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Jackknife Estimation in an Exponential Model[1]

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Abstract

Parametric estimation of truncated point in a truncated exponential distribution will be considered. The MLE, bias reducing estimator and the ordinary jackknife estimator of the truncated parameter will be compared by mean square errors. And the MME and MLE of mean parameter and estimations of the right tail probability in the distribution will be compared by their MSE's.

Keywords: Jackknife estimation, MLE, Right tail probability

1. Introduction

Here we shall consider a right truncated exponential distribution with the following pdf:

$$f(x;\eta)=\frac{e^{-x}}{1-e^{-\eta}}, \qquad 0<x<\eta, \quad \text{where} \quad \eta>0. \tag{1.1}$$

The truncated exponential distribution has an increasing failure rate, ideally suited for use as a survival distribution for biological and industrial data.

For values of η that are corresponding to small amounts of truncation, the hazard rate function increases very slowly up to a certain time and then asymptotically climbs to infinity at η.

The general right truncated exponential pdf of X, $f(x;\theta,T)=\frac{\theta e^{-\theta x}}{1-e^{-\theta T}}$, $0<x<T$ (1.2)(see Hannon & Dahiya(1999)) comes the pdf (1.1) when Y= θX if the parameter θ is known.

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Hannon & Dahiya(1999) examined the general and asymptotic properties of all estimators in a right truncated exponential distribution with the pdf (1.2).

Here the MLE, a bias reducing estimator and the ordinary jackknife estimator of η will be considered, and we shall compare each other in a sense of MSE. And the MLE and MME of mean parameter and estimations of the right tail probability in the right truncated exponential distribution will be compared each other in a sense of MSE

2. Parametric estimations

From the pdf (1.1) of the right truncated exponential distribution, its mgf can be obtained:

$$M_X(t) = \frac{\tau^{-1}(\eta)}{t-1} \cdot (e^{\eta(t-1)} - 1), \text{ if } t \neq 1, \quad \text{where } \tau(\eta) = 1 - e^{-\eta}.$$

Existence of the mgf guarantees all orders of moment in the right truncated distribution, especially the expectation and variance of the distribution

$$\mu = E(X) = \eta + (1 - \eta \cdot \tau^{-1}(\eta))$$

and

$$\sigma^2 = Var(X) = 1 - \eta^2 \cdot e^{-\eta} / \tau^2(\eta).$$

Let $X_1, X_2, \ldots, X_n$ be a simple random sample from the right truncated exponential distribution with the pdf (1.1). Then from the factorization Theorem in Rohatgi(1976), the largest order statistics $X_{(n)}$ is sufficient statistic of η and its pdf is:

$$f_{X_{(n)}}(x) = n \cdot \tau^{-n}(\eta) \cdot e^{-x} \cdot (1 - e^{-x})^{n-1}, \quad 0 < x < \eta. \tag{2.2}$$

From the formulas 3.381(1) & 8.352(1) in Gradsheyn & Ryzhik(1965), we can show the following indefinite integral by finite sums.

Fact 1. Let $I(p, q, r; \eta) \equiv \int_0^{\eta} x^p (1 - e^{-x})^q e^{-rx} dx$, where p and q are non-negative integers. Then

$$I(p, q, r; \eta) = p! \cdot \sum_{i=0}^{q} (-1)^i \binom{q}{i} (i + r)^{-p-1} (1 - e^{-(i+r)\eta} \cdot \sum_{m=0}^{p} \frac{(i+r)^m}{m!} \eta^m).$$

2-1. Estimation of a truncated parameter

The following estimators of η are given as the followings:

$$\hat{\eta} = X_{(n)}, \text{ the MLE of } \eta,$$

$\tilde{\eta} = 2X_{(n)} - X_{(n-1)}$, a bias reducing estimator of η(Hannon & Dahiya (1999)),

and $J(\hat{\eta}) = \frac{2n-1}{n} X_{(n)} - \frac{n-1}{n} X_{(n-1)}$, the ordinary jackknife estimator of η (Gray & Schucany(1972)).

From the pdf (2.2) and Fact 1, we can represent the expectation and variance of MLE by the integral forms in Fact 1.

$$E(\hat{\eta}) = n \cdot \tau^{-n}(\eta) \cdot I(1, n-1, 1; \eta),$$
$$Var(\hat{\eta}) = n \cdot \tau^{-n}(\eta) \cdot I(2, n-1, 1; \eta) - n^2 \cdot \tau^{-2n}(\eta) \cdot I^2(1, n-1, 1; \eta). \quad (2.3)$$

Since the pdf of (n-1)-th order statistic $X_{(n-1)}$ is

$$f_{X_{(n-1)}}(x) = n(n-1) \cdot \tau^{-n}(\eta) \cdot (1-e^{-x})^{n-2} \cdot (e^{-x} - e^{-\eta}) \cdot e^{-x}, \ 0<x<\eta, \quad (2.4)$$

from Fact 1, the pdfs (2.2) and (2.4), we can represent the expectations of the bias reducing estimator $\tilde{\eta}$ and the ordinary jackknife estimator $J(\hat{\eta})$ by the integral forms.

$$E(\tilde{\eta}) = n(n+1) \cdot \tau^{-n}(\eta) \cdot I(1, n-1, 1; \eta) - n(n-1) \cdot \tau^{1-n}(\eta) \cdot I(1, n-2, 1; \eta)$$
$$E(J(\hat{\eta})) = n^2 \cdot \tau^{-n} \cdot I(1, n-1, 1; \eta) - (n-1)^2 \cdot \tau^{1-n}(\eta) \cdot I(1, n-2, 1; \eta). \quad (2.5)$$

Since the joint pdf of (n-1)-th and n-th order statistics is

$$f_{X_{(n-1)}, X_{(n)}}(x, y) = n(n-1) \cdot \tau^{-n}(\eta) \cdot (1-e^{-x})^{n-2} e^{-x} e^{-y}, \ 0<x<y<\eta, \quad (2.6)$$

from Fact 1, the pdfs (2.2), (2.4) and (2.6), the second moments of $\tilde{\eta}$ and $J(\hat{\eta})$ can be represented by the inyegral forms to evaluate their variances.

$$E(\tilde{\eta}^2) = 4n \cdot \tau^{-n} \cdot I(2, n-1, 1; \eta) - n(n-1)\tau^{-n}(\eta)[I(2, n-1, 1; \eta) - \tau(\eta) \cdot I(2, n-2, 1; \eta)] - 4n(n-1) \cdot \tau^{-n}(\eta)[I(2, n-2, 2; \eta) + I(1, n-2, 2; \eta) - (\eta+1)e^{-\eta} \cdot I(1, n-2, 1; \eta)],$$

$$E(J^2(\hat{\eta})) = \frac{(2n-1)^2}{n}\tau^{-n}(\eta)\cdot I(2,n-1,1;\eta)$$
$$-\frac{(n-1)^3}{n}\tau^{-n}(\eta)[I(2,n-1,1;\eta)-\tau(\eta)\cdot I(2,n-2,1;\eta)]$$
$$-\frac{2(n-1)^2(2n-1)}{n}\tau^{-n}(\eta)[I(2,n-2,2;\eta)+I(1,n-2,2;\eta)$$
$$-(1+\eta)e^{-\eta}\cdot I(1,n-2,1;\eta)]. \quad (2.7)$$

Table 1 shows numerical values of MSE of the MLE, the bias reducing estimator and the ordinary jackknife estimator of the truncated parameter in the right truncated exponential distribution by using the results (2.3), (2.5) and (2.7) when η=1 and n=10(10)50.

Through Table 1, the ordinary jackknife estimator of η is more efficient than other estimators in a sense of MSE when η=1.

Table 1. MSE of the MLE, a bias reducing estimator and the ordinary jackknife estimator of η when η=1(units are 10^{-4}).

sample size	$\hat{\eta}$	$\tilde{\eta}$	$J(\hat{\eta})$
10	31.7587	30.7676	28.3443
20	11.7595	10.6787	10.3189
30	5.9845	5.5160	5.3160
40	3.0159	3.0101	2.9375
50	2.0306	1.9135	1.8776

To find an asymptotic confidence interval of η, we need the following asymptotic properties:

By corollary 2 of Hannon & Dahiya(1999), asymptotic Theorems in Rohatgi(1976) and the continuity of F(x), we can obtain the following:

Fact 2. (a) The MLE $X_{(n)}$ converges to η in probability.

(b) $F(x;\hat{\eta})$ converges to $F(x;\eta)$ in probability at every continuous point x of $F(x;\eta)=\dfrac{1-e^{-x}}{1-e^{-\eta}}$, $0<x<\eta$.

By Theorem 1 of Hannon & Dahiya(1999) and Fact 2, we can obtain the following an asymptotic property to obtain an asymptotic confidence interval of η.

Fact 3. $\dfrac{\eta-X_{(n)}}{\hat{\eta}-F^{-1}(1-\frac{1}{n};\hat{\eta})}$ converges to a random variable X in distribution, where X follows an exponential distribution with mean 1, and

$F(x;\eta)=\dfrac{1-e^{-x}}{1-e^{-\eta}}$, $0<x<\eta$.

2-2. Estimation of mean paramemter

From (2.1), the MLE $\hat{\mu}$ and the MME $\tilde{\mu}$ of μ are

$$\hat{\mu}=1-X_{(n)}\cdot e^{-X_{(n)}}/(1-e^{-X_{(n)}}),\ \text{and}$$

$$\tilde{\mu}=\frac{1}{n}\sum_{i=1}^{n}X_i\ ,\ \text{respectively.} \tag{2.8}$$

From Fact 1 and the pdf (2.2) of $X_{(n)}$, the expectation and variance of $\hat{\mu}$ can be represented by the integral forms in Fact 1.

$$E(\hat{\mu})=1-n\cdot\tau^{-n}(\eta)\cdot I(1,n-2,2;\eta)\ \text{and}$$

$$Var(\hat{\mu})=n\cdot\tau^{-n}(\eta)\cdot I(2,n-3,3;\eta)-n^2\tau^{-2n}(\eta)\cdot I^2(1,n-2,2;\eta). \tag{2.9}$$

From the result (2.1), the MME $\tilde{\mu}$ is an unbiased estimator of μ and

$$Var(\tilde{\mu})=\frac{1}{n}[1-\eta^2e^{-\eta}/\tau^2(\eta)]. \tag{2.10}$$

Table 2 shows numerical values of MSE of the MLE and MME of mean parameter in the right truncated exponential distribution by using the results (2.9) and (2.10) when η=1 and n=10(10)50.

Through Table 2, the MLE is more efficient than the MME of mean parameter in the right truncated exponential distribution when η=1.

Table 2. MSE of the MLE and MME of mean parameter in the right truncated exponential distribution(units are 10^{-4}).

Sample size	MLE	MME
10	4.213	7.814
20	1.507	4.127
30	0.626	2.611
40	0.402	1.915
50	0.222	1.603

While, since the mean parameter $\mu=\mu(\eta)=2-\dfrac{\eta}{1-e^{-\eta}}$, $\eta>0$, $\lim_{\eta\to 0}\mu(\eta)=1$

and $\frac{d}{d\eta}\mu(\eta) = \frac{e^{-\eta}(1+\eta)-1}{(1-e^{-\eta})^2}$ is negative, and hence $\mu(\eta)$ is a monotone decreasing function of η.

Therefore, inference on η is equivalence to inference on $\mu(\eta)$(see McCool(1991)), and so it's sufficient for us to estimate η instead of estimating $\mu(\eta)$(see section 2-1). Here we could recommend the ordinary jackknife estimator of η to estimate $\mu(\eta)$ by the result in Section 2-1.

2-3. Estimation of right tail probability

Based on the pdf (1.1) of the right truncated exponential distribution, the right tail probability is

$$R(t) = P(X > t) = 1 - \tau(t) \cdot \tau^{-1}(\eta).$$

Since $\frac{d}{d\eta}R(t;\eta) = \tau(t) \cdot e^{-\eta} \cdot \tau^{-2}(\eta)$ is positive , $R(t;\eta)$ is a monotone increas- ing function of η.

Therefore, inference on η is equivalence to inference on $R(t;\eta)$(see McCool(1991)), and so it's sufficient for us to estimate η instead of estimating $R(t;\eta)$(see section 2-1).

Here we could recommend the ordinary jackknife estimator of η to estimate $R(t;\eta)$ by the result in Section 2-1.

Next we shall consider estimation of $F(t;\eta) = \frac{1-e^{-t}}{1-e^{-\eta}}$ instead of estimating $R(t;\eta)$=1- $F(t;\eta)$. From the MLE of η, the MLE $\widehat{F}(t)$ of F(t) is given by:

$$\widehat{F}(t) = \tau(t) \cdot (1-e^{-X_{(n)}})^{-1},\ 0<t<\eta.$$

To evaluate the expectation and variance of $\widehat{F}(t)$, from the integral of Fact 1 we need the following result.

Fact 4. $I(0, n-k, 1; \eta) = \frac{1}{n-k+1}(1-e^{-\eta})^{n-k+1}$, k=1,2,....,n.

Proof. By transformation of variable in the definition of I(0,n-k,1 ; η), we can obtain

$$I(0, n-k, 1; \eta) = B(1, n-k+1) - B_{e^{-\eta}}(1, n-k+1),$$

where B(a,b) and B_x(a,b) are beta and incomplete beta functions, respectively.

By the formulas (6.62) and (6.6.4) in Abramowitz & Stegun(1972), we can obtain

$$B_{e^{-\eta}}(1, n-k+1) = \frac{1}{n-k+1}(1-(1-(1-e^{-\eta})^{n-k+1}).$$

Since B(a,b)= $\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$, we have done.

From Fact 4 and the pdf (2.2) of $X_{(n)}$, we can obtain the expectation and variance of $\widehat{F}(t)$:

$$E(\widehat{F}(t)) = \frac{n}{n-1} F(t) \quad \text{and}$$

$$Var(\widehat{F}(t)) = \frac{n}{(n-1)^2(n-2)} F^2(t). \tag{2.11}$$

From the expectation in the result (2.11),

$$\widetilde{F}(t) \equiv \frac{n-1}{n} \tau(t) \cdot (1-e^{-X_{(n)}})^{-1}$$

is an unbiased estimator of F(t), and its variance is

$$Var(\widetilde{F}(t)) = \frac{1}{n(n-2)} F^2(t). \tag{2.12}$$

From the results (2.11) and (2.12), $MSE(\widetilde{F}(t))$ is less than $MSE(\widehat{F}(t))$.

Fact 5. An unbiased estimator $\widetilde{F}(t)$ is more efficient than the MLE $\widehat{F}(t)$ in a sense of MSE.

For nonparametric estimation F(t), we have well known the followings in Rohatgi(1976)

$$\overline{F}(t) = \#\{ X_i ; X_i \le t, \text{ for i=1,2,...,n } \} /n$$

$$E(\overline{F}(t)) = F(t) \text{ and } Var(\overline{F}(t)) = F(t)(1-F(t))/n. \tag{2.13}$$

From the results (2.12) and (2.13), we can obtain the following result.

Fact 6. An unbiased estimator $\widetilde{F}(t)$ is more efficient than the nonparametric estimator $\overline{F}(t)$ in a sense of MSE for all t satisfying F(t)< $\frac{n-2}{n-1}$, vice versa for else t.

2-4. An application(see David (1981, p.171)

To test $H_o ; \eta_1 = \eta_2$ against a general alternative, assume two independent random variables $X_1, X_2, \ldots, X_m \sim f(x;\eta_1)$ and $Y_1, Y_2, \ldots, Y_n \sim f(x;\eta_1)$, where $f(x;\eta) = \tau^{-1}(\eta) \cdot e^{-x}$, $0 < x < \eta$, $\tau(\eta) = 1 - e^{-\eta}$.

Let $Z_{(m+n)} = \max\{ X_1, \ldots, X_m, Y_1, \ldots, Y_n \}$. Then the test statistics

$\Lambda(X_i{}'s, Y_j{}'s) = -2\log\frac{\tau^m(X_{(m)})\cdot\tau^n(Y_{(n)})}{\tau^{m+n}(Z_{(m+n)})}$ follows χ^2-distribution with df 2.

where $X_{(m)}$ and $Y_{(n)}$ are the largest order statistics of $X_1, X_2, \ldots, X_m \sim f(x;\eta_1)$ and $Y_1, Y_2, \ldots, Y_n \sim f(x;\eta_1)$, respectively.

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