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Tests for Uniformity : A Comparative Study

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Abstract

The subject of assessing whether a data set is from a specific distribution has received a good deal of attention. This topic is critically important for uniform distributions. Several parametric tests are compared. These tests also can be used in testing randomness of a sample. Anderson-Darling A^2 statistic is found to be most powerful.

KeyWords : Approximate Pearson Chi-square statistic, Cramer-Von Misses W2 statistic, Exact Pearson Chi-square statistic, Watson U2 statistic.

1. INTRODUCTION

The subject of assessing whether a data set is from a specific distribution has received a good deal of attention. This topic is critically important for uniform distributions and is different from usual tests for randomness. There are different parametric and nonparametric tests for randomness. A nonparametric textbook such as Daniel(1990) and Gibbons and Chakraborti(2003) would provide extensive references. Marsaglia(2003) and L'Ecuyer and Hellekalek(1998), and the references therein provide a group of tests meant for testing goodness of different random number generators. Here we compare eight different commonly used parametric goodness of fit tests for uniformity through simulation. Let us consider that $X_1, X_2, \cdots, X_n$ be a random sample taken from a Uniform(0,1) distribution. We will first explain all the eight tests. In Section 2 we will provide the simulation results. In Section 3 we will give a brief conclusion based on the simulation results.

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1.1. APPROXIMATE χ_U^2 GOODNESS OF FIT

By grouping the data into g equal groups such that each group has expected frequency of at least five and the number of groups is not too large, we can calculate χ_U^2 goodness of fit statistic as

$$\chi_U^2 = \sum_{i=1}^{g} \frac{(O_{Ui} - E_{Ui})^2}{E_{Ui}}, \qquad (1)$$

where O_{Ui} is the observed number of values in the i^{th} group and $E_{Ui} = n/g$ is the expected frequency in the i^{th} group assuming that the sample is from the Uniform(0,1) distribution. The χ_U^2 statistic will follow approximate Chi-square distribution with $g-1$ degrees of freedom.

1.2. APPROXIMATE χ_U^2 GOODNESS OF FIT VIA NORMAL

Box and Muller (1958) showed that if X_1 and X_2 are two independent uniform random variables on (0,1), then $Z_1 = \sqrt{-2\ln(X_1)}\sin(2\pi X_2)$ and $Z_2 = \sqrt{-2\ln(X_1)}\cos(2\pi X_2)$ are independent standard normal variates. After using the Box-Muller transformation, we can compute χ_N^2 statistic as in equation (1). The expected frequencies are computed after computing the cell probabilities using the standard normal table (here we used MATLAB mathematical computational software) and then multiplied by n. It is to be noted that in simulation we have used two independent random samples of size n from Uniform (0,1) distribution to transform to the standard normal but in practice the sample should be divided into two equal groups before applying the Box-Muller transformation. The groups are determined such that the expected frequencies are more than five and are equal.

1.3. EXACT χ_Z^2 GOODNESS OF FIT VIA NORMAL

As we know that the sum of squared standard normal variates follow the Chi-square distribution with n degrees of freedom. So, the test statistic is

$$\chi_Z^2 = \sum_{i=1}^{n} Z_i^2 ,$$

where Z_i's are the standard normal variate after the Box-Muller transformation as in Section 1.2.

1.4. THE CRAMER-VON MISES W^2 TEST

A distribution function test suggested by van Soest (1969), is known as the Cramer-von Mises W^2 test. The Cramer-von Mises statistic is computed as

$$W^2 = \left(W^{*2} - \frac{4}{10n} + \frac{6}{10n^2} \right)\left(1 + \frac{1}{n}\right) ,$$

where

$$W^{*2} = \left\{ \sum_{i=1}^{n} \left(Z_i - \frac{2i-1}{2n} \right)^2 + \frac{1}{12n} \right\}$$

and $Z_i = X_{i:n}$, the ordered data from the smallest to the largest. The test was modified so that the percentiles are independent of n. In Table 1, the percentiles are displayed. In Table 1, the first line indicates the upper tail probabilities and the second line represents the corresponding quantiles. These are recently recomputed and displayed by D'Agastino and Stephens (1986, p.105).

Table 1 : Upper tail percentiles for cramer-von Mises W^2 test

0.250	0.150	0.100	0.050	0.025	0.010	0.005	0.001
0.209	0.284	0.347	0.461	0.581	0.743	0.869	1.167

1.5. THE WATSON U^2 TEST

A distribution function test is suggested by Watson (1961). The Watson U^2 statistic is computed as

$$U^2 = \left(U^{*2} - \frac{1}{10n} + \frac{1}{10n^2} \right)\left(1 + \frac{8}{10n}\right) ,$$

where

$$U^{*2} = \left\{ W^{*2} - n\left(\bar{Z} - \frac{1}{2}\right)^2 \right\},$$

W^{*2} and Z_i's are defined above and $\overline{Z} = \sum_{i=1}^{n} Z_i/n$. The percentiles are displayed in Table 2. In Table 2, the first line indicates the upper tail probabilities and the second line represents the corresponding quantiles. The test was modified so that the percentiles are independent of n and are given in D'Agastino and Stephens (1986, p.105).

Table 2: Upper tail percentiles for Cramer-von Mises U2 test

0.250	0.150	0.100	0.050	0.025	0.010	0.005	0.001
0.105	0.131	0.152	0.187	0.222	0.268	0.304	0.385

1.6. THE ANDERSON-DARLING A2 TEST

A distribution function test is suggested by Anderson and Darling (1952). The Anderson-Darling A^2 statistic is computed as

$$A^2 = -n - \frac{1}{n}\sum_{i=1}^{n}\{(2i-1)\ln Z_i + (2n+1-2i)\ln(1-Z_i)\},$$

where Z_i's are as above. The percentiles are given in Table 3. In Table 3, the first line indicates the upper tail probabilities and the second line represents the corresponding quantiles. The percentiles are independent of n and are from D'Agastino and Stephens (1986, p.105).

Table 3: Upper tail percentiles for Anderson-Darling A2 test

0.250	0.150	0.100	0.050	0.025	0.010	0.005	0.001
1.248	1.610	1.933	2.492	3.070	3.880	4.500	6.000

1.7. APPROXIMATE χ_E^2 GOODNESS OF FIT VIA EXPONENTIAL

We can easily convert a uniform (0,1) random variate X to a standard exponential variate Y using the transformation $Y = -\ln(1-X)$. Then we can compute the Chi-square statistic as in Section 1.1. Here the expected frequencies are computed by multiplying the group probabilities using the standard exponential CDF(cumulative distribution function). The groups are determined such that the expected frequencies are more than five and are equal.

1.8. EXACT χ_G^2 GOODNESS OF FIT VIA EXPONENTIAL

We know that $2\sum_{i=1}^{n} Y_i$ has a Chi-square distribution with $2n$ degrees of freedom and can be used as a test statistic. Where Y_i's are defined as in Section 1.7.

2. Simulation Results

One hundred thousand samples are taken for each of 10, 20, 30, 40, 50, and 100 sample sizes. Then the proportions of rejections are computed for 1% and 5% levels of significances. Samples are taken from Uniform (0,1) (U(0,1)) to compute the empirical levels of significances to compare with the true levels of significances. Then samples are also taken from standard normal (N(0,1)) distribution as a representative from a symmetric class of distributions and standard exponential (Exp(1)) distribution as a representative from an asymmetric class of distributions to assess the powers of the tests. In the power computations, the samples are transformed such that the range of the data is between 0 and 1 to compute all the statistics mentioned above. The simulation results are given in Table 4.

3. Conclusion

All eight tests are consistently estimating the levels of significances, showing that the distributions under the assumption of uniformity are pretty accurate. In comparison for powers, A^2 statistic outperform all the tests regardless of the distribution normal or exponential from which samples are taken. χ_Z^2 test has higher power compared to χ_G^2 test when the samples are from N(0,1) and the role is reversed when the samples are from Exp(1). χ_U^2, χ_N^2, χ_E^2, W^2, and U^2 tests have higher powers when the samples are from Exp(1) compared to the samples from N(0,1).

The findings here are consistent with Marsaglia and Zaman (1993) regarding inferiority of Pearsons Chi-square tests and consistent with L'Ecuyer and Hellekalek (1998) regarding superiority of the A^2 statistic. Note that L'Ecuyer and Hellekalek (1998) did not present any comparative study with Pearson Chi-square tests and with U^2 and W^2 tests. Here we also compared between different exact and approximate Pearson Chi-square statistics.

Table 4: Rejection Proportions

n	g	Test	U(0,1) Samples $\alpha = 0.01$	U(0,1) Samples $\alpha = 0.05$	Exp(1) Samples $\alpha = 0.01$	Exp(1) Samples $\alpha = 0.05$	N(0,1) Samples $\alpha = 0.01$	N(0,1) Samples $\alpha = 0.05$
10	2	χ_U^2	0.0017	0.0208	0.0000	0.1837	0.0000	0.0294
		χ_N^2	0.0019	0.0217	0.0000	0.1847	0.0013	0.0413
		χ_Z^2	0.0095	0.0490	0.0417	0.1305	0.8991	0.8992
		W^2	0.0100	0.0491	0.2069	0.3996	0.0113	0.0552
		U^2	0.0092	0.0493	0.2032	0.3937	0.0113	0.0585
		A^2	0.0102	0.0494	0.2370	0.4342	0.9990	0.9991
		χ_E^2	0.0017	0.0208	0.0000	0.1837	0.0000	0.0294
		χ_G^2	0.0097	0.0496	0.0002	0.0031	0.0121	0.0689
20	3	χ_U^2	0.0076	0.0556	0.5840	0.8011	0.0470	0.2127
		χ_N^2	0.0072	0.0552	0.4069	0.7018	0.0273	0.1285
		χ_Z^2	0.0097	0.0497	0.1817	0.3616	0.9503	0.9502
		W^2	0.0105	0.0503	0.7429	0.8712	0.0416	0.1353
		U^2	0.0099	0.0497	0.6157	0.7851	0.0801	0.2186
		A^2	0.0105	0.0516	0.7445	0.8720	1.0000	1.0000
		χ_E^2	0.0076	0.0556	0.5839	0.8011	0.0470	0.2127
		χ_G^2	0.0102	0.0501	0.1507	0.4257	0.0090	0.0445
30	5	χ_U^2	0.0099	0.0455	0.8590	0.9349	0.1708	0.3565
		χ_N^2	0.0095	0.0460	0.8324	0.9325	0.0631	0.1623
		χ_Z^2	0.0102	0.0507	0.3671	0.5749	0.9661	0.9661
		W^2	0.0102	0.0503	0.9455	0.9808	0.0956	0.2627
		U^2	0.0099	0.0504	0.8635	0.9427	0.2887	0.5041
		A^2	0.0101	0.0503	0.9442	0.9799	1.0000	1.0000
		χ_E^2	0.0099	0.0455	0.8590	0.9349	0.1708	0.3565
		χ_G^2	0.0100	0.0497	0.7145	0.8702	0.0141	0.0665
40	6	χ_U^2	0.0092	0.0464	0.9589	0.9844	0.3517	0.5693
		χ_N^2	0.0094	0.0469	0.9695	0.9905	0.0950	0.2024
		χ_Z^2	0.0101	0.0498	0.5382	0.7256	0.9750	0.9750
		W^2	0.0102	0.0496	0.9910	0.9976	0.1790	0.4330
		U^2	0.0099	0.0495	0.9906	0.9873	0.5498	0.7454
		A^2	0.0102	0.0493	0.9908	0.9976	1.0000	1.0000
		χ_E^2	0.0092	0.0464	0.9589	0.9844	0.3519	0.5693
		χ_G^2	0.0096	0.0492	0.9318	0.9775	0.0356	0.1163

Table 4 (Continued): Rejection Proportions

n	g	Test	U(0,1) Samples		Exp(1) Samples		N(0,1) Samples	
			$\alpha=0.01$	$\alpha=0.05$	$\alpha=0.01$	$\alpha=0.05$	$\alpha=0.01$	$\alpha=0.05$
30	5	χ_U^2	0.0110	0.0462	0.9872	0.9958	0.5145	0.7105
		χ_N^2	0.0097	0.0457	0.9943	0.9986	0.1446	0.2790
		χ_Z^2	0.0105	0.0510	0.6704	0.8212	0.9803	0.9803
		W^2	0.0101	0.0495	0.9985	0.9998	0.2962	0.6051
		U^2	0.0097	0.0484	0.9901	0.9974	0.7522	0.8842
		A^2	0.0108	0.0506	0.9985	0.9998	1.0000	1.0000
		χ_E^2	0.0110	0.0462	0.9872	0.9958	0.5145	0.7105
		χ_G^2	0.0101	0.0504	0.9865	0.9965	0.0696	0.1751
40	6	χ_U^2	0.0103	0.0503	1.0000	1.0000	0.9629	0.9890
		χ_N^2	0.0106	0.0494	1.0000	1.0000	0.4273	0.6212
		χ_Z^2	0.0101	0.0500	0.9370	0.9727	0.9897	0.9897
		W^2	0.0102	0.0507	1.0000	1.0000	0.8930	0.9834
		U^2	0.0100	0.0498	1.0000	1.0000	0.9967	0.9992
		A^2	0.0105	0.0510	1.0000	1.0000	1.0000	1.0000
		χ_E^2	0.0103	0.0503	1.0000	1.0000	0.9629	0.9890
		χ_G^2	0.0104	0.0508	1.0000	1.0000	0.2862	0.4365

4. Bibliography

1. T.W. Anderson and D.A. Darling (1952). Asymptotic theory of certain goodness-of-fit criteria based on stochastic processes, *Annals of Mathematical Statistics*, 23, 193-212.
2. G.E.P. Box and M.E. Muller (1958). A note on the generation of random normal deviates. *Annals of Mathematical Statistics*, 29, 610-611.
3. R.B. D'Agostino and M.A. Stephens (Eds.) (1986). *Goodness-of-Fit Techniques*, Marcel Dekker, New York.
4. W.W. Daniel (1990). *Applied Nonparametric Statistics*, Second Edition, Duxbury Thomson Learning, Pacific Grove, CA.
5. J.D. Gibbons and S. Chakraborti (2003). *Nonparametric Statistical inference*, Marcel Dekker, Inc., New York.
6. P. L'Ecuyer and P. Hellekalek (1998). Random Number Generators: Selection Criteria and Testing, in *Random and Quasi-Random Point sets, Lecture Notes in Statistics*, no. 138, Springer, 223-266.
7. G. Marsaglia and A. Zaman (1993). Monkey tests for random number

generators, *Computers and Mathematics with Applications*, 26(9), 1-10.
8. G. Marsaglia (2003). Random Number Generators. *Journal of Modern Applied Statistical Methods*, 2(1), 2-13.
9. J. van Soest (1969). Some goodness of fit tests for the exponential distribution, *Statistica Neerlandica*, 23, 41-51.
10. G.S. Watson (1961). Goodness-of-fit tests on a circle, *Biometrika*, 48, 109-114.

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