribution function with parameters r and s. We now obtain the RBEs of means (θ_1, θ_2) subject to the restriction $\theta_1 < \theta_2$.

2.1 Case of inverted gamma prior

Assume that the joint prior pdf of (θ_1, θ_2) is

$$p_1(\theta_1, \theta_2) = \frac{\Pi^2_{i=1} g(\theta_i; \alpha_i, \beta_i)}{I\left(\frac{\beta_2}{\beta_1+\beta_2}; \alpha_2, \alpha_1\right)}, \quad \theta_1 < \theta_2, \tag{4}$$

where parameters $\alpha_i > 0$ and $\beta_i > 0$ are chosen to reflect prior knowledge. It is clear that formula (4) is in the form of a ratio of the unrestricted joint pdf to the probability that $\theta_1 < \theta_2$.

Under the assumption of prior pdf (4), the joint posterior pdf of (θ_1, θ_2) is obtained as

$$\begin{aligned} h_1(\theta_1, \theta_2 | s_1, s_2) &= c(s_1, s_2)\Pi^2_{i=1} g(\theta_i; \alpha_i, \beta_i) l(\theta_1, \theta_2; s_1, s_2) \\ &= c(s_1, s_2)\Pi^2_{i=1} g(\theta_i; n_i+\alpha_i, s_i+\beta_1) \quad \theta_1 < \theta_2, \end{aligned} \tag{5}$$

where the constant of proportionality $c(s_1, s_2)$ can easily be found from the requirement that $h_1(\theta_1, \theta_2 \mid s_1, s_2)$ must be a density function. From (5),

$$\begin{aligned} (c(s_1, s_2))^{-1} &= \int_0^\infty \int_{\theta_1}^\infty \Pi^2_{i=1} g(\theta_i; n_i+\alpha_i, s_i+\beta_i) d\theta_2 d\theta_1 \\ &= P(\theta_1 < \theta_2) \\ &= I\left(\frac{s_2+\beta_2}{s_1+s_2+\beta_1+\beta_2}; n_2+\alpha_2, n_1+\alpha_1\right). \end{aligned}$$

Therefore, the joint posterior pdf can be written as

$$h_1(\theta_1, \theta_2 | s_1, s_2) = \frac{\Pi^2_{i=1} g(\theta_i; n_i+\alpha_i, s_i+\beta_i)}{I\left(\frac{s_2+\beta_2}{s_1+s_2+\beta_1+\beta_2}; n_2+\alpha_2, n_1+\alpha_1\right)}, \quad \theta_1 < \theta_2. \tag{6}$$

Since both (4) and (6) belong to the same family of distributions, (4) is a conjugate prior under the restriction $\theta_1 < \theta_2$. It can be easily shown that under a quadratic loss function the unrestricted Bayes estimator(UBE) of θ_i is

$$\tilde{\theta}_i = \frac{S_i + \beta_i}{n_i + \alpha_i - 1}, \quad i = 1, 2. \tag{7}$$

See Bhattacharya(1967). The RBEs are obtained in the following theorem.

Theorem 1. The RBE of θ_i with respect to inverted gamma prior (4) and a quadratic loss function is

$$\widetilde{\theta_{iR}} = \tilde{\theta}_i \cdot \rho_{i,n}, \quad i = 1, 2, \tag{8}$$

where

$$\rho_{i,n} = \frac{I\left(\frac{s_2 + \beta_2}{s_1 + s_2 + \beta_1 + \beta_2}; n_2 + \alpha_2 - i + 1, n_1 + \alpha_1 + i - 2\right)}{I\left(\frac{s_2 + \beta_2}{s_1 + s_2 + \beta_1 + \beta_2}; n_2 + \alpha_2, n_1 + \alpha_1\right)},$$

and $n = n_1 + n_2$.

Proof: Using (6), under a quadratic loss function, the Bayes estimate of θ_1 is

$$\begin{aligned} \widetilde{\theta_{1R}} &= E[\theta_1 | s_1, s_2] \\ &= \frac{\int_0^\infty \int_{\theta_1}^\infty \theta_1 \Pi_{i=1}^2 g(\theta_i; n_i + \alpha_i, s_i + \beta_i) d\theta_2 d\theta_1}{I\left(\frac{s_2 + \beta_2}{s_1 + s_2 + \beta_1 + \beta_2}; n_2 + \alpha_2, n_1 + \alpha_1\right)}. \end{aligned} \tag{9}$$

From (3) and the identity

$$\begin{aligned} \theta_1 g(\theta_1; n_1 + \alpha_1, s_1 + \beta_1) &= \theta_1 \frac{(s_1 + \beta_1)^{n_1 + \alpha_1}}{\Gamma(n_1 + \alpha_1)} \theta_1^{-(n_1 + \alpha_1 + 1)} e^{-\frac{s_1 + \beta_1}{\theta_1}} \\ &= \frac{s_1 + \beta_1}{n_1 + \alpha_1 - 1} \cdot \frac{(s_1 + \beta_1)^{-(n_1 + \alpha_1 - 1)}}{\Gamma(n_1 + \alpha_1 - 1)} \theta_1^{-(n_1 + \alpha_1)} e^{-\frac{s_1 + \beta_1}{\theta_1}} \\ &= \tilde{\theta}_1 g(\theta_1; n_1 + \alpha_1 - 1, s_1 + \beta_1), \end{aligned}$$

the numerator of the right-hand side(rhs) of (9) can be rewritten as

$$\tilde{\theta}_1 I\left(\frac{s_2 + \beta_2}{s_1 + s_2 + \beta_1 + \beta_2}; n_2 + \alpha_2, n_1 + \alpha_1 - 1\right).$$

That is, $\widetilde{\theta_{1R}} = \tilde{\theta}_1 \cdot \rho_{1,n}$. Similarly, we can see that

$$\widetilde{\theta_{2R}} = \tilde{\theta}_2 \cdot \rho_{2,n}.$$

2.2 Case of noninformative prior

For the situation where the experimenter has no prior information about θ_i, we may use noninformative prior with order restriction;

$$p_2(\theta_1, \theta_2) \propto \frac{1}{\theta_1\theta_2}, \quad \theta_1 < \theta_2. \tag{10}$$

The Bayes estimators of θ_i with respect to prior (10) and a quadratic loss function can be obtained by putting $\alpha_i = \beta_i = 0$ in Theorem 1.

2.3 Case of uniform prior

It frequently happens that the experimenter knows in advance that the probable values of θ_i lie over a finite range (c_i, d_i) but he does not have any strong opinion about any subset of values over this range. In such a case the following distribution over (c_i, d_i) may be a good approximation;

$$p_3(\theta_1, \theta_2) \propto \frac{1}{(d_1 - c_1)(d_2 - c_2)}, 0 < c_i < \theta_i < d_i, \quad \theta_1 < \theta_2, \tag{11}$$

Where we assume that $c_1 \le c_2 < d_1 \le d_2$. The case of $c_1 > c_2$ and/or $d_1 > d_2$ is unrealistic because it is known apriori that $\theta_1 < \theta_2$. Inequality $c_2 \ge d_1$ implies $\theta_1 < \theta_2$, so that the RBEs of (θ_1, θ_2) have the same form as the ones obtained in the absence of the restriction. Therefore the assumption of $c_1 \le c_2 < d_1 \le d_2$ is natural. From formulas (1) and (11), and the Bayes theorem, the posterior pdf of (θ_1, θ_2) is

$$h_3(\theta_1, \theta_2 | s_1, s_2) = A(s_1, s_2, n_1, n_2; \omega) \frac{\exp(-s_1/\theta_1 - s_2/\theta_2)}{\theta_1^{n_1}\theta_2^{n_2}}, \tag{12}$$

where $\theta_1 < \theta_2$, $0 < c_i < \theta_i < d_i$, $\omega = \{(c_1, c_2, d_1, d_2); c_1 \le c_2 < d_1 \le d_2\}$, and $A(s_1, s_2, n_1, n_2; \omega)$ is the normalizing constant given by

$$(A(s_1, s_2, n_1, n_2; \omega))^{-1}$$

$$= \int_{c_2}^{d_1}\int_{c_1}^{\theta_2} \frac{\exp(-s_1/\theta_1 - s_2/\theta_2)}{\theta_1^{n_1}\theta_2^{n_2}}\, d\theta_1 d\theta_2 + \int_{d_1}^{d_2}\int_{c_1}^{d_1} \frac{\exp(-s_1/\theta_1 - s_2/\theta_2)}{\theta_1^{n_1}\theta_2^{n_2}}\, d\theta_1 d\theta_2$$

$$= \frac{\Gamma(n_1-1)}{s_1^{n_1-1}(s_1+s_2)^{n_2-1}} \cdot \sum_{i=0}^{n_1-2} \frac{s_1^i}{\Gamma(i+1)(s_1+s_2)^i}$$

$$\cdot \left\{ \Gamma\left(\frac{s_1+s_2}{c_2}; n_2+i-1\right) - \Gamma\left(\frac{s_1+s_2}{d_1}; n_2+i-1\right) \right\}$$

$$- \frac{1}{s_1^{n_1-1} s_2^{n_2-1}} \Bigg[\Gamma(n_1-1) \left\{ \Gamma\left(\frac{s_2}{c_2}; n_2-1\right) - \Gamma\left(\frac{s_2}{d_1}; n_2-1\right) \right\}$$

$$- \Gamma\left(\frac{s_1}{c_1}; n_1-1\right) \left\{ \Gamma\left(\frac{s_2}{c_2}; n_2-1\right) - \Gamma\left(\frac{s_2}{d_2}; n_2-1\right) \right\}$$

$$+ \Gamma\left(\frac{s_1}{d_1}; n_1-1\right) \left\{ \Gamma\left(\frac{s_2}{d_1}; n_2-1\right) - \Gamma\left(\frac{s_2}{d_2}; n_2-1\right) \right\} \Bigg],$$

where $\Gamma(x; a) = \int_0^x u^{a-1}\exp(-u)du$, $a > 0$ is the incomplete gamma function. Bhattacharya(1967) obtained the UBE of θ_i as

$$\tilde{\theta}_i = s_i \frac{\Gamma(S_i/c_i; n_i-2) - \Gamma(S_i/d_i; n_i-2)}{\Gamma(S_i/c_i; n_i-1) - \Gamma(S_i/d_i; n_i-1)}, \quad i=1,2. \tag{13}$$

In the following theorem, we obtain the RBEs.

Theorem 2. The RBEs of θ_i with respect to prior (11) and a quadratic loss function is

$$\widetilde{\theta_{iR}} = \frac{A(S_1, S_2, n_1, n_2; \omega)}{A(S_1, S_2, n_1+i-2, n_2-i+1; \omega)}, \quad i=1,2. \tag{14}$$

Proof: The posterior expectation of $\theta_1^i \theta_2^j$ under the restriction $\theta_1 < \theta_2$ is given by

$$E[\theta_1^j\theta_2^k|s_1, s_2]$$

$$= A(s_1, s_2, n_1, n_2; \omega)\Big[\int_{c_2}^{d_1}\int_{c_1}^{\theta_2}\frac{\exp(-s_1/\theta_1 - s_2/\theta_2)}{\theta_1^{n_1-j}\theta_2^{n_2-k}}\,d\theta_1 d\theta_2$$

$$+\int_{d_1}^{d_2}\int_{c_1}^{d_1}\frac{\exp(-s_1/\theta_1 - s_2/\theta_2)}{\theta_1^{n_1-j}\theta_2^{n_2-k}}\,d\theta_1 d\theta_2\Big]$$

$$= \frac{A(s_1, s_2, n_1, n_2; \omega)}{A(s_1, s_2, n_1-j, n_2-k; \omega)}\,.$$

This completes the proof.

3. Large Sample Properties

In this section we consider large sample properties of the order restricted estimators of (θ_1, θ_2) under noninformative and inverted gamma prior distributions.

Lemma 1. Let $n = n_1 + n_2 \to \infty$ such that $n_2/n \to \eta$, $0 < \eta < 1$ as $n \to \infty$. Then,

$$\frac{I(x; n_2+k_2, n_1+k_1)}{I(x; n_2, n_1)} \to \left(\frac{x}{\eta}\right)^{k_2}\left(\frac{1-x}{1-\eta}\right)^{k_1} I[x < \eta] + I[x \geq \eta]$$

uniformly on any compact subset of $(0, 1]$ where $I[\cdot]$ denotes the usual indicator function.

Proof. See Hong(1996).

Lemma 2. Let $n = n_1 + n_2 \to \infty$ such that $n_2/n \to \eta$, $0 < \eta < 1$ and $X_{n_1, n_2} \xrightarrow{p} q$, $0 < q < 1$. Then,

$$\frac{I(X_{n_1, n_2}; n_2+k_2, n_1+k_1)}{I(X_{n_1, n_2}; n_2, n_1)} \xrightarrow{p} \begin{cases} 1, & \text{if } q \geq \eta, \\ \left(\frac{q}{\eta}\right)^{k_2}\left(\frac{1-q}{1-\eta}\right)^{k_1}, & \text{if } q < \eta. \end{cases} \tag{15}$$

Proof. Let

$$f(x) = \frac{x^{k_2}(1-x)^{k_1}}{\eta^{k_2}(1-\eta)^{k_1}} I[x < \eta] + I[x \geq \eta]$$

We must show that

$$P\left\{\left|\frac{I(X_{n_1,n_2}; n_2+k_2, n_1+k_1)}{I(X_{n_1,n_2}; n_2, n_1)} - f(q)\right| > \varepsilon\right\} \to 0$$

as $n \to \infty$. Choose $\xi > 0$ so that $[q-\xi, q+\xi] \subset (0,1]$ and $|f(x)-f(q)| < \varepsilon/2$ if $[x-q] < \xi$. From Lemma 1, we can choose $N > 0$ such that

$$\left|\frac{I(x; n_2+k_2, n_1+k_1)}{I(x; n_2, n_1)} - f(x)\right| < \varepsilon/2 \quad \text{on} \quad [q-\xi, q+\xi].$$

If $n \geq N$,

P{| $P\{|X_{n_1,n_2} - q| < \xi\}$- q | < ε}

$$\leq P\{|f(X_{n_1,n_2}) - f(q)| < \varepsilon/2 \text{ and } \left|\frac{I(X_{n_1,n_2}; n_2+k_2, n_1+k_1)}{I(X_{n_1,n_2}; n_2, n_1)} - f(X_{n_1,n_2})\right| < \varepsilon/2\ \}$$

$$\leq P\left\{\left|\frac{I(X_{n_1,n_2}; n_2+k_2, n_1+k_1)}{I(X_{n_1,n_2}; n_2, n_1)} - f(q)\right| < \varepsilon\right\}.$$

Hence

$$P\left\{\left|\frac{I(X_{n_1,n_2}; n_2+k_2, n_1+k_1)}{I(X_{n_1,n_2}; n_2, n_1)} - f(q)\right| \geq \varepsilon\right\} \leq P\{|X_{n_1,n_2} - q|\} \to 0.$$

as $n \to \infty$. Therefore,

$$\frac{I(X_{n_1,n_2}; n_2+k_2, n_1+k_1)}{I(X_{n_1,n_2}; n_2, n_1)} \xrightarrow{p} f(q).$$

This completes the proof.

Lemma 2 can be used to establish the asymptotic properties of RBE of θ_i for the inverted gamma and noninformative priors. Consistency and asymptotic normality of RBE of θ_i are shown in the following theorem. We note that these properties can be easily established for the MLE and UBE. For RMLE $\widehat{\theta}_{iR,n}$ of θ_i, we have

$$\widehat{\theta}_{iR,n} \xrightarrow{p} \theta_i,$$

$$\sqrt{n}(\widehat{\theta}_{iR,n} - \widehat{\theta}_{i,n}) \xrightarrow{p} 0$$

because $\widehat{\theta}_{2,n} - \widehat{\theta}_{1,n} \xrightarrow{p} \theta_2 - \theta_1 > 0$ as $n \to \infty$, where $\widehat{\theta}_{i,n}$ denotes MLE of θ_1. Hence, $\widehat{\theta}_{iR,n}$ and $\widehat{\theta}_{i,n}$ have the same limiting distribution. Write $\widetilde{\theta}_{i,n}$ and $\widetilde{\theta}_{iR,n}$ for the UBE and RBE of θ_i.

Theorem 3. Let $n \to \infty$ such that $n_2/n \to \eta$. $0 < \eta < 1$. If $\theta_1 < \theta_2$, then i) $\widetilde{\theta_{iR,n}} \xrightarrow{p} \theta_i$,

ii) $\sqrt{n}(\widetilde{\theta_{iR,n}} - \theta_i) \xrightarrow{L} Z_i$,

where $Z_i \sim N(0, \sigma_i^2)$, $\sigma_1^2 = \theta_1^2/(1-\eta)$, and $\sigma_2^2 = \theta_2^2/\eta$.

Proof. We can show that

$$\frac{S_2+\beta_2}{S_1+S_2+\beta_1+\beta_2} \xrightarrow{p} q_0,$$

where $q_0 = \dfrac{\eta\theta_2}{(1-\eta)\theta_1+\eta\theta_2}$. Since $q_0 - \eta = \dfrac{\eta(1-\eta)(\theta_2-\theta_1)}{(1-\eta)\theta_1+\eta\theta_2} > 0$, Theorem 1 and Lemma 2 give $\sqrt{n}(\rho_{i,n}-1) \xrightarrow{p} 0$, and hence $\rho_{i,n} \xrightarrow{p} 1$ and $\widetilde{\theta_{iR,n}} \xrightarrow{p} \theta_1$. That is, the RBE in Theorem 1 is consistent for θ_i. From (7), it can be easily shown that

$$\sqrt{n}(\widetilde{\theta_{i,n}} - \theta_i) \xrightarrow{L} Z_i.$$

Hence,

$$\sqrt{n}(\widetilde{\theta_{iR,n}} - \rho_{i,n}\theta_i) = \rho_{i,n}\sqrt{n}(\widetilde{\theta_{i,n}} - \theta_i) \xrightarrow{L} Z_i.$$

Therefore,

$$\sqrt{n}(\widetilde{\theta_{iR,n}} - \theta_i) = \sqrt{n}(\widetilde{\theta_{iR,n}} - \rho_{i,n}\theta_i) + \sqrt{n}(\rho_{i,n}-1)\theta_i \xrightarrow{L} Z_i.$$

From Theorem 3, we note that the MLE, RMLE, UBE, and RBE of θ_i are all consistent and have the same limiting distribution.

4. Simulation Comparisons

In this section we evaluate the performance of the RBEs of (θ_1, θ_2) by a Monte Carlo simulation. The RBEs are compared with the UBEs for selected combinations of prior distributions, parmeters, and sample sizes. Samples of sizes n_1 and n_2 from two exponential distributions with means θ_1 and $\theta_2 (\theta_1 < \theta_2)$, respectively, were generated. The experiment was repeated 10,000 times. The biases and/or MSEs for each estimator are calculated by averaging over 10,000 replications.

Tables 1 and 2 give the estimated MSEs of the estimators of (θ_1, θ_2), where each MSE is the sum of the two MSEs for θ_1 and θ_2. The parameter values of prior distribution are chosen by solving $E[\Theta_i] = \theta_i$ and $Var[\Theta_i] = \theta_i^2$, simultaneously, where Θ_i is the random variable of prior distribution. In general, as expected, the RBEs have smaller MSEs than UBEs. The RBEs under the inverted gamma priors with $\alpha_i = 3.0$ and $\beta_i = 2\theta_i$ are shown to be preferable to the RMLEs. The estimated MSEs tend to decrease when sample size increases. The simulation results indicate that a prior knowledge about parameter ordering should be incorporated to obtain more accurate estimates of parameters.

Table 1. Estimated MSEs of estimators for (θ_1, θ_2)

$\theta_1 = 1.0$ and $n_1 = n_2$

θ_2	n_1	inverted gamma(*)		noninformative		uniform(**)		RMLE	MLE
		RBE	UBE	RBE	UBE	RBE	UBE		
1.5	10	0.1863	0.2238	0.3897	0.4391	0.2995	0.3467	0.2845	0.3223
	20	0.1079	0.1322	0.1561	0.1860	0.1389	0.1472	0.1473	0.1600
	30	0.0798	0.0953	0.1016	0.1199	0.0926	0.1011	0.1034	0.1085
	50	0.0522	0.0598	0.0601	0.0687	0.0572	0.0648	0.0637	0.0647
2.0	10	0.3021	0.3453	0.6116	0.6784	0.4873	0.5321	0.4782	0.4972
	20	0.1852	0.2036	0.2606	0.2864	0.2528	0.2687	0.2432	0.2463
	30	0.1392	0.1491	0.1744	0.1851	0.1603	0.1710	0.1668	0.1674
	50	0.0901	0.0919	0.1032	0.1054	0.0981	0.1014	0.0994	0.0994
3.0	10	0.6651	0.6923	1.3031	1.3624	1.0387	1.0845	0.9933	0.9970
	20	0.4022	0.4074	0.5634	0.5733	0.5025	0.5288	0.4928	0.4929
	30	0.2941	0.2950	0.3694	0.3712	0.3398	0.3407	0.3357	0.3357
	50	0.1835	0.1835	0.2105	0.2105	0.2007	0.2077	0.1985	0.1985
5.0	10	1.7961	1.8030	3.5227	3.5512	3.0561	3.0623	2.5961	2.5963
	20	1.0592	1.0595	1.4902	1.4913	1.3508	1.3514	1.2820	1.2820
	30	0.7683	0.7683	0.9667	0.9668	0.8980	0.8982	0.8741	0.8741
	50	0.4767	0.4767	0.5467	0.5467	0.5256	0.5256	0.5156	0.5156

(*) $\alpha_i = 3.0$ and $\beta_i = 2\theta_i$, $i = 1, 2$.

(**) $c_i = 0.0$ and $d_i = (1+\sqrt{3})\theta_i$, $i = 1, 2$.

Table 2. Estimated MSEs of estimators for $(\theta_1, \theta_2)=(1.0, 1.5)$

n_1, n_2	inverted gamma(*)		noninformative		uniform(**)		RMLE	MLE
	RBE	UBE	RBE	UBE	RBE	UBE		
10, 5	0.2515	0.3133	0.5651	0.6587	0.4942	0.5825	0.5064	0.5801
10, 10	0.1863	0.2238	0.3897	0.4391	0.2995	0.3467	0.2845	0.3223
10, 15	0.1567	0.1750	0.1954	0.2268	0.1752	0.1998	0.1687	0.1785
10, 20	0.1247	0.1348	0.1487	0.1623	0.1358	0.1524	0.1298	0.1354
20, 15	0.1164	0.1334	0.1524	0.1742	0.1374	0.1498	0.1386	0.1477
20, 20	0.1079	0.1322	0.1561	0.1860	0.1389	0.1472	0.1473	0.1600
20, 25	0.0963	0.1180	0.1394	0.1661	0.1240	0.1314	0.1315	0.1429
20, 30	0.0858	0.1051	0.1241	0.1479	0.1104	0.1170	0.1171	0.1272

(*) $\alpha_i=3.0$ and $\beta_i=2\theta_i$, $i=1,2$.

(**) $c_i=0.0$ and $d_i=(1+\sqrt{3})\theta_i$, $i=1,2$.

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