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OPPOSITE SKEW COPAIRED HOPF ALGEBRAS

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Abstract. Let A be a Hopf algebra with a linear form $\sigma\colon k \to A\otimes A$, which is convolution invertible, such that $\sigma_{21}(\Delta\otimes id)\tau(\sigma(1)) = \sigma_{32}(id\otimes\Delta)\tau(\sigma(1))$. We define Hopf algebras, $(A_\sigma, m, u, \Delta_\sigma, \varepsilon, S_\sigma)$. If B and C are opposite skew copaired Hopf algebras and $A = B\otimes_k C$ then we find Hopf algebras, $(A_{[\sigma]}, m_B\otimes m_C, u_B\otimes u_C, \Delta_{[\sigma]}, \varepsilon_B\otimes\varepsilon_C, S_{[\sigma]})$. Let H be a finite dimensional commutative Hopf algebra with dual basis $\{h_i\}$ and $\{h_i^*\}$, and let $A = H^{op}\otimes H^*$. We show that if we define $\sigma\colon k \to H^{op}\otimes H^*$ by $\sigma(1) = \sum h_i\otimes h_i^*$ then $(A_{[\sigma]}, m_A, u_A, \Delta_{[\sigma]}, \varepsilon_A, S_{[\sigma]})$ is the dual space of Drinfeld double, $D(H)^*$, as Hopf algebra.

Let k be a field. All unadorned tensor products are over k and all maps are k-linear. For $f, g \in Hom(C, A)$, where C is a coalgebra and A is an algebra, $f * g$ is its convolution product $m_A(f\otimes g)\Delta_C$. Let $\tau\colon V\otimes W \to W\otimes V$ be the twist map given by $\tau(v\otimes w) = w\otimes v$. We use the sigma notation [4]; for $c\in C$, $\Delta(c) = \sum c_{(1)}\otimes c_{(2)}$. If $(B, m_B, u_B, \Delta_B, \varepsilon_B, S_B)$ and $(C, m_C, u_C, \Delta_C, \varepsilon_C, S_C)$ are Hopf algebras then $(B\otimes_k C, m_{B\otimes C}, u_B\otimes u_C, \Delta_{B\otimes C}, \varepsilon_B\otimes\varepsilon_C, S_B\otimes S_C)$ has the Hopf algebra structure of the usual tensor product of algebras and the usual tensor product of coalgebras.

Proposition 1. *If $(A, m, u, \Delta, \varepsilon, S)$ is a Hopf algebra with a convolution invertible linear form, $\sigma : k \to A\otimes A$, such that*

$$(*)\qquad \begin{aligned}&\sum \sigma_{i2}(1)(\sigma_{j2}(1))_{(1)}\otimes\sigma_{i1}(1)(\sigma_{j2}(1))_{(2)}\otimes\sigma_{j1}(1)\\ &\quad=\sum \sigma_{j2}(1)\otimes\sigma_{i2}(1)(\sigma_{j1}(1))_{(1)}\otimes\sigma_{i1}(1)(\sigma_{j1}(1))_{(2)},\end{aligned}$$

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where $\sigma(1)=\sum\sigma_{i1}(1)\otimes\sigma_{i2}(1)$ *and* $\sigma^{-1}(1)=\sum\sigma^{-1}_{j1}(1)\otimes\sigma^{-1}_{j2}(1)$, *then*

(1) $\sum\sigma_{i1}(1)\sigma^{-1}_{j1}(1)\otimes\sigma_{i2}(1)\sigma^{-1}_{j2}(1)=1\otimes 1$,
$\sum\sigma^{-1}_{j1}(1)\sigma_{i1}(1)\otimes\sigma^{-1}_{j2}(1)\sigma_{i2}(1)=1\otimes 1$.

(2) $\sum\varepsilon(\sigma_1(1))\sigma_2(1)=1, \quad \sum\sigma_1(1)\varepsilon(\sigma_2(1))=1$.

(3) $\sum(\sigma^{-1}_{i2}(1))_{(1)}\sigma^{-1}_{j2}(1)\otimes(\sigma^{-1}_{i2}(1))_{(2)}\sigma^{-1}_{j1}(1)\otimes\sigma^{-1}_{i1}(1)$
$=\sum\sigma^{-1}_{i2}(1)\otimes(\sigma^{-1}_{i1}(1))_{(1)}\sigma^{-1}_{j2}(1)\otimes(\sigma^{-1}_{i1}(1))_{(2)}\sigma^{-1}_{j1}(1)$.

(4) $\sum\varepsilon(\sigma^{-1}_1(1))\sigma^{-1}_2(1)=1, \quad \sum\sigma^{-1}_1(1)\varepsilon(\sigma^{-1}_2(1))=1$.

(5) $\sum\sigma_{i1}(1)\varepsilon(\sigma^{-1}_{j1}(1))\otimes\sigma^{-1}_{j2}(1)\varepsilon(\sigma_{i2}(1))=1\otimes 1$,
$\sum\sigma^{-1}_{j1}(1)\varepsilon(\sigma_{i1}(1))\otimes\sigma_{i2}(1)\varepsilon(\sigma^{-1}_{j2}(1))=1\otimes 1$.

(6) $\sum S(\sigma_{i2}(1))\sigma_{i1}(1)S^{-1}(\sigma^{-1}_{j2}(1))\sigma^{-1}_{j1}(1)=1$,
$\sum S^{-1}(\sigma^{-1}_{j2}(1))\sigma^{-1}_{j1}(1)S(\sigma_{i2}(1))\sigma_{i1}(1)=1$.

Proof. (1) : Since σ^{-1} is a convolution inverse of σ,

$1\otimes 1 = u_{A\otimes A}\varepsilon_k(1) = (\sigma*\sigma^{-1})(1) = (m\circ(\sigma\otimes\sigma^{-1})\circ\Delta)(1) = \sigma(1)\sigma^{-1}(1)$,

$1\otimes 1 = u_{A\otimes A}\varepsilon_k(1) = (\sigma^{-1}*\sigma)(1) = \sigma^{-1}(1)\sigma(1)$.

(2) : Applying $id\otimes\varepsilon\otimes id$ to both sides of $(*)$, we have

$\sum\sigma_{i2}(1)(\sigma_{j2}(1))_{(1)}\otimes\varepsilon(\sigma_{i1}(1))\varepsilon((\sigma_{j2}(1))_{(2)})\otimes\sigma_{j1}(1)$
$=\sum\sigma_{j2}(1)\otimes\varepsilon(\sigma_{i2}(1))\varepsilon((\sigma_{j1}(1))_{(1)})\otimes\sigma_{i1}(1)(\sigma_{j1}(1))_{(2)}$.

Therefore,

$$\sum\sigma_{i2}(1)\sigma_{j2}(1)\otimes\varepsilon(\sigma_{i1}(1))\otimes\sigma_{j1}(1)=\sum\sigma_{j2}(1)\otimes\varepsilon(\sigma_{i2}(1))\otimes\sigma_{i1}(1)\sigma_{j1}(1).$$

Multiplying $\sum\sigma^{-1}_{k2}(1)\otimes 1\otimes\sigma^{-1}_{k1}(1)$ on the right,

$$(\sum\varepsilon(\sigma_{i1}(1))\sigma_{i2}(1))\otimes 1\otimes 1 = 1\otimes 1\otimes(\sum\sigma_{i1}(1)\varepsilon(\sigma_{i2}(1))),$$

by (1). Thus

$$\sum\sigma_{i1}(1)\varepsilon(\sigma_{i2}(1))=1=\sum\varepsilon(\sigma_{i1}(1))\sigma_{i2}(1).$$

(3): The condition $(*)$ can be expressed as

$$\sigma_{21}(\Delta \otimes id)\tau(\sigma(1)) = \sigma_{32}(id \otimes \Delta)\tau(\sigma(1)),$$

where $\sigma_{21} = \sum \sigma_2(1) \otimes \sigma_1(1) \otimes 1$ and $\sigma_{32} = \sum 1 \otimes \sigma_2(1) \otimes \sigma_1(1)$.

Since

$$1 = (\Delta \otimes id)(\tau(\sigma^{-1} * \sigma)(1)) = [(\Delta \otimes id)\tau(\sigma^{-1}(1))][(\Delta \otimes id)\tau(\sigma(1))]$$

and

$$[(\Delta \otimes id)\tau(\sigma^{-1}(1))][(\Delta \otimes id)\tau(\sigma(1))] = 1,$$

it follows that

$$(\Delta \otimes id)\tau(\sigma^{-1}(1)) = [(\Delta \otimes id)\tau(\sigma(1))]^{-1}.$$

Similarly,

$$(id \otimes \Delta)\tau(\sigma^{-1}(1)) = [(id \otimes \Delta)\tau(\sigma(1))]^{-1}.$$

Hence

$$\begin{aligned}
(\Delta \otimes id)\tau(\sigma^{-1}(1)) \cdot \sigma_{21}^{-1} &= [(\Delta \otimes id)\tau(\sigma(1))]^{-1} \cdot \sigma_{21}^{-1} \\
&= [\sigma_{21} \cdot (\Delta \otimes id)\tau(\sigma(1))]^{-1} \\
&= [\sigma_{32} \cdot (id \otimes \Delta)\tau(\sigma(1))]^{-1} \\
&= [(id \otimes \Delta)\tau(\sigma(1))]^{-1} \cdot \sigma_{32}^{-1} \\
&= (id \otimes \Delta)\tau(\sigma^{-1}(1)) \cdot \sigma_{32}^{-1},
\end{aligned}$$

where third equality follows from $(*)$.

(4) : It is proved in a same way of the proof of (2), using (3).

(5) : From (2) and (4).

(6) : Using (1) and (5),

$\sum S(\sigma_{i2}(1))\sigma_{i1}(1)S^{-1}(\sigma_{j2}^{-1}(1))\sigma_{j1}^{-1}(1)$
$= \sum S(\sigma_{i2}(1))\sigma_{i1}(1)S^{-1}(\sigma_{j2}^{-1}(1))\sigma_{j1}^{-1}(1)\varepsilon(\sigma_{l1}^{-1}(1)\sigma_{k1}(1))\varepsilon(\sigma_{l2}^{-1}(1)$
$\sigma_{k2}(1))$
$= \sum S(\sigma_{i2}(1)\varepsilon(\sigma_{l2}^{-1}(1)))\sigma_{i1}(1)\varepsilon(\sigma_{l1}^{-1}(1))S^{-1}(\sigma_{j2}^{-1}(1)\varepsilon(\sigma_{k2}(1)))\sigma_{j1}^{-1}(1)$
$\varepsilon(\sigma_{k1}(1))$
$= 1.$
Similarly, $\sum S^{-1}(\sigma_2^{-1}(1))\sigma_1^{-1}(1)S(\sigma_2(1))\sigma_1(1) = 1.$ □

Example 1. Every Hopf algebra satisfies the condition $(*)$ by setting $\sigma(1) = 1 \otimes 1$. Let kZ_2 be written multiplicatively as $\{1, g\}$, and assume *char* $k \neq 2$. Now let

$$\sigma(1) = \frac{1}{2}(1 \otimes 1 + 1 \otimes g + g \otimes 1 - g \otimes g).$$

Then one can easily check that σ satisfies the property $(*)$.

We will define a new comultiplication Δ_σ and a new antipode S_σ on a Hopf algebra $(A, m, u, \Delta, \varepsilon, S)$. Then the following shows that $(A, m, u, \Delta_\sigma, \varepsilon, S_\sigma)$ is a Hopf algebra.

Theorem 1. *Let $(A, m, u, \Delta, \varepsilon, S)$ be a Hopf algebra with an invertible linear form, $\sigma : k \to A \otimes A$. Assume that σ satisfies the condition $(*)$. Define $A_\sigma = A$, as an algebra. If we define the coproduct Δ_σ and the antipode S_σ by*
$\Delta_\sigma(a) = \sum \sigma_{i2}(1)a_{(1)}\sigma_{j2}^{-1}(1) \otimes \sigma_{i1}(1)a_{(2)}\sigma_{j1}^{-1}(1),$
$S_\sigma(a) = \sum S(\sigma_{i2}(1))\sigma_{i1}(1)S(a)S^{-1}(\sigma_{j2}^{-1}(1))\sigma_{j1}^{-1}(1), \quad a \in A,$ *then* $(A_\sigma, m, u, \Delta_\sigma, \varepsilon, S_\sigma)$ *is a Hopf algebra.*

Proof. By the property $(*)$ and Proposition 1, (3),
$(\Delta_\sigma \otimes id)\Delta_\sigma(a)$
$= (\Delta_\sigma \otimes id)(\sum \sigma_{i2}(1)a_{(1)}\sigma_{j2}^{-1}(1) \otimes \sigma_{i1}(1)a_{(2)}\sigma_{j1}^{-1}(1))$
$= \sum \Delta_\sigma(\sigma_{i2}(1)a_{(1)}\sigma_{j2}^{-1}(1)) \otimes \sigma_{i1}(1)a_{(2)}\sigma_{j1}^{-1}(1)$
$= \sum \sigma_{k2}(1)(\sigma_{i2}(1)a_{(1)}\sigma_{j2}^{-1}(1))_{(1)}\sigma_{l2}^{-1}(1) \otimes \sigma_{k1}(1)(\sigma_{i2}(1)a_{(1)}\sigma_{j2}^{-1}(1))_{(2)}$

$\sigma_{l1}^{-1}(1) \otimes \sigma_{i1}(1)a_{(2)}\sigma_{j1}^{-1}(1)$

$= \sum \sigma_{k2}(1)(\sigma_{i2}(1))_{(1)}(a_{(1)})_{(1)}\ (\sigma_{j2}^{-1}(1))_{(1)}\sigma_{l2}^{-1}(1) \otimes \sigma_{k1}(1)(\sigma_{i2}(1))_{(2)}$

$(a_{(1)})_{(2)}(\sigma_{j2}^{-1}(1))_{(2)}\sigma_{l1}^{-1}(1) \otimes\ \sigma_{i1}(1)a_{(2)}\sigma_{j1}^{-1}(1)$

$= \sum \sigma_{i2}(1)a_{(1)}\sigma_{j2}^{-1}(1) \otimes \sigma_{k2}(1)(\sigma_{i1}(1))_{(1)}\ (a_{(2)})_{(1)}\ (\sigma_{j1}^{-1}(1))_{(1)}\sigma_{l2}^{-1}(1)$

$\otimes\ \sigma_{k1}(1)(\sigma_{i1}(1))_2\ (a_{(2)})_{(2)}\ (\sigma_{j1}^{-1}(1))_{(2)}\sigma_{l1}^{-1}(1)$

$= \sum \sigma_{i2}(1)a_{(1)}\sigma_{j2}^{-1}(1)\ \otimes\ \sigma_{k2}(1)(\sigma_{i1}(1)a_{(2)}\sigma_{j1}^{-1}(1))_{(1)}\sigma_{l2}^{-1}(1)$

$\otimes\ \sigma_{k1}(1)(\sigma_{i1}(1)a_{(2)}\sigma_{j1}^{-1}(1))_{(2)}\sigma_{l1}^{-1}(1)$

$= \sum \sigma_{i2}(1)a_{(1)}\sigma_{j2}^{-1}(1) \otimes \Delta_\sigma(\sigma_{i1}(1)a_{(2)}\sigma_{j1}^{-1}(1))$

$= (id\ \otimes\ \Delta_\sigma)(\sum \sigma_{i2}(1)a_{(1)}\sigma_{j2}^{-1}(1)\ \otimes\ \sigma_{i1}(1)a_{(2)}\sigma_{j1}^{-1}(1))$

$= (id\ \otimes\ \Delta_\sigma)\Delta_\sigma(a)\ \ a, b \in A.$

Thus

$$(\Delta_\sigma \otimes id)\Delta_\sigma = (id\ \otimes\ \Delta_\sigma)\Delta_\sigma.$$

By Proposition 1, (2) and (4),

$\sum \varepsilon(\sigma_{i2}(1)\ a_{(1)}\ \sigma_{j2}^{-1}(1))\sigma_{i1}(1)\ a_{(2)}\sigma_{j1}^{-1}(1)$

$= \sum \varepsilon(\sigma_{i2}(1))\varepsilon(a_{(1)})\varepsilon(\sigma_{j2}^{-1}(1))\sigma_{i1}(1)\ a_{(2)}\sigma_{j1}^{-1}(1) = a, \ \ a \in A.$

Thus

$$(\varepsilon \otimes id)\Delta_\sigma = 1 \otimes .$$

Similarly,

$$(id \otimes \varepsilon)\Delta_\sigma = \otimes 1.$$

Hence $(A_\sigma, \Delta_\sigma, \varepsilon)$ is coassociative and counitary. By Proposition 1, (1),

$$
\begin{aligned}
&\Delta_\sigma(ab)\\
&= \sigma_{i2}(1)(ab)_{(1)}\sigma_{j2}^{-1}(1)\otimes\sigma_{i1}(1)(ab)_{(2)}\sigma_{j1}^{-1}(1)\\
&= \sigma_{i2}(1)a_{(1)}b_{(1)}\sigma_{j2}^{-1}(1)\otimes\sigma_{i1}(1)a_{(2)}b_{(2)}\sigma_{j1}^{-1}(1)\\
&= \sum\sigma_{i2}(1)a_{(1)}\sigma_{j2}^{-1}(1)\sigma_{k2}(1)b_{(1)}\sigma_{l2}^{-1}(1)\otimes\sigma_{i1}(1)a_{(2)}\sigma_{j1}^{-1}(1)\sigma_{k1}(1)\\
&\qquad b_{(2)}\sigma_{l1}^{-1}(1)\\
&= (\sum\sigma_{i2}(1)a_{(1)}\sigma_{j2}^{-1}(1)\otimes\sigma_{i1}(1)a_{(2)}\sigma_{j1}^{-1}(1))(\sum\sigma_{k2}(1)b_{(1)}\sigma_{l2}^{-1}(1)\\
&\qquad \otimes\sigma_{k1}(1)b_{(2)}\sigma_{l1}^{-1}(1))\\
&= \Delta_\sigma(a)\Delta_\sigma(b), a, b\in A.
\end{aligned}
$$

Thus

$$\Delta_\sigma(ab) = \Delta_\sigma(a)\Delta_\sigma(b),\quad a, b\in A.$$

And $\Delta_\sigma(1) = \sigma_2(1)1_A\sigma_2^{-1}(1)\otimes\sigma_1(1)1_A\sigma_1^{-1}(1) = 1_A\otimes 1_A$.

Hence Δ_σ and ε are algebra homomorphisms, i.e., $(A_\sigma, m, u, \Delta_\sigma, \varepsilon)$ is a bialgebra. We can show that S_σ is an antipode of the bialgebra $(A_\sigma, m, u, \Delta_\sigma, \varepsilon)$:

$$
\begin{aligned}
&(S_\sigma * id)(a)\\
&= m\circ(S_\sigma\otimes id)(\Delta_\sigma(a))\\
&= \sum S_\sigma(\sigma_{k2}(1)a_{(1)}\sigma_{l2}^{-1}(1))(\sigma_{k1}(1)a_{(2)}\sigma_{l1}^{-1}(1))\\
&= \sum S(\sigma_{i2}(1))\sigma_{i1}(1)S(\sigma_{k2}(1)a_{(1)}\sigma_{l2}^{-1}(1))S^{-1}(\sigma_{j2}^{-1}(1))\sigma_{j1}^{-1}(1)\sigma_{k1}(1)\\
&\qquad a_{(2)}\sigma_{l1}^{-1}(1)\\
&= \sum S(\sigma_{i2}(1))\sigma_{i1}(1)S(\sigma_{l2}^{-1}(1))S(a_{(1)})S(\sigma_{k2}(1))S^{-1}(\sigma_{j2}^{-1}(1))\sigma_{j1}^{-1}(1)\\
&\qquad \sigma_{k1}(1)a_{(2)}\sigma_{l1}^{-1}(1)\\
&= \sum S(\sigma_{i2}(1))\sigma_{i1}(1)S(\sigma_{l2}^{-1}(1))S(a_{(1)})S(\sigma_{k2}(1))S^{-1}(\sigma_{j2}^{-1}(1))\sigma_{j1}^{-1}(1)\\
&\qquad \sigma_{k1}(1)a_{(2)}\sigma_{l1}^{-1}(1)\varepsilon(\sigma_{m1}(1)\sigma_{n1}^{-1}(1))\varepsilon(\sigma_{m2}(1)\sigma_{n2}^{-1}(1))\\
&= \sum S(\sigma_{i2}(1))\varepsilon(\sigma_{n2}^{-1}(1))\sigma_{i1}(1)\varepsilon(\sigma_{n1}^{-1}(1))S(\sigma_{l2}^{-1}(1))S(a_{(1)})S(\sigma_{k2}(1))\\
&\qquad S^{-1}(\sigma_{j2}^{-1}(1)\varepsilon(\sigma_{m2}(1)))\sigma_{j1}^{-1}(1)\varepsilon(\sigma_{m1}(1))\sigma_{k1}(1)a_{(2)}\sigma_{l1}^{-1}(1)\\
&= \sum S(\sigma_{l2}^{-1}(1))S(a_{(1)})S(\sigma_{k2}(1))\sigma_{k1}(1)a_{(2)}\sigma_{l1}^{-1}(1)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum S(\sigma_{l2}^{-1}(1))S(a_{(1)})S(\sigma_{k2}(1))\sigma_{k1}(1)a_{(2)}\sigma_{l1}^{-1}(1)\varepsilon(\sigma_{j1}^{-1}(1)\sigma_{i1}(1)) \\
&\qquad \varepsilon(\sigma_{j2}^{-1}(1)\sigma_{i2}(1)) \\
&= \sum S(\sigma_{l2}^{-1}(1))\varepsilon(\sigma_{i2}(1))S(a_{(1)})S(\sigma_{k2}(1))\varepsilon(\sigma_{j2}^{-1}(1))\sigma_{k1}(1)\varepsilon(\sigma_{j1}^{-1}(1)) \\
&\qquad a_{(2)}\sigma_{l1}^{-1}(1)\varepsilon(\sigma_{i1}(1)) \\
&= \sum S(a_{(1)})a_{(2)} \\
&= \varepsilon(a)1_A \\
&= (u\circ\varepsilon)(a), \quad a\in A, \text{ by Proposition 1, (5). Similarly,}
\end{aligned}
$$

$$\sum(id * S_\sigma) = u\circ\varepsilon.$$

Hence S_σ is an antipode of A_σ. Therefore, $(A_\sigma, m, u, \Delta_\sigma, \varepsilon, S_\sigma)$ is a Hopf algebra. □

Let B and C be bialgebras. Recall the definition of skew copairing in [3], which is the dual concept of skew pairing in [5].

Definition 1. Let B and C be bialgebras. We say that B and C are *skew copaired* if there exists a k-linear map $\sigma\colon k\to B\otimes C$, $\sigma(1)=\sum\sigma_1(1)\otimes\sigma_2(1)$ *(called the skew copairing)* such that the diagrams below commute :

$$
\begin{array}{ccccc}
k & \xrightarrow{\sigma} & B\otimes C & \xrightarrow{id\otimes\Delta_C} & B\otimes C\otimes C \\
{\scriptstyle\Delta_k}\downarrow & & & & \uparrow{\scriptstyle m_B\otimes id\otimes id} \\
k\otimes k & \xrightarrow{\sigma\otimes\sigma} & B\otimes C\otimes B\otimes C & \xrightarrow{id\otimes\tau\otimes id} & B\otimes B\otimes C\otimes C
\end{array}
$$

$$
\begin{array}{ccccc}
k & \xrightarrow{\sigma} & B\otimes C & \xrightarrow{\Delta_B\otimes id} & B\otimes B\otimes C \\
{\scriptstyle\Delta_k}\downarrow & & & & \uparrow{\scriptstyle id\otimes id\otimes m_C} \\
k\otimes k & \xrightarrow{\sigma\otimes\sigma} & B\otimes C\otimes B\otimes C & \xrightarrow{id\otimes\tau\otimes id} & B\otimes B\otimes C\otimes C
\end{array}
$$

$$
\begin{array}{ccccc}
k & \xrightarrow{id} & k & \xrightarrow{id} & k \\
\downarrow{\scriptstyle 1_B\otimes} & & \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle\otimes 1_C} \\
B\otimes k & \xleftarrow[id\otimes\varepsilon_C]{} & B\otimes C & \xrightarrow{\varepsilon_B\otimes id} & k\otimes C
\end{array}
$$

The commutativity of the diagrams above can be expressed equationally in the following way:

$$\sum \sigma_1(1) \otimes (\sigma_2(1))_{(1)} \otimes (\sigma_2(1))_{(2)} = \sum \sigma_{i1}(1)\sigma_{j1}(1) \otimes \sigma_{i2}(1) \otimes \sigma_{j2}(1) \cdots (1)$$

$$\sum (\sigma_1(1))_{(1)} \otimes (\sigma_1(1))_{(2)} \otimes \sigma_2(1) = \sum \sigma_{i1}(1) \otimes \sigma_{j1}(1) \otimes \sigma_{i2}(1)\sigma_{j2}(1) \cdots (2)$$

$$\sum \varepsilon(\sigma_1(1)) \otimes \sigma_2(1) = 1 \otimes 1, \ \sum \sigma_1(1) \otimes \varepsilon(\sigma_2(1)) = 1 \otimes 1 \cdots (3)$$

where $\sigma(1) = \sum \sigma_1(1) \otimes \sigma_2(1) \in B \otimes C$.

EXAMPLE 2. ([3]) Let H be a finite dimensional Hopf algebra. Define σ as the coevaluation map

$$\sigma \ : \ k \to H^{op} \otimes H^*, \quad \sigma(1) \ = \ \sum h_i \ \otimes \ h_i^*,$$

where $\{h_i\}, \{h_i^*\}$ are dual bases of H and H^*. Then we see that σ is a skew copairing on H^{op} and H^*.

Recall the definition of quasitrianular Hopf algebra in [1] and [2]. If B and C are skew copaired Hopf algebras with an invertible skew copairing then some properties are similar to those in quasitriangular Hopf algebras.

PROPOSITION 2. *If B and C are skew copaired Hopf algebras with an invertible skew copairing, $\sigma\colon k \to B \otimes C$, which is convolution invertible where $\sigma^{-1}(1) = \sum \sigma_1^{-1}(1) \otimes \sigma_2^{-1}(1)$, then*

(1) $\sigma(1)\sigma^{-1}(1) = 1_B \otimes 1_C, \ \sigma^{-1}(1)\sigma(1) = 1_B \otimes 1_C$

(2) $\sum (\sigma_1^{-1}(1))_{(1)} \otimes (\sigma_1^{-1}(1))_{(2)} \otimes \sigma_2^{-1}(1)$
$= \sum \sigma_{i1}^{-1}(1) \otimes \sigma_{j1}^{-1}(1) \otimes \sigma_{j2}^{-1}(1)\sigma_{i2}^{-1}(1)$

(3) $\sum \sigma_1^{-1}(1) \otimes (\sigma_2^{-1}(1))_{(1)} \otimes (\sigma_2^{-1}(1))_{(2)}$
$= \sum \sigma_{i1}^{-1}(1)\sigma_{j1}^{-1}(1) \otimes \sigma_{j2}^{-1}(1) \otimes \sigma_{i2}^{-1}(1)$

(4) $\sum \varepsilon_B(\sigma_1^{-1}(1))\sigma_2^{-1}(1) = 1, \ \sum \sigma_1^{-1}(1)\varepsilon_C(\sigma_2^{-1}(1)) = 1.$

Proof. (1) : Since σ^{-1} is a convolution inverse of σ,

$$\begin{aligned} 1\otimes 1 &= u_{B\otimes C}\varepsilon_k(1) = (\sigma * \sigma^{-1})(1) \\ &= (m_{B\otimes C}\circ(\sigma\otimes\sigma^{-1})\circ\Delta)(1) \\ &= \sigma(1)\sigma^{-1}(1). \end{aligned}$$

Similarly,

$$\sigma^{-1}(1)\sigma(1) = 1\otimes 1.$$

(2) : Since

$$\begin{aligned} 1 &= (\Delta_B\otimes id)((\sigma * \sigma^{-1})(1)) \\ &= [(\Delta_B\otimes id)\sigma(1)][(\Delta_B\otimes id)\sigma^{-1}(1)], \end{aligned}$$

we have

$$[(\Delta_B\otimes id)\sigma(1)]^{-1} = (\Delta_B\otimes id)\sigma^{-1}(1).$$

By Definition 1, (2), $(\Delta_B\otimes id)\sigma(1) = \sigma_{13}\sigma_{23}$, where $\sigma_{13} = \sum\sigma_1(1)\otimes 1_B\otimes\sigma_2(1)$ and $\sigma_{23} = \sum 1_B\otimes\sigma_1(1)\otimes\sigma_2(1)$ in $B\otimes B\otimes C$, so $(\Delta_B\otimes id)\sigma^{-1}(1) = [(\Delta_B\otimes id)\sigma(1)]^{-1} = \sigma_{23}^{-1}\sigma_{13}^{-1}$.

Thus

$$\begin{aligned} (\Delta_B\otimes id)\sigma^{-1}(1) &= \sigma_{23}^{-1}\sigma_{13}^{-1} \\ &= \sum\sigma_{i1}^{-1}(1)\otimes\sigma_{j1}^{-1}(1)\otimes\sigma_{j2}^{-1}(1)\sigma_{i2}^{-1}(1). \end{aligned}$$

(3) : Similarly,

$$\begin{aligned} (id\otimes\Delta_C)\sigma^{-1}(1) &= \sigma_{13}^{-1}\sigma_{12}^{-1} \\ &= \sum\sigma_{i1}^{-1}(1)\sigma_{j1}^{-1}(1)\otimes\sigma_{j2}^{-1}(1)\otimes\sigma_{i2}^{-1}(1). \end{aligned}$$

(4) : Since

$$\begin{aligned} 1\otimes 1 &= (id\otimes\varepsilon_C)((\sigma * \sigma^{-1})(1)) \\ &= (id\otimes\varepsilon_C)(\sigma(1)\sigma^{-1}(1)) \\ &= [(id\otimes\varepsilon_C)\sigma(1)][(id\otimes\varepsilon_C)\sigma^{-1}(1)], \end{aligned}$$

$1 \otimes 1 = [(id \otimes \varepsilon_C)\sigma(1)]^{-1} = (id \otimes \varepsilon_C)\sigma^{-1}(1) = \sum \sigma_1^{-1}(1) \otimes \varepsilon_C(\sigma_2^{-1}(1)$ by Definition 1,(3). Similarly, $\sum \varepsilon_B(\sigma_1^{-1}(1))\sigma_2^{-1}(1) = 1$. □

LEMMA 1. *If B and C are skew copaired bialgebras with a skew copairing $\sigma\colon k \to B \otimes C$, then B^{op} and C^{cop} are skew copaired bialgebras with same skew copairing $\sigma\colon k \to B^{op} \otimes C^{cop}$.*

Proof. Since B and C are skew copaired bialgebras

$$\begin{aligned}
&((id \otimes \Delta_C^{op})\sigma)(1)\\
&= (id \otimes \tau_C \circ \Delta_C)\sigma(1)\\
&= (id \otimes \tau_C)((id \otimes \Delta_C)\sigma(1))\\
&= (id \otimes \tau_C)((m_B \otimes id \otimes id)(id \otimes \tau \otimes id)(\sigma \otimes \sigma)(1))\\
&= (id \otimes \tau_C)(\sum \sigma_{i1}(1)\sigma_{j1}(1) \otimes \sigma_{i2}(1) \otimes \sigma_{j2}(1))\\
&= \sum \sigma_{i1}(1)\sigma_{j1}(1) \otimes \sigma_{j2}(1) \otimes \sigma_{i2}(1)\\
&= (m_B^{op} \otimes id \otimes id)(id \otimes \tau \otimes id)(\sigma \otimes \sigma)(1),
\end{aligned}$$

and

$$((\Delta_B \otimes id)\sigma)(1) = (id \otimes id \otimes m_C)(id \otimes \tau \otimes id)(\sigma \otimes \sigma)(1)$$

by definition. □

EXAMPLE 3. Let H be a finite dimensional Hopf algebra. As in Example 2, the coevaluation map $\sigma : k \to H^{op} \otimes H^*$, $\sigma(1) = \sum h_i \otimes h_i^*$ is a skew copairing on H^{op} and H^*. By Lemma 1, $\sigma\colon k \to (H^{op})^{op} \otimes (H^*)^{cop}$ is a skew copairing on H and $(H^*)^{cop}$.

PROPOSITION 3. *Let B and C be skew copaired Hopf algebras with bijective antipodes S_B, S_C respectively and with an invertible skew copairing $\sigma\colon k \to B \otimes C$. Then*

$$\sigma^{-1}(1) = \sum \sigma_1(1) \otimes S_C(\sigma_2(1)), \quad \sigma(1) = \sum S_B^{-1}(\sigma_1^{-1}(1)) \otimes \sigma_2^{-1}(1).$$

Proof. By the Definition 1, (1) and (3),

$$
\begin{aligned}
&\sigma(1)(id \otimes S_C)(\sigma(1)) \\
&= \sum \sigma_{i1}(1)\sigma_{j1}(1) \otimes \sigma_{i2}(1)S_C(\sigma_{j2}(1)) \\
&= (id \otimes m)(id \otimes id \otimes S_C)(\sum \sigma_{i1}(1)\sigma_{j1}(1) \otimes \sigma_{i2}(1) \otimes \sigma_{j2}(1)) \\
&= (id \otimes m)(id \otimes id \otimes S_C)(\sum \sigma_1(1) \otimes (\sigma_2(1))_{(1)} \otimes (\sigma_2(1))_{(2)}) \\
&= (id \otimes m)(id \otimes id \otimes S_C)((id \otimes \Delta_C)(\sigma(1)) \\
&= (id \otimes \varepsilon)(\sigma(1)) \\
&= 1 \otimes 1.
\end{aligned}
$$

Therefore

$$\sigma^{-1}(1) = \sum \sigma_1(1) \otimes S_C(\sigma_2(1)).$$

Since S^{-1} is an anti-coalgebra morphism

$$
\begin{aligned}
&\sigma^{-1}(1)(S_B^{-1} \otimes id)(\sigma^{-1}(1)) \\
&= \sum \sigma_{j1}^{-1}(1)S_B^{-1}(\sigma_{i1}^{-1}(1)) \otimes \sigma_{j2}^{-1}(1)\sigma_{i2}^{-1}(1) \\
&= (m_B \otimes id)(id \otimes S_B^{-1} \otimes id)(\sum \sigma_{j1}^{-1}(1) \otimes \sigma_{i1}^{-1}(1) \otimes \sigma_{j2}^{-1}(1)\sigma_{i2}^{-1}(1)) \\
&= (m_B \otimes id)(id \otimes S_B^{-1} \otimes id)(\sum (\sigma_1^{-1}(1))_{(2)} \otimes (\sigma_1^{-1}(1))_{(1)} \otimes \sigma_2^{-1}(1)) \\
&= \sum (\sigma_1^{-1}(1))_{(2)} S_B^{-1}((\sigma_1^{-1}(1))_{(1)}) \otimes \sigma_2^{-1}(1) \\
&= (\varepsilon \otimes id)(\sigma^{-1}(1)) \\
&= 1 \otimes 1,
\end{aligned}
$$

by Proposition 2, (2) and (4). Therefore

$$\sigma(1) = \sum S_B^{-1}(\sigma_1^{-1}(1)) \otimes \sigma_2^{-1}(1),$$

as desired. □

COROLLARY 1. *Let B and C be skew copaired Hopf algebras with bijective antipodes S_B, S_C respectively and with an invertible skew copairing $\sigma\colon k \to B \otimes C$. Then*

$$\sigma^{-1}(1) = \sum \sigma_1(1) \otimes S_C^{-1}(\sigma_2(1)).$$

Proof. If $(B, m_B, u_B, \Delta_B, \varepsilon_B, S_B)$ and $(C, m_C, u_C, \Delta_C, \varepsilon_C, S_C)$ are Hopf algebras then $(B^{op}, m_B^{op}, u_B, \Delta_B, \varepsilon_B, S_B^{-1})$ and $(C^{cop}, m_C, u_C, \Delta_C^{op}, \varepsilon_C, S_C^{-1})$ are Hopf algebras with antipode S_B^{-1}, and S_C^{-1} respectively [1]. By Lemma 1, B^{op} and C^{cop} are skew copaired Hopf algebras with skew copairing σ. By Proposition 3, $\sigma^{-1}(1) = \sum \sigma_1(1) \otimes S_C^{-1}(\sigma_2(1))$. □

EXAMPLE 4. Let $(H, m, u, \Delta, \varepsilon, S)$ be a finite dimensional Hopf algebra with basis $\{h_i\}$. Define σ as the coevaluation map $\sigma : k \to H^{op} \otimes H^*$, $\sigma(1) = \sum h_i \otimes h_i^*$ where $\{h_i^*\}$ is the dual basis of $\{h_i\}$. Then we see that σ is an invertible skew copairing on H^{op} and H^*. By Proposition 3 and Corollary 1, $\sigma^{-1}(1) = \sum h_i \otimes S^{-1}(h_i^*)$ and $\sigma(1) = \sum S^{-1}(\sigma_1^{-1}(1)) \otimes \sigma_2^{-1}(1)$. So $\sigma(1) = \sum S^{-1}(h_i) \otimes S^{-1}(h_i^*)$. Hence $\sigma^{-1}(1) = \sum \sigma_1(1) \otimes S(\sigma_2(1)) = \sum S^{-1}(h_i) \otimes S(S^{-1}(h_i^*)) = \sum S^{-1}(h_i) \otimes h_i^*$.

DEFINITION 2. Bialgebras B and C are *opposite skew copaired* if there exists a skew coparing $\sigma\colon k \to B \otimes C$ (is called the *opposite skew coparing on B and C*) such that $\sum \sigma_{i1}(1)\sigma_{j1}(1) = \sum \sigma_{j1}(1)\sigma_{i1}(1)$ where $\sigma(1) = \sum \sigma_{i1}(1) \otimes \sigma_{i2}(1) \in B \otimes C$.

The opposite skew copairity can be expressed equationally in the following way:

$$\sum \sigma_1(1) \otimes (\sigma_2(1))_{(1)} \otimes (\sigma_2(1))_{(2)} = \sum \sigma_{j1}(1)\sigma_{i1}(1) \otimes \sigma_{i2}(1) \otimes \sigma_{j2}(1) \cdots (1)$$

$$\sum (\sigma_1(1))_{(1)} \otimes (\sigma_1(1))_{(2)} \otimes \sigma_2(1) = \sum \sigma_{i1}(1) \otimes \sigma_{j1}(1) \otimes \sigma_{i2}(1)\sigma_{j2}(1) \cdots (2)$$

$$\sum \varepsilon(\sigma_1(1)) \otimes \sigma_2(1) = 1 \otimes 1, \quad \sum \sigma_1(1) \otimes \varepsilon(\sigma_2(1)) = 1 \otimes 1 \cdots (3)$$

where $\sigma(1) = \sum \sigma_{i1}(1) \otimes \sigma_{i2}(1) \in B \otimes C$.

EXAMPLE 5. Let H_4 be Sweedler's four-dimensional Hopf algebra over k, and assume *char* $k \neq 2$. As an algebra over k, H_4 is generated by g and x with relations

$$g^2 = 1, \;\; x^2 = 0, \;\; xg = -gx.$$

The coalgebra structure and antipode are determine by

$$\Delta(g) \;=\; g \otimes g, \qquad \Delta(x) \;=\; (x \otimes g) + (1 \otimes x),$$

$$\varepsilon(g) = 1, \quad \varepsilon(x) = 0, \quad S(g) = g = g^{-1}, \quad S(x) = gx.$$

H_4 has a basis $\{1, g, x, gx\}$. Let kZ_2 be written multiplicatively as $\{1, a\}$, and assume *char* $k \neq 2$. Now let

$$\sigma(1) = \frac{1}{2}(1 \otimes 1 + 1 \otimes a + g \otimes 1 - g \otimes a) \in H_4 \otimes kZ_2.$$

Then one can easily check that σ is an invertible opposite skew copairing of $(H_4,\; kZ_2)$ with $\sigma^{-1} = \sigma$. Therefore (H_4, kZ_2) are opposite skew copaired Hopf algebras.

PROPOSITION 4. *Let σ be an invertible opposite skew copairing on (B, C) and let $A = B \otimes_k C$, the usual tensor product of algebras. Then the linear form $[\sigma]$ on A defined by*

$$[\sigma] \colon k \mapsto (B \otimes C) \otimes (B \otimes C),$$

$1 \mapsto \sum [\sigma]_{(1)} \otimes [\sigma]_{(2)} = \sum (\sigma_1(1) \otimes 1) \otimes (1 \otimes \sigma_2(1))$ *satisfies* $(*)$ *with inverse* $[\sigma]^{-1}(1) = \sum (\sigma_1^{-1}(1) \otimes 1) \otimes (1 \otimes \sigma_2^{-1}(1))$.

Proof. By the Definition 2, (1) and (2),

$$\begin{aligned}
&\sum[\sigma]_{i2}(1)([\sigma]_{j2}(1))_{(1)}\otimes[\sigma]_{i1}(1)([\sigma]_{j2}(1))_{(2)}\otimes[\sigma]_{j1}(1)\\
&=\sum(1\otimes\sigma_{i2}(1))(1\otimes\sigma_{j2}(1))_{(1)}\otimes(\sigma_{i1}(1)\otimes 1)(1\otimes\sigma_{j2}(1))_{(2)}\otimes(\sigma_{j1}(1)\otimes 1)\\
&=\sum(1\otimes\sigma_{i2}(1))(1\otimes(\sigma_{j2}(1))_{(1)})\otimes(\sigma_{i1}(1)\otimes 1)(1\otimes(\sigma_{j2}(1))_{(2)})\otimes(\sigma_{j1}(1)\otimes 1)\\
&=\sum(1\otimes\sigma_{i2}(1))(1\otimes\sigma_{j2}(1))\otimes(\sigma_{i1}(1)\otimes 1)(1\otimes\sigma_{k2}(1))\otimes(\sigma_{k1}(1)\sigma_{j1}(1)\otimes 1)\\
&=\sum(1\otimes\sigma_{i2}(1)\sigma_{j2}(1))\otimes(1\otimes\sigma_{k2}(1))(\sigma_{i1}(1)\otimes 1)\otimes(\sigma_{k1}(1)\otimes 1)(\sigma_{j1}(1)\otimes 1)\\
&=\sum(1\otimes\sigma_{j2}(1))\otimes(1\otimes\sigma_{k2}(1))((\sigma_{j1}(1))_{(1)}\otimes 1)\otimes(\sigma_{k1}(1)\otimes 1)((\sigma_{j1}(1))_{(2)}\otimes 1)\\
&=\sum(1\otimes\sigma_{j2}(1))\otimes(1\otimes\sigma_{i2}(1))(\sigma_{j1}(1)\otimes 1)_{(1)}\otimes(\sigma_{i1}(1)\otimes 1)(\sigma_{j1}(1)\otimes 1)_{(2)}\\
&=\sum[\sigma]_{j2}(1)\otimes[\sigma]_{i2}(1)([\sigma]_{j1}(1))_{(1)}\otimes[\sigma]_{i1}(1)([\sigma]_{j1}(1))_{(2)},
\end{aligned}$$

as desired. □

The following theorem shows that $(A_{[\sigma]}, m, u, \Delta_{[\sigma]}, \varepsilon, S_{[\sigma]})$ are new Hopf algebras.

THEOREM 2. *Let $(B, m_B, u_B, \Delta_B, \varepsilon_B)$ and $(C, m_C, u_C, \Delta_C, \varepsilon_C)$ be opposite skew copaired bialgebras with an invertible opposite skew copairing σ. Let $[\sigma]$ be the linear form in Proposition 4. There exists a bialgebra structure on the vector space $B\otimes_k C$, such that $A_{[\sigma]} = B\otimes C$, the usual tensor product of algebras as algebra with unit $u_B\otimes u_C$ and its comultiplication is given by*

$$\Delta_{[\sigma]}(b\otimes c)=\sum(b_{(1)}\otimes\sigma_2(1)c_{(1)}\sigma_2^{-1}(1))\otimes(\sigma_1(1)b_{(2)}\sigma_1^{-1}(1)\otimes c_{(2)}),$$

and its counit by

$$\varepsilon(b\otimes c)=\varepsilon_B(b)\varepsilon_C(c)$$

for all $b\in B, c\in C$.

If the bialgebras B and C have antipodes, respectively denote S_B and S_C, then the bialgebra $(A_{[\sigma]}, m_B\otimes m_C, u_B\otimes u_C, \Delta_{[\sigma]}, \varepsilon)$ is a Hopf *algebra with antipode $S_{[\sigma]}$ given by*

$$S_{[\sigma]}(b\otimes c)=\sum\sigma_1(1)S_B(b)\sigma_1^{-1}(1)\otimes S_C(\sigma_2(1))S_C(c)S_C^{-1}(\sigma_2^{-1}(1)).$$

Proof. By Theorem 1, and by Proposition 4, $(A_{[\sigma]},m,u,\Delta_{[\sigma]},\varepsilon)$ is a bialgebra, if B and C are Hopf algebras, then $(A_{[\sigma]},m,u,\Delta_{[\sigma]},\varepsilon,S_{[\sigma]})$ is a Hopf algebra since $A_{[\sigma]}=B\otimes C$ is an algebra by the definition of A_σ. And if let $[\sigma]\colon k\to(B\otimes C)\otimes(B\otimes C),\ 1\mapsto\sum[\sigma]_1(1)\otimes[\sigma]_2(1)$ then

$$\begin{aligned}
&\Delta_{[\sigma]}(b\otimes c)\\
&=\sum[\sigma]_2(1)(b\otimes c)_{(1)}[\sigma]_2^{-1}(1)\otimes[\sigma]_1(1)(b\otimes c)_{(2)}[\sigma]_1^{-1}(1)\\
&=\sum(1\otimes\sigma_2(1))(b_{(1)}\otimes c_{(1)})(1\otimes\sigma_2^{-1}(1))\otimes(\sigma_1(1)\otimes 1)(b_{(2)}\otimes c_{(2)})(\sigma_1^{-1}(1)\otimes 1)\\
&=\sum(b_{(1)}\otimes\sigma_2(1)c_{(1)}\sigma_2^{-1}(1))\otimes(\sigma_1(1)b_{(2)}\sigma_1^{-1}(1)\otimes c_{(2)}),
\end{aligned}$$

and

$$\begin{aligned}
&S_{[\sigma]}(b\otimes c)\\
&=\sum S([\sigma]_2(1))[\sigma]_1(1)S(b\otimes c)S^{-1}([\sigma]_2^{-1}(1))[\sigma]_1^{-1}(1)\\
&=\sum S(1\otimes\sigma_2(1))(\sigma_1(1)\otimes 1)(S(b)\otimes S(c))S^{-1}(1\otimes\sigma_2^{-1}(1))(\sigma_1^{-1}(1)\otimes 1)\\
&=\sum(S_B(1)\otimes S_C(\sigma_2(1)))(\sigma_1(1)\otimes 1)(S_B(b)\otimes S_C(c))(S_B^{-1}(1)\otimes S_C^{-1}(\sigma_2^{-1}(1)))\\
&\qquad(\sigma_1^{-1}(1)\otimes 1)\\
&=\sum\sigma_1(1)S_B(b)\sigma_1^{-1}(1)\otimes S_C(\sigma_2(1))S_C(c)S_C^{-1}(\sigma_2^{-1}(1)),
\end{aligned}$$

as desired. □

Recall the definition of coquasitriangular Hopf algebra in [2]. The dual space of Drinfeld double $D(H)^*$ is an interesting object for the study of coquasitriangular Hopf algebras. We are going to show that $(A_{[\sigma]},m_A,u_A,\Delta_{[\sigma]},\varepsilon_A,S_{[\sigma]})$ is the dual space of Drinfeld double, $D(H)^*$, as Hopf algebra. So we find a constuction of coquasitriangular Hopf algebras when H is a finite dimensional commutative Hopf algebra.

THEOREM 3. *Let $(H, m, u, \Delta, \varepsilon, S)$ be a finite dimensional commutative Hopf algebra with dual basis $\{h_i\}$ and $\{h_i^*\}$. Let $A = H^{op} \otimes H^*$. If we define $\sigma\colon k \to H^{op} \otimes H^*$ by $\sigma(1) = \sum h_i \otimes h_i^*$ then the Hopf algebra $(A_{[\sigma]}, m_A, u_A, \Delta_{[\sigma]}, \varepsilon_A, S_{[\sigma]})$ is the dual space of Drinfeld double, $D(H)^*$, as Hopf algebra.*

Proof. Since H is commutative, the skew copairing σ is an opposite skew coparing of bialgebras H^{op} and H^* with inverse $\sigma^{-1}(1) = \sum S^{-1}(h_i) \otimes h_i^*$ as in Example 4. As an algebra, $A = H^{op} \otimes H^*$ has the same product with $D(H)^*$, the usual tensor product of algebras. The comultiplication on $A_{[\sigma]}$ is

$$\begin{aligned}&\Delta_{[\sigma]}(h \otimes f)\\ &= \sum (h_{(1)} \otimes \sigma_2(1) f_{(1)} \sigma_2^{-1}(1)) \otimes (\sigma_1^{-1}(1) h_{(2)} \sigma_1(1) \otimes f_{(2)})\\ &= \sum (h_{(1)} \otimes h_s^* f_{(1)} h_t^*) \otimes (S^{-1}(h_t) h_{(2)} h_s \otimes f_{(2)})\end{aligned}$$

in $H^{op} \otimes H^*$ by Theorem 2. Therefore the bialgebra $A_{[\sigma]}$ is the dual space of Drinfeld double, $D(H)^*$, as bialgebra. By Proposition 3, and Corollary 1, $\sum h_t \otimes S(h_t^*) = \sigma^{-1}(1) = \sum h_i \otimes S^{-1}(h_i^*)$ and $\sigma(1) = \sum S^{-1}(\sigma_1^{-1}(1)) \otimes \sigma_2^{-1}(1)$. Hence $\sigma(1) = \sum S^{-1}(h_i) \otimes S^{-1}(h_i^*)$. The antipode on $A_{[\sigma]}$ is

$$\begin{aligned}&S_{[\sigma]}(h \otimes f)\\ &= \sum \sigma_1^{-1}(1) S(h) \sigma_1(1) \otimes S(\sigma_2(1)) S(f) S^{-1}(\sigma_2^{-1}(1))\\ &= \sum h_t S(h) S^{-1}(h_i) \otimes S(S^{-1}(h_i^*)) S(f) S^{-1}(S(h_t^*))\\ &= \sum h_t S(h) S^{-1}(h_i) \otimes h_i^* S(f) h_t^*,\end{aligned}$$

by Theorem 2. Therefore the Hopf algebra $A_{[\sigma]}$ is the dual space of Drinfeld double, $D(H)^*$, as Hopf algebra. □

COROLLARY 2. *Let* $(H, m, u, \Delta, \varepsilon, S)$ *be a finite dimensional commutative Hopf algebra with dual basis* $\{h_i\}$ *and* $\{h_i^*\}$*. Let* $A = H^{op} \otimes H^*$*. If we define* $\sigma\colon k \to H^{op} \otimes H^*$ *by* $\sigma(1) = \sum h_i \otimes h_i^*$ *then the Hopf algebra* $(A_{[\sigma]}, m_A, u_A, \Delta_{[\sigma]}, \varepsilon_A, S_{[\sigma]})$ *is coquasitriangular.*

Proof. By [2, Proposition 10.3.14], the dual space of Drinfeld double, $D(H)^*$ is a coquasitriangular Hopf algebra with braiding $< | >$: $D(H)^* \otimes D(H)^* \to k$ is given by $< h \otimes f | k \otimes g >= \varepsilon(k) f(1) < g, h >$, as desired. □

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