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STABILITY OF CYCLIC FUNCTIONAL EQUATIONS IN BANACH MODULES OVER A C^*–ALGEBRA

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Abstract. We prove the Cauchy-Rassias stability of cyclic functional equations in Banach modules over a unital C^*-algebra.

1. Introduction

Recently, T. Trif [6, Theorem 2.1] proved that, for vector spaces V and W, a mapping $f : V \to W$ with $f(0) = 0$ satisfies the functional equation

$$n\ {}_{n-2}C_{k-2} f(\frac{x_1 + \cdots + x_n}{n}) + {}_{n-2}C_{k-1} \sum_{i=1}^{n} f(x_i) = k \sum_{1 \le i_1 < \cdots < i_k \le n} f(\frac{x_{i_1} + \cdots + x_{i_k}}{k})$$

for all $x_1, \cdots, x_n \in V$ if and only if the mapping $f : V \to W$ satisfies the additive Cauchy equation $f(x + y) = f(x) + f(y)$ for all $x, y \in V$.

In [2], the author conjectured the following, and gave a partial answer for the conjecture.

Conjecture. Let p be an integer greater than 1. A mapping $f : V \to W$ with $f(0) = 0$ satisfies the functional equation

$$p^n f(\frac{x_1 + \cdots + x_{p^n}}{p^n}) + (pk - p) \sum_{i=1}^{p^{n-1}} f(\frac{x_{pi-p+1} + \cdots + x_{pi}}{p}) = k \sum_{i=1}^{p^n} f(\frac{x_i + \cdots + x_{i+k-1}}{k}) \quad \text{(C)}$$

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for all $x_1 = x_{p^n+1}, \cdots, x_{k-1} = x_{p^n+k-1}, x_k, \cdots, x_{p^n} \in V$ if and only if the mapping $f : V \to W$ satisfies the additive Cauchy equation $f(x+y) = f(x) + f(y)$ for all $x, y \in V$.

Throughout this paper, let A be a unital C^*-algebra with norm $|\cdot|$ and $\mathcal{U}(A)$ the unitary group of A. Let ${}_A\mathcal{B}$ and ${}_A\mathcal{C}$ be left Banach A-modules with norms $||\cdot||$ and $\|\cdot\|$, respectively. Let d, r and p be positive integers and b an integer greater than 1.

The main purpose of this paper is to prove the Cauchy-Rassias stability of the functional equation (C) in Banach modules over a unital C^*-algebra for a special case.

2. Cyclic functional equations

In this section, we are going to solve the conjecture for a special case.

THEOREM 2.1. *A mapping $f : V \to W$ with $f(0) = 0$ satisfies the functional equation*

$$(bp)^n f(\frac{x_1 + \cdots + x_{(bp)^n}}{(bp)^n}) + bp(bp-2) \sum_{i=1}^{(bp)^{n-1}} f(\frac{x_{bpi-bp+1} + \cdots + x_{bpi}}{bp})$$

$$= (bp-1) \sum_{i=1}^{(bp)^n} f(\frac{x_i + \cdots + x_{i+bp-2}}{bp-1}) \tag{2.i}$$

for all $x_1 = x_{(bp)^n+1}, \cdots, x_{bp-2} = x_{(bp)^n+bp-2}, x_{bp-1}, \cdots, x_{(bp)^n} \in V$ if and only if the mapping $f : V \to W$ satisfies the additive Cauchy equation $f(x+y) = f(x) + f(y)$ for all $x, y \in V$.

Proof. Assume a mapping $f : V \to W$ satisfies (2.i). Put $x_{bi-b+1} = x_1, x_{bi-b+2}, \cdots, x_{bi} = x_b$ in (2.i) for all $i = 1, \cdots, p(bp)^{n-1}$. Then

$$(bp)^n f(\frac{x_1 + \cdots + x_b}{b}) + (bp-2)(bp)^n f(\frac{x_1 + \cdots + x_b}{b})$$

$$= (bp-1)(bp)^{n-1} \sum_{i=1}^{b} p f(\frac{\sum_{m=1}^{b} p x_m - x_i}{bp-1}) \tag{2.1}$$

for all $x_1, \cdots, x_b \in V$. Put $x_1 = x$ and $x_2 = \cdots = x_b = y$ in (2.1). Then we get

$$\begin{aligned}(bp-1)(bp)^n f(\frac{x+(b-1)y}{b}) \\ =(bp-1)(bp)^{n-1}(pf(\frac{(p-1)x+(b-1)py}{bp-1}) \\ + (b-1)pf(\frac{px+(bp-p-1)y}{bp-1}))\end{aligned} \tag{2.2}$$

for all $x, y \in V$. Put $x = 0$ in (2.2). Replacing x and y by $-(bp-p-1)x$ and px in (2.2), respectively, we get

$$(bp-1)(bp)^n f(\frac{x}{b}) = (bp-1)(bp)^{n-1} pf(x)$$

for all $x \in V$. So we get

$$f(\frac{1}{b}x) = \frac{1}{b} f(x) \tag{2.3}$$

for all $x \in V$.

By (2.3), it follows from (2.1) that

$$\begin{aligned}&(bp-1)p(bp)^{n-1} f(x_1 + \cdots + x_b) \\ &= (bp-1)(bp)^{n-1} \sum_{i=1}^{b} pf(\frac{\sum_{m=1}^{b} px_m - x_i}{bp-1})\end{aligned}$$

for all $x_1, \cdots, x_b \in V$. Let $v_i = \frac{\sum_{m=1}^{b} px_m - x_i}{bp-1}$ for all $i = 1, \cdots, b$. Then we get

$$\begin{aligned}(bp-1)p(bp)^{n-1} \sum_{i=1}^{b} f(v_i) &= (bp-1)p(bp)^{n-1} f(x_1 + \cdots + x_b) \\ &= (bp-1)p(bp)^{n-1} f(v_1 + \cdots + v_b)\end{aligned}$$

for all $v_1, \cdots, v_b \in V$. Let $v_3 = \cdots = v_b = 0$. Then we get $f(v_1) + f(v_2) = f(v_1 + v_2)$ for all $v_1, v_2 \in V$. So the mapping $f : V \to W$ satisfies the additive Cauchy equation $f(x + y) = f(x) + f(y)$ for all $x, y \in V$.

The converse is obvious. □

3. Stability of cyclic functional equations in Banach modules over a C^*-algebra associated with its unitary group

We are going to prove the Cauchy-Rassias stability of the functional equation (C) in Banach modules over a unital C^*-algebra associated with its unitary group for a special case.

THEOREM 3.1. *Let $f : {}_A\mathcal{B} \to {}_A\mathcal{C}$ be a mapping with $f(0) = 0$ for which there exists a function $\varphi : {}_A\mathcal{B}^{(bp)^n} \to [0, \infty)$ such that*

$$\widetilde{\varphi}(x_1, \cdots, x_{(bp)^n}) := \sum_{j=0}^{\infty} (\frac{d}{(bp)^n r})^j \varphi((\frac{(bp)^n r}{d})^j x_1, \cdots, (\frac{(bp)^n r}{d})^j x_{(bp)^n}) < \infty \tag{3.i}$$

$$\begin{aligned} \|D_u f(x_1, \cdots, x_{(bp)^n})\| = \| & \frac{d}{r} u f(\frac{r x_1 + \cdots + r x_{(bp)^n}}{d}) \\ & + bp(bp-2) \sum_{i=1}^{(bp)^{n-1}} u f(\frac{x_{bpi-bp+1} + \cdots + x_{bpi}}{bp}) \\ & - (bp-1) \sum_{i=1}^{(bp)^n} f(\frac{u x_i + \cdots + u x_{i+bp-2}}{bp-1}) \| \\ & \le \varphi(x_1, \cdots, x_{(bp)^n}) \end{aligned} \tag{3.ii}$$

for all $x_1 = x_{(bp)^n+1}, \cdots, x_{bp-2} = x_{(bp)^n+bp-2}, x_{bp-1}, \cdots, x_{(bp)^n} \in {}_A\mathcal{B}$ and all $u \in \mathcal{U}(A)$. Then there exists a unique A-linear mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{C}$ such that

$$\|f(x) - T(x)\| \le \frac{1}{(bp)^n} \widetilde{\varphi}(x, \cdots, x) \tag{3.iii}$$

for all $x \in {}_A\mathcal{B}$.

Proof. Put $u = 1 \in \mathcal{U}(A)$. Let $x_1 = \cdots = x_{(bp)^n} = x$ in (3.ii). Then we get

$$\| \frac{d}{r} \cdot f(\frac{(bp)^n r}{d} x) + (bp-2)(bp)^n f(x) - (bp-1)(bp)^n f(x) \| \le \varphi(x, \cdots, x)$$

for all $x \in {}_A\mathcal{B}$. So one can obtain

$$\|f(x) - \frac{d}{(bp)^n r} f(\frac{(bp)^n r}{d} x)\| \le \frac{1}{(bp)^n} \varphi(x, \cdots, x),$$

hence

$$\begin{aligned}\|(\frac{d}{(bp)^n r})^j f((\frac{(bp)^n r}{d})^j x) - (\frac{d}{(bp)^n r})^{j+1} f((\frac{(bp)^n r}{d})^{j+1} x)\| \\ \le \frac{1}{(bp)^n} (\frac{d}{(bp)^n r})^j \varphi((\frac{(bp)^n r}{d})^j x, \cdots, (\frac{(bp)^n r}{d})^j x)\end{aligned}$$

for all $x \in {}_A\mathcal{B}$. So we get

$$\begin{aligned}&\|f(x) - (\frac{d}{(bp)^n r})^j f((\frac{(bp)^n r}{d})^j x)\| \\ &\le \frac{1}{(bp)^n} \sum_{m=0}^{j-1} (\frac{d}{(bp)^n r})^m \varphi((\frac{(bp)^n r}{d})^m x, \cdots, (\frac{(bp)^n r}{d})^m x) \qquad (3.1)\end{aligned}$$

for all $x \in {}_A\mathcal{B}$.

Let x be an element in ${}_A\mathcal{B}$. For positive integers l and m with $l > m$,

$$\begin{aligned}\|(\frac{d}{(bp)^n r})^l f((\frac{(bp)^n r}{d})^l x) - (\frac{d}{(bp)^n r})^m f((\frac{(bp)^n r}{d})^m x)\| \\ \le \frac{1}{(bp)^n} \sum_{j=m}^{l-1} (\frac{d}{(bp)^n r})^j \varphi((\frac{(bp)^n r}{d})^j x, \cdots, (\frac{(bp)^n r}{d})^j x),\end{aligned}$$

which tends to zero as $m \to \infty$ by (3.i). So $\{(\frac{d}{(bp)^n r})^j f((\frac{(bp)^n r}{d})^j x)\}$ is a Cauchy sequence for all $x \in {}_A\mathcal{B}$. Since ${}_A\mathcal{C}$ is complete, the sequence $\{(\frac{d}{(bp)^n r})^j f((\frac{(bp)^n r}{d})^j x)\}$ converges for all $x \in {}_A\mathcal{B}$. We can define a mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{C}$ by

$$T(x) = \lim_{j\to\infty} (\frac{d}{(bp)^n r})^j f((\frac{(bp)^n r}{d})^j x) \qquad (3.2)$$

for all $x \in {}_A\mathcal{B}$.

By (3.i) and (3.2), we get

$$\begin{aligned}
&\|D_1T(x_1,\cdots,x_{(bp)^n})\| \\
&\quad = \lim_{j\to\infty}(\frac{d}{(bp)^n r})^j \|D_1 f((\frac{(bp)^n r}{d})^j x_1,\cdots,(\frac{(bp)^n r}{d})^j x_{(bp)^n})\| \\
&\quad \le \lim_{j\to\infty}(\frac{d}{(bp)^n r})^j \varphi((\frac{(bp)^n r}{d})^j x_1,\cdots,(\frac{(bp)^n r}{d})^j x_{(bp)^n}) = 0
\end{aligned}$$

for all $x_1,\cdots,x_{(bp)^n} \in {}_A\mathcal{B}$. Hence $D_1T(x_1,\cdots,x_{(bp)^n}) = 0$ for all $x_1,\cdots,x_{(bp)^n} \in {}_A\mathcal{B}$. Put $x_1 = \cdots = x_{(bp)^n} = x$ in $D_1T(x_1,\cdots,x_{(bp)^n})$. Then

$$\frac{d}{r}T(\frac{(bp)^n r}{d}x) + (bp-2)(bp)^n T(x) - (bp-1)(bp)^n T(x) = 0$$

for all $x \in {}_A\mathcal{B}$. So

$$T(\frac{(bp)^n r}{d}x) = \frac{(bp)^n r}{d}T(x)$$

for all $x \in {}_A\mathcal{B}$. Since

$$\begin{aligned}
\frac{d}{r}T(\frac{rx_1+\cdots+rx_{(bp)^n}}{d}) &= \frac{d}{r}T(\frac{(bp)^n r(x_1+\cdots+x_{(bp)^n})}{(bp)^n d}) \\
&= \frac{d}{r}\frac{(bp)^n r}{d}T(\frac{x_1+\cdots+x_{(bp)^n}}{(bp)^n}) \\
&= (bp)^n T(\frac{x_1+\cdots+x_{(bp)^n}}{(bp)^n})
\end{aligned}$$

for all $x_1,\cdots,x_{(bp)^n} \in {}_A\mathcal{B}$,

$$\begin{aligned}
(bp)^n T(\frac{x_1+\cdots+x_{(bp)^n}}{(bp)^n}) + bp(bp-2)\sum_{i=1}^{(bp)^{n-1}} T(\frac{x_{bpi-bp+1}+\cdots+x_{bpi}}{bp}) \\
= (bp-1)\sum_{i=1}^{(bp)^n} T(\frac{x_i+\cdots+x_{i+bp-2}}{bp-1})
\end{aligned}$$

for all $x_1 = x_{(bp)^n+1}, \cdots, x_{bp-2} = x_{(bp)^n+bp-2}, x_{bp-1}, \cdots, x_{(bp)^n} \in {}_A\mathcal{B}$. By Theorem 2.1, T is additive. Moreover, by passing to the limit in (3.1) as $j \to \infty$, we get (3.iii).

Now let $L : {}_A\mathcal{B} \to {}_A\mathcal{C}$ be another additive mapping satisfying

$$\|f(x) - L(x)\| \le \frac{1}{(bp)^n}\widetilde{\varphi}(x, \cdots, x)$$

for all $x \in {}_A\mathcal{B}$.

$$\begin{aligned}
\|T(x) - L(x)\| =& (\frac{d}{(bp)^n r})^j \|T((\frac{(bp)^n r}{d})^j x) - L((\frac{(bp)^n r}{d})^j x)\| \\
\le& (\frac{d}{(bp)^n r})^j \|T((\frac{(bp)^n r}{d})^j x) - f((\frac{(bp)^n r}{d})^j x)\| \\
&+ (\frac{d}{(bp)^n r})^j \|f((\frac{(bp)^n r}{d})^j x) - L((\frac{(bp)^n r}{d})^j x)\| \\
\le& \frac{2}{(bp)^n}(\frac{d}{(bp)^n r})^j \widetilde{\varphi}((\frac{(bp)^n r}{d})^j x, \cdots, (\frac{(bp)^n r}{d})^j x),
\end{aligned}$$

which tends to zero as $j \to \infty$ by (3.i). Thus $T(x) = L(x)$ for all $x \in {}_A\mathcal{B}$. This proves the uniqueness of T.

By the assumption, for each $u \in \mathcal{U}(A)$,

$$\begin{aligned}
&(\frac{d}{(bp)^n r})^j \|D_u f((\frac{(bp)^n r}{d})^j x, \cdots, (\frac{(bp)^n r}{d})^j x)\| \\
&\qquad \le (\frac{d}{(bp)^n r})^j \varphi((\frac{(bp)^n r}{d})^j x, \cdots, (\frac{(bp)^n r}{d})^j x)
\end{aligned}$$

for all $x \in {}_A\mathcal{B}$, and

$$(\frac{d}{(bp)^n r})^j \|D_u f((\frac{(bp)^n r}{d})^j x, \cdots, (\frac{(bp)^n r}{d})^j x)\| \to 0$$

as $j \to \infty$ for all $x \in {}_A\mathcal{B}$. So

$$D_u T(x, \cdots, x) = \lim_{j\to\infty}(\frac{d}{(bp)^n r})^j D_u f((\frac{(bp)^n r}{d})^j x, \cdots, (\frac{(bp)^n r}{d})^j x) = 0$$

for all $u \in \mathcal{U}(A)$ and all $x \in {}_A\mathcal{B}$. Hence

$$D_u T(x, \cdots, x) = \frac{d}{r} \cdot uT(\frac{(bp)^n r}{d} x) + (bp-2)(bp)^n uT(x) - (bp-1)(bp)^n T(ux) = 0$$

for all $u \in \mathcal{U}(A)$ and all $x \in {}_A\mathcal{B}$. So

$$uT(x) = T(ux)$$

for all $u \in \mathcal{U}(A)$ and all $x \in {}_A\mathcal{B}$.

Now let $a \in A$ $(a \neq 0)$ and M an integer greater than $4|a|$. Then

$$|\frac{a}{M}| = \frac{1}{M}|a| < \frac{|a|}{4|a|} = \frac{1}{4} < 1 - \frac{2}{3} = \frac{1}{3}.$$

By [1, Theorem 1], there exist three elements $u_1, u_2, u_3 \in \mathcal{U}(A)$ such that $3\frac{a}{M} = u_1 + u_2 + u_3$. And $T(x) = T(3 \cdot \frac{1}{3}x) = 3T(\frac{1}{3}x)$ for all $x \in {}_A\mathcal{B}$. So $T(\frac{1}{3}x) = \frac{1}{3}T(x)$ for all $x \in {}_A\mathcal{B}$. Thus

$$\begin{aligned} T(ax) &= T(\frac{M}{3} \cdot 3\frac{a}{M}x) = M \cdot T(\frac{1}{3} \cdot 3\frac{a}{M}x) = \frac{M}{3}T(3\frac{a}{M}x) \\ &= \frac{M}{3}T(u_1x + u_2x + u_3x) = \frac{M}{3}(T(u_1x) + T(u_2x) + T(u_3x)) \\ &= \frac{M}{3}(u_1 + u_2 + u_3)T(x) = \frac{M}{3} \cdot 3\frac{a}{M}T(x) \\ &= aT(x) \end{aligned}$$

for all $x \in {}_A\mathcal{B}$. Obviously, $T(0x) = 0T(x)$ for all $x \in {}_A\mathcal{B}$. Hence

$$T(ax + by) = T(ax) + T(by) = aT(x) + bT(y)$$

for all $a, b \in A$ and all $x, y \in {}_A\mathcal{B}$. So the unique additive mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{C}$ is an A-linear mapping, as desired. □

THEOREM 3.2. *Let* $f : {}_A\mathcal{B} \to {}_A\mathcal{C}$ *be a mapping with* $f(0) = 0$ *for which there exists a function* $\varphi : {}_A\mathcal{B}^{(bp)^n} \to [0, \infty)$ *such that*

$$\widetilde{\varphi}(x_1, \cdots, x_{(bp)^n}) := \sum_{j=0}^{\infty} (\frac{d}{(bp-1)r})^j \varphi((\frac{(bp-1)r}{d})^j x_1, \cdots,$$

$$(\frac{(bp-1)r}{d})^j x_{(bp)^n}) \quad < \infty \tag{3.iv}$$

$$\|D_u f(x_1, \cdots, x_{(bp)^n})\| = \|(bp)^n u f(\frac{x_1 + \cdots + x_{(bp)^n}}{(bp)^n})$$

$$+ bp(bp-2) \sum_{i=1}^{(bp)^{n-1}} u f(\frac{x_{bpi-bp+1} + \cdots + x_{bpi}}{bp}) \tag{3.v}$$

$$-\frac{d}{r} \sum_{i=1}^{(bp)^n} f(\frac{rux_i + \cdots + rux_{i+bp-2}}{d})\| \le \varphi(x_1, \cdots, x_{(bp)^n})$$

for all $x_1 = x_{(bp)^n+1}, \cdots, x_{bp-2} = x_{(bp)^n+bp-2}, x_{bp-1}, \cdots, x_{(bp)^n} \in {}_A\mathcal{B}$ *and all* $u \in \mathcal{U}(A)$. *Then there exists a unique A-linear mapping* $T : {}_A\mathcal{B} \to {}_A\mathcal{C}$ *such that*

$$\|f(x) - T(x)\| \le \frac{1}{(bp-1)(bp)^n} \widetilde{\varphi}(x, \cdots, x) \tag{3.vi}$$

for all $x \in {}_A\mathcal{B}$.

Proof. Put $u = 1 \in \mathcal{U}(A)$. Let $x_1 = \cdots = x_{(bp)^n} = x$ in (3.v). Then we get

$$\|(bp)^n f(x) + (bp-2)(bp)^n f(x) - \frac{(bp)^n d}{r} f(\frac{(bp-1)r}{d} x)\| \le \varphi(x, \cdots, x)$$

for all $x \in {}_A\mathcal{B}$. So one can obtain

$$\|f(x) - \frac{d}{(bp-1)r} f(\frac{(bp-1)r}{d} x)\| \le \frac{1}{(bp-1)(bp)^n} \varphi(x, \cdots, x),$$

hence

$$\|(\frac{d}{(bp-1)r})^j f((\frac{(bp-1)r}{d})^j x) - (\frac{d}{(bp-1)r})^{j+1} f((\frac{(bp-1)r}{d})^{j+1} x)\|$$

$$\le \frac{1}{(bp-1)(bp)^n} (\frac{d}{(bp-1)r})^j \varphi((\frac{(bp-1)r}{d})^j x, \cdots, (\frac{(bp-1)r}{d})^j x)$$

for all $x \in {}_A\mathcal{B}$. So we get

$$\|f(x)-(\frac{d}{(bp-1)r})^j f((\frac{(bp-1)r}{d})^j x)\| \le \frac{1}{(bp-1)(bp)^n}$$
$$\sum_{m=0}^{j-1}(\frac{d}{(bp-1)r})^m \varphi((\frac{(bp-1)r}{d})^m x, \cdots, (\frac{(bp-1)r}{d})^m x) \tag{3.3}$$

for all $x \in {}_A\mathcal{B}$.

Let x be an element in ${}_A\mathcal{B}$. For positive integers l and m with $l > m$,

$$\|(\frac{d}{(bp-1)r})^l f((\frac{(bp-1)r}{d})^l x) - (\frac{d}{(bp-1)r})^m f(\frac{(bp-1)r}{d})^m x)\|$$
$$\le \frac{1}{(bp-1)(bp)^n} \sum_{j=m}^{l-1}(\frac{d}{(bp-1)r})^j \varphi((\frac{(bp-1)r}{d})^j x, \cdots, (\frac{(bp-1)r}{d})^j x),$$

which tends to zero as $m \to \infty$ by (3.iv). So $\{(\frac{d}{(bp-1)r})^j f((\frac{(bp-1)r}{d})^j x)\}$ is a Cauchy sequence for all $x \in {}_A\mathcal{B}$. Since ${}_A\mathcal{C}$ is complete, the sequence $\{(\frac{d}{(bp-1)r})^j f((\frac{(bp-1)r}{d})^j x)\}$ converges for all $x \in {}_A\mathcal{B}$. We can define a mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{C}$ by

$$T(x) = \lim_{j\to\infty}(\frac{d}{(bp-1)r})^j f((\frac{(bp-1)r}{d})^j x) \tag{3.4}$$

for all $x \in {}_A\mathcal{B}$.

By (3.iv) and (3.4), we get

$$\|D_1T(x_1, \cdots, x_{(bp)^n})\|$$
$$= \lim_{j\to\infty}(\frac{d}{(bp-1)r})^j \|D_1 f((\frac{(bp-1)r}{d})^j x_1, \cdots, (\frac{(bp-1)r}{d})^j x_{(bp)^n})\|$$
$$\le \lim_{j\to\infty}(\frac{d}{(bp-1)r})^j \varphi((\frac{(bp-1)r}{d})^j x_1, \cdots, (\frac{(bp-1)r}{d})^j x_{(bp)^n}) = 0$$

for all $x_1, \cdots, x_{(bp)^n} \in {}_A\mathcal{B}$. Hence $D_1T(x_1, \cdots, x_{(bp)^n}) = 0$ for all $x_1, \cdots, x_{(bp)^n} \in {}_A\mathcal{B}$. Put $x_1 = \cdots = x_{(bp)^n} = x$ in $D_1T(x_1, \cdots, x_{(bp)^n})$. Then

$$(bp)^n T(x) + (bp-2)\ (bp)^n T(x) - \frac{(bp)^n d}{r} T(\frac{(bp-1)r}{d} x) = 0$$

for all $x \in {}_A\mathcal{B}$. So

$$T(\frac{(bp-1)r}{d} x) = \frac{(bp-1)r}{d} T(x)$$

for all $x \in {}_A\mathcal{B}$. Since

$$\begin{aligned}
\frac{d}{r} T(\frac{rx_i + \cdots + rx_{i+bp-2}}{d}) &= \frac{d}{r} T(\frac{(bp-1)r(x_i + \cdots + x_{i+bp-2})}{(bp-1)d}) \\
&= \frac{d}{r} \frac{(bp-1)r}{d} T(\frac{x_i + \cdots + x_{i+bp-2}}{bp-1}) \\
&= (bp-1)\ T(\frac{x_i + \cdots + x_{i+bp-2}}{bp-1})
\end{aligned}$$

for all $x_i, \cdots, x_{i+bp-2} \in {}_A\mathcal{B}$,

$$\begin{aligned}
(bp)^n T(\frac{x_1 + \cdots + x_{(bp)^n}}{(bp)^n}) + bp(bp-2) \sum_{i=1}^{(bp)^{n-1}} T(\frac{x_{bpi-bp+1} + \cdots + x_{bpi}}{bp}) \\
= (bp-1) \sum_{i=1}^{(bp)^n} T(\frac{x_i + \cdots + x_{i+bp-2}}{bp-1})
\end{aligned}$$

for all $x_1 = x_{(bp)^n+1}, \cdots, x_{bp-2} = x_{(bp)^n+bp-2}, x_{bp-1}, \cdots, x_{(bp)^n} \in {}_A\mathcal{B}$. By Theorem 2.1, T is additive. Moreover, by passing to the limit in (3.3) as $j \to \infty$, we get (3.vi).

Now let $L : {}_A\mathcal{B} \to {}_A\mathcal{C}$ be another additive mapping satisfying

$$\|f(x) - L(x)\| \leq \frac{1}{(bp-1)(bp)^n} \widetilde{\varphi}(x, \cdots, x)$$

for all $x \in {}_A\mathcal{B}$.

$$\begin{aligned}\|T(x) - L(x)\| =&(\frac{d}{(bp-1)r})^j \|T((\frac{(bp-1)r}{d})^j x) - L((\frac{(bp-1)r}{d})^j x)\| \\ \leq&(\frac{d}{(bp-1)r})^j \|T((\frac{(bp-1)r}{d})^j x) - f((\frac{(bp-1)r}{d})^j x)\| \\ &+(\frac{d}{(bp-1)r})^j \|f((\frac{(bp-1)r}{d})^j x) - L((\frac{(bp-1)r}{d})^j x)\| \\ \leq \frac{2}{(bp-1)(bp)^n}&(\frac{d}{(bp-1)r})^j \widetilde{\varphi}((\frac{(bp-1)r}{d})^j x, \cdots, (\frac{(bp-1)r}{d})^j x),\end{aligned}$$

which tends to zero as $j \to \infty$ by (3.iv). Thus $T(x) = L(x)$ for all $x \in {}_A\mathcal{B}$. This proves the uniqueness of T.

By the assumption, for each $u \in \mathcal{U}(A)$,

$$\begin{aligned}&(\frac{d}{(bp-1)r})^j \|D_u f((\frac{(bp-1)r}{d})^j x, \cdots, (\frac{(bp-1)r}{d})^j x)\| \\ &\qquad \leq (\frac{d}{(bp-1)r})^j \varphi((\frac{(bp-1)r}{d})^j x, \cdots, (\frac{(bp-1)r}{d})^j x)\end{aligned}$$

for all $x \in {}_A\mathcal{B}$, and

$$(\frac{d}{(bp-1)r})^j \|D_u f((\frac{(bp-1)r}{d})^j x, \cdots, (\frac{(bp-1)r}{d})^j x)\| \to 0$$

as $j \to \infty$ for all $x \in {}_A\mathcal{B}$. So

$$\begin{aligned}&D_u T(x, \cdots, x) \\ &\quad = \lim_{j\to\infty} (\frac{d}{(bp-1)r})^j D_u f((\frac{(bp-1)r}{d})^j x, \cdots, (\frac{(bp-1)r}{d})^j x) = 0\end{aligned}$$

for all $u \in \mathcal{U}(A)$ and all $x \in {}_A\mathcal{B}$. Hence

$$\begin{aligned}&D_u T(x, \cdots, x) \\ &\quad = (bp)^n \ uT(x) + (bp-2)(bp)^n uT(x) - \frac{(bp)^n d}{r} T(\frac{(bp-1)r}{d} ux) \\ &\quad = 0\end{aligned}$$

for all $u \in \mathcal{U}(A)$ and all $x \in {}_A\mathcal{B}$. So

$$uT(x) = T(ux)$$

for all $u \in \mathcal{U}(A)$ and all $x \in {}_A\mathcal{B}$.

The rest of the proof is the same as the proof of Theorem 3.1. □

THEOREM 3.3. *Let $f : {}_A\mathcal{B} \to {}_A\mathcal{C}$ be a mapping with $f(0) = 0$ for which there exists a function $\varphi : {}_A\mathcal{B}^{(bp)^n} \to [0, \infty)$ such that*

$$\begin{aligned} \widetilde{\varphi}(x_1, \cdots, x_{(bp)^n}) = &\sum_{j=0}^{\infty} (bp)^j \varphi(-\frac{bp-2}{(bp)^j} x_1, \frac{1}{(bp)^j} x_2, \cdots, \\ &-\frac{bp-2}{(bp)^j} x_{(bp)^n-bp+1}, \frac{1}{(bp)^j} x_{(bp)^n-bp+2}, \cdots, \\ &\frac{1}{(bp)^j} x_{(bp)^n}) < \infty \end{aligned} \tag{3.vii}$$

$$\begin{aligned} \|D_u f(x_1, \cdots, x_{(bp)^n})\| = &\|(bp)^n u f(\frac{x_1 + \cdots + x_{(bp)^n}}{(bp)^n}) \\ &+ bp(bp-2) \sum_{i=1}^{(bp)^{n-1}} u f(\frac{x_{bpi-bp+1} + \cdots + x_{bpi}}{bp}) \\ &- (bp-1) \sum_{i=1}^{(bp)^n} f(\frac{ux_i + \cdots + ux_{i+bp-2}}{bp-1})\| \\ &\le \varphi(x_1, \cdots, x_{(bp)^n}) \end{aligned} \tag{3.viii}$$

for all $x_1 = x_{(bp)^n+1}, \cdots, x_{bp-2} = x_{(bp)^n+bp-2}, x_{bp-1}, \cdots, x_{(bp)^n} \in {}_A\mathcal{B}$ and all $u \in \mathcal{U}(A)$. Then there exists a unique A-linear mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{C}$ such that

$$\|f(x) - T(x)\| \le \frac{1}{(bp-1)(bp)^{n-1}} \widetilde{\varphi}(x, \cdots, x) \tag{3.ix}$$

for all $x \in {}_A\mathcal{B}$.

Proof. Put $u = 1 \in \mathcal{U}(A)$. Let $x_{bpi-bp+1} = x$ and $x_{bpi-bp+2} = \cdots = x_{bpi} = y$ in (3.viii) for all $i = 1, \cdots, (bp)^{n-1}$. Then we get

$$\begin{aligned}\|(bp)^n f(\frac{x+(bp-1)y}{bp}) &+ (bp-2)(bp)^n f(\frac{x+(bp-1)y}{bp}) \\ &- (bp-1)(bp)^{n-1} f(y) \\ &- (bp-1)(bp-1)(bp)^{n-1} f(\frac{x+(bp-2)y}{bp-1})\| \\ &\leq \varphi(x, y, y, \cdots, x, y, \cdots, y)\end{aligned} \tag{3.5}$$

for all $x, y \in {}_A\mathcal{B}$. Replacing x and y by $-(bp-2)x$ and x in (3.5), respectively, we get

$$\begin{aligned}\|(bp-1)(bp)^n f(\frac{x}{bp}) &- (bp-1)(bp)^{n-1} f(x)\| \\ &\leq \varphi(-(bp-2)x, x, x, \cdots, -(bp-2)x, x, \cdots, x)\end{aligned}$$

for all $x \in {}_A\mathcal{B}$. So one can obtain

$$\begin{aligned}&\|f(x) - bpf(\frac{x}{bp})\| \\ &\leq \frac{1}{(bp-1)(bp)^{n-1}} \varphi(-(bp-2)x, x, x, \cdots, -(bp-2)x, x, \cdots, x),\end{aligned}$$

hence

$$\begin{aligned}\|(bp)^j f(\frac{1}{(bp)^j}x) - (bp)^{j+1} f(\frac{1}{(bp)^{j+1}}x)\| &\leq \frac{(bp)^j}{(bp-1)(bp)^{n-1}} \varphi(-\frac{bp-2}{(bp)^j}x, \frac{1}{(bp)^j}x, \\ &\quad \frac{1}{(bp)^j}x, \cdots, -\frac{bp-2}{(bp)^j}x, \frac{1}{(bp)^j}x, \cdots, \frac{1}{(bp)^j}x)\end{aligned}$$

for all $x \in {}_A\mathcal{B}$. So we get

$$\begin{aligned}&\|f(x) - (bp)^j f(\frac{1}{(bp)^j}x)\| \\ &\leq \frac{1}{(bp-1)(bp)^{n-1}} \sum_{m=0}^{j-1} (bp)^m \varphi(-\frac{bp-2}{(bp)^m}x, \frac{1}{(bp)^m}x, \\ &\quad \frac{1}{(bp)^m}x, \cdots, -\frac{bp-2}{(bp)^m}x, \frac{1}{(bp)^m}x, \cdots, \frac{1}{(bp)^m}x)\end{aligned} \tag{3.6}$$

for all $x \in {}_A\mathcal{B}$.

Let x be an element in ${}_A\mathcal{B}$. For positive integers l and m with $l > m$,

$$\begin{aligned}
&\|(bp)^l f(\frac{1}{(bp)^l}x) - (bp)^m f(\frac{1}{(bp)^m}x)\| \\
&\qquad \le \frac{1}{(bp-1)(bp)^{n-1}} \sum_{j=m}^{l-1} (bp)^j \varphi(-\frac{bp-2}{(bp)^j}x, \\
&\qquad \frac{1}{(bp)^j}x, \frac{1}{(bp)^j}x, \cdots, -\frac{bp-2}{(bp)^j}x, \frac{1}{(bp)^j}x, \cdots, \frac{1}{(bp)^j}x),
\end{aligned}$$

which tends to zero as $m \to \infty$ by (3.vii). So $\{(bp)^j f(\frac{1}{(bp)^j}x)\}$ is a Cauchy sequence for all $x \in {}_A\mathcal{B}$. Since ${}_A\mathcal{C}$ is complete, the sequence $\{(bp)^j f(\frac{1}{(bp)^j}x)\}$ converges for all $x \in {}_A\mathcal{B}$. We can define a mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{C}$ by

$$T(x) = \lim_{j\to\infty} (bp)^j f(\frac{1}{(bp)^j}x) \tag{3.7}$$

for all $x \in {}_A\mathcal{B}$.

By (3.vii) and (3.7), we get

$$\begin{aligned}
\|D_1 T(x_1, \cdots, x_{(bp)^n})\| &= \lim_{j\to\infty} (bp)^j \|D_1 f(\frac{1}{(bp)^j}x_1, \cdots, \frac{1}{(bp)^j}x_{(bp)^n})\| \\
&\le \lim_{j\to\infty} (bp)^j \varphi(\frac{1}{(bp)^j}x_1, \cdots, \frac{1}{(bp)^j}x_{(bp)^n}) = 0
\end{aligned}$$

for all $x_1, \cdots, x_{(bp)^n} \in {}_A\mathcal{B}$. Hence $D_1 T(x_1, \cdots, x_{(bp)^n}) = 0$ for all $x_1, \cdots, x_{(bp)^n} \in {}_A\mathcal{B}$. By Theorem 2.1, T is additive. Moreover, by passing to the limit in (3.6) as $j \to \infty$, we get (3.ix).

Now let $L : {}_A\mathcal{B} \to {}_A\mathcal{C}$ be another additive mapping satisfying

$$\|f(x) - L(x)\| \le \frac{1}{(bp-1)(bp)^{n-1}} \widetilde{\varphi}(x, \cdots, x)$$

for all $x \in {}_A\mathcal{B}$.

$$\begin{aligned}\|T(x) - L(x)\| =&(bp)^j \|T(\frac{1}{(bp)^j}x) - L(\frac{1}{(bp)^j}x)\| \\ \leq&(bp)^j \|T(\frac{1}{(bp)^j}x) - f(\frac{1}{(bp)^j}x)\| \\ &+ (bp)^j \|f(\frac{1}{(bp)^j}x) - L(\frac{1}{(bp)^j}x)\| \\ \leq&\frac{2}{(bp-1)(bp)^{n-1}}(bp)^j \widetilde{\varphi}(\frac{1}{(bp)^j}x, \cdots, \frac{1}{(bp)^j}x),\end{aligned}$$

which tends to zero as $j \to \infty$ by (3.vii). Thus $T(x) = L(x)$ for all $x \in {}_A\mathcal{B}$. This proves the uniqueness of T.

By the assumption, for each $u \in \mathcal{U}(A)$,

$$(bp)^j \|D_u f(\frac{1}{(bp)^j}x, \cdots, \frac{1}{(bp)^j}x)\| \leq (bp)^j \varphi(\frac{1}{(bp)^j}x, \cdots, \frac{1}{(bp)^j}x)$$

for all $x \in {}_A\mathcal{B}$, and

$$(bp)^j \|D_u f(\frac{1}{(bp)^j}x, \cdots, \frac{1}{(bp)^j}x)\| \to 0$$

as $j \to \infty$ for all $x \in {}_A\mathcal{B}$. So

$$D_u T(x, \cdots, x) = \lim_{j\to\infty} (bp)^j D_u f(\frac{1}{(bp)^j}x, \cdots, \frac{1}{(bp)^j}x) = 0$$

for all $u \in \mathcal{U}(A)$ and all $x \in {}_A\mathcal{B}$. Hence

$$\begin{aligned}&D_u T(x, \cdots, x) \\ &\quad= (bp)^n uT(x) + (bp-2)(bp)^n uT(x) - (bp-1)(bp)^n T(ux) = 0\end{aligned}$$

for all $u \in \mathcal{U}(A)$ and all $x \in {}_A\mathcal{B}$. So

$$uT(x) = T(ux)$$

for all $u \in \mathcal{U}(A)$ and all $x \in {}_A\mathcal{B}$.

The rest of the proof is the same as the proof of Theorem 3.1. $\square$

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