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STABILITY OF ADDITIVE $(n, 2)$–MAPPINGS

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Abstract. We define an additive $(n, 2)$-mapping, and prove the stability of additive $(n, 2)$-mappings.

1. Additive $(n, 2)$-mappings

In [1, Definition 4.2.3], the authors defined a linear 2-functional. We introduce an additive $(n, 2)$-mapping that is weaker than a linear 2-functional.

Definition 1. Let B be a Banach algebra, and let ${}_B\mathcal{A}$ and ${}_B\mathcal{C}$ be left B-modules. Let ${}_B\mathcal{D}$ be a left Banach B-module with norm $\|\cdot\|$. An *additive* $(n, 2)$-*mapping* $f : {}_B\mathcal{A} \times {}_B\mathcal{C} \to {}_B\mathcal{D}$ is a mapping such that

$$\text{(i)} \qquad f(\sum_{i=1}^{n} x_i, \sum_{j=1}^{n} z_j) = \sum_{i,j=1}^{n} f(x_i, z_j)$$

for all $x_1, \cdots, x_n \in {}_B\mathcal{A}$ and all $z_1, \cdots, z_n \in {}_B\mathcal{C}$.

Th.M. Rassias [3] proved the stability of linear mappings in Banach spaces. If B is unital and we add the condition

$$\text{(ii)} \quad f(ax, bz) = abf(x, z), \ \textit{for all } a, b \in B, \ x \in {}_B\mathcal{A}, \ z \in {}_B\mathcal{C},$$

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to (i), then it is a B-bilinear mapping. But (i) is a weaker condition than

(i') $f(x_1+x_2, z_1+z_2) = f(x_1, z_1) + f(x_1, z_2) + f(x_2, z_1) + f(x_2, z_2)$.

In this paper we call $f : {}_B\mathcal{A} \times {}_B\mathcal{C} \to {}_B\mathcal{D}$ *a linear* $(n, 2)$*-mapping* if it satisfies (i) and (ii) rather than (i') and (ii) to emphasize the additivity on the sum of n elements.

The main purpose of this paper is to prove the stability of additive $(n, 2)$-mappings.

THEOREM 1. *Let* $f : {}_B\mathcal{A} \times {}_B\mathcal{C} \to {}_B\mathcal{D}$ *be a mapping for which there exist a nonnegative function* $\varphi : {}_B\mathcal{A}^n \times {}_B\mathcal{C}^n \to [0, \infty)$ *for some integer* $n > 1$ *such that*

$$\widetilde{\varphi}(x_1, \cdots, x_n, z_1, \cdots, z_n) :$$

$$\text{(iii)} \qquad = \sum_{k=0}^{\infty} \frac{1}{n^{2k}} \varphi(n^k x_1, \cdots, n^k x_n, n^k z_1, \cdots, n^k z_n) < \infty$$

$$\text{(iv)} \qquad \|f(\sum_{i=1}^{n} x_i, \sum_{j=1}^{n} z_j) - \sum_{i,j=1}^{n} f(x_i, z_j)\| \le \varphi(x_1, \cdots, x_n, z_1, \cdots, z_n)$$

for all all $x_1, \cdots, x_n \in {}_B\mathcal{A}$ *and all* $z_1, \cdots, z_n \in {}_B\mathcal{C}$*. Then there exists a unique additive* $(n, 2)$*-mapping* $T : {}_B\mathcal{A} \times {}_B\mathcal{C} \to {}_B\mathcal{D}$ *such that*

$$\text{(v)} \qquad \|f(x, z) - T(x, z)\| \le \frac{1}{n^2} \widetilde{\varphi}(x, \cdots, x, z, \cdots, z)$$

for all $x \in {}_B\mathcal{A}$ *and all* $z \in {}_B\mathcal{C}$*. If, in addition,* f *satisfies*

$$\text{(vi)} \qquad \|f(ax, bz) - ab f(x, z)\| \le \varphi(x, \cdots, x, z, \cdots, z)$$

for all $a, b \in B$*, all* $x \in {}_B\mathcal{A}$ *and all* $z \in {}_B\mathcal{C}$*, then* T *satisfies* (ii) *too.*

Proof. For each integer $k \ge 0$, let $x_1 = \cdots = x_n = n^k x$ and $z_1 = \cdots = z_n = n^k z$ in (iv). Then we get

$$\|f(n^{k+1}x, n^{k+1}z) - n^2 f(n^k x, n^k z)\| \le \varphi(n^k x, \cdots, n^k x, n^k z, \cdots, n^k z)$$

for all $x \in {}_B\mathcal{A}$ and all $z \in {}_B\mathcal{C}$. A division by n^{2k+2} gives

$$\begin{aligned}&\|\frac{1}{n^{2k}}f(n^kx,n^kz)-\frac{1}{n^{2k+2}}f(n^{k+1}x,n^{k+1}z)\|\\&\qquad\leq\frac{1}{n^{2k+2}}\varphi(n^kx,\cdots,n^kx,n^kz,\cdots,n^kz)\end{aligned}$$

for all $x \in {}_B\mathcal{A}$ and all $z \in {}_B\mathcal{C}$. So we get

$$\begin{aligned}&\|f(x,z)-\frac{1}{n^{2k}}f(n^kx,n^kz)\|\\&\qquad\leq\sum_{d=0}^{k-1}\frac{1}{n^{2d+2}}\varphi(n^dx,\cdots,n^dx,n^dz,\cdots,n^dz)\end{aligned}\tag{1}$$

for all $x \in {}_B\mathcal{A}$ and all $z \in {}_B\mathcal{C}$.

Let $x \in {}_B\mathcal{A}$ and $z \in {}_B\mathcal{C}$. For positive integers l and d with $l > d$,

$$\begin{aligned}&\|\frac{1}{n^{2l}}f(n^lx,n^lz)-\frac{1}{n^{2d}}f(n^dx,n^dz)\|\\&\qquad\leq\frac{1}{n^2}\sum_{j=d}^{l-1}\frac{1}{n^{2j}}\varphi(n^jx,\cdots,n^jx,n^jz,\cdots,n^jz),\end{aligned}$$

which tends to zero as $d \to \infty$ by (iii). So $\{\frac{1}{n^{2j}}f(n^jx,n^jz)\}$ is a Cauchy sequence for all $x \in {}_B\mathcal{A}$ and all $z \in {}_B\mathcal{C}$. Since ${}_B\mathcal{D}$ is complete, the sequence $\{\frac{1}{n^{2j}}f(n^jx,n^jz)\}$ converges for all $x \in {}_B\mathcal{A}$ and all $z \in {}_B\mathcal{C}$. We can define a mapping $T : {}_B\mathcal{A} \times {}_B\mathcal{C} \to {}_B\mathcal{D}$ by

$$T(x,z)=\lim_{j\to\infty}\frac{1}{n^{2j}}f(n^jx,n^jz)\tag{2}$$

for all $x \in {}_B\mathcal{A}$ and all $z \in {}_B\mathcal{C}$. By passing to the limit in (1) as $k \to \infty$, we get the inequality (v).

By (2), it follows from (iv) and (vi) that the mapping $T : {}_B\mathcal{A} \times {}_B\mathcal{C} \to {}_B\mathcal{D}$ satisfies (i) and (ii) respectively.

Now let $L : {}_B\mathcal{A} \times {}_B\mathcal{C} \to {}_B\mathcal{D}$ be another additive $(n, 2)$-mapping satisfying

$$\|f(x,z) - L(x,z)\| \le \frac{1}{n^2}\widetilde{\varphi}(x,\cdots,x,z,\cdots,z)$$

for all $x \in {}_B\mathcal{A}$ and all $z \in {}_B\mathcal{C}$.

$$\begin{aligned}\|T(x,z) - L(x,z)\| =& \frac{1}{n^{2j}}\|T(n^jx,n^jz) - L(n^jx,n^jz)\| \\ \le& \frac{1}{n^{2j}}\|T(n^jx,n^jz) - f(n^jx,n^jz)\| \\ &+ \frac{1}{n^{2j}}\|f(n^jx,n^jz) - L(n^jx,n^jz)\| \\ \le& \frac{2}{n^2}\frac{1}{n^{2j}}\widetilde{\varphi}(n^jx,\cdots,n^jx,n^jz,\cdots,n^jz),\end{aligned}$$

which tends to zero as $j \to \infty$ by (iii). Thus $T(x,z) = L(x,z)$ for all $x \in {}_B\mathcal{A}$ and all $z \in {}_B\mathcal{C}$. This proves the uniqueness of T. □

THEOREM 2. *Let $f : {}_B\mathcal{A} \times {}_B\mathcal{C} \to {}_B\mathcal{D}$ be a mapping for which there exists a function $\varphi : {}_B\mathcal{A}^n \times {}_B\mathcal{C}^n \to [0,\infty)$ for some integer $n > 1$ such that*

$$\widetilde{\varphi}(x_1,\cdots,x_n,z_1,\cdots,z_n) := \sum_{k=1}^{\infty} n^{2k}\varphi(\frac{x_1}{n^k},\cdots,$$

(vii) $$\frac{x_n}{n^k},\frac{z_1}{n^k},\cdots,\frac{z_n}{n^k}) < \infty$$

(viii)
$$\|f(\sum_{i=1}^{n} x_i, \sum_{j=1}^{n} z_j) - \sum_{i,j=1}^{n} f(x_i,z_j)\| \le \varphi(x_1,\cdots,x_n,z_1,\cdots,z_n)$$

for all $x_1, x_2\cdots,x_n \in {}_B\mathcal{A}$ and all $z_1, z_2\cdots,z_n \in {}_B\mathcal{C}$. Then there exists a unique additive $(n,2)$-mapping $T : {}_B\mathcal{A} \times {}_B\mathcal{C} \to {}_B\mathcal{D}$ such that

(ix) $$\|f(x,z) - T(x,z)\| \le \frac{1}{n^2}\widetilde{\varphi}(x,\cdots,x,z,\cdots,z)$$

for all $x \in {}_B\mathcal{A}$ and all $z \in {}_B\mathcal{C}$. If, in addition, f satisfies

$$\text{(x)} \qquad \|f(ax,bz) - abf(x,z)\| \le \varphi(x,\cdots,x,z,\cdots,z)$$

for all $a,b \in B$, all $x \in {}_B\mathcal{A}$ and all $z \in {}_B\mathcal{C}$, then T is a linear $(n,2)$-mapping.

Proof. The proof is similar to the proof of Theorem 2. □

2. Linear (n,m)-mappings

DEFINITION 2. Let B be a Banach algebra, and ${}_B\mathcal{A}_i$ left B-modules, $i = 1,\cdots,m$. Let ${}_B\mathcal{D}$ be a left Banach B-module with norm $\|\cdot\|$. A *linear (n,m)-mapping* $f : \prod_{i=1}^m {}_B\mathcal{A}_i \to {}_B\mathcal{D}$ is a mapping such that

(a) $f(\sum_{i=1}^n x_{1i},\cdots,\sum_{i=1}^n x_{mi}) = \sum_{i_1,\cdots,i_m=1}^n f(x_{1i_1},\cdots,x_{mi_m})$ for all $x_{ji_j} \in {}_B\mathcal{A}_j$ $(1 \le i_j \le n)$, $j = 1,2,\cdots,m$,

(b) $f(a_1x_1,\cdots,a_mx_m) = a_1\cdots a_m \cdot f(x_1,\cdots,x_m)$ for all $a_1,\cdots,a_m \in B$ and all $x_i \in {}_B\mathcal{A}_i$, $i = 1,2,\cdots,m$.

By a similar method to the proof of Theorem 2, one can obtain the following theorems. We need only the additivity on the sum of n elements in the proof of the existence and the uniqueness.

THEOREM 3. *Let $f : \prod_{i=1}^m {}_B\mathcal{A}_i \to {}_B\mathcal{D}$ be a mapping for which there exists a function $\varphi : \prod_{i=1}^m {}_B\mathcal{A}_i^n \to [0,\infty)$ for some integer $n > 1$ such that*

$$\widetilde{\varphi}(x_{11},\cdots,x_{1n},\cdots,x_{m1},\cdots,x_{mn})$$

$$\text{(xi)} \qquad = \sum_{k=0}^{\infty} \frac{1}{n^{mk}} \varphi(n^k x_{11},\cdots,n^k x_{1n},\cdots,n^k x_{m1},\cdots,n^k x_{mn}) < \infty$$

$$\|f(\sum_{i=1}^n x_{1i},\cdots,\sum_{i=1}^n x_{mi}) - \sum_{i_1,\cdots,i_m=1}^n f(x_{1i_1},\cdots,x_{mi_m})\| \le \varphi(x_{11},\cdots,x_{1n},\cdots,x_{m1},\cdots,x_{mn})$$

$$\|f(a_1x_1,\cdots,a_mx_m)-a_1\cdots a_m\cdot f(x_1,\cdots,x_m)\| \\ \leq \varphi(x_1,\cdots,x_1,\cdots,x_m,\cdots,x_m)$$

for all $a_1,\cdots,a_m\in B$ *and all* $x_i,x_{i1},\cdots,x_{in}\in {}_B\mathcal{A}_i$, $i=1,2,\cdots,m$. *Then there exists a unique linear* (n,m)*-mapping* $T:\prod_{i=1}^m {}_B\mathcal{A}_i\to {}_B\mathcal{D}$ *such that*

$$\|f(x_1,\cdots,x_m)-T(x_1,\cdots,x_m)\| \\ \leq \frac{1}{n^m}\widetilde{\varphi}(x_1,\cdots,x_1,\cdots,x_m,\cdots,x_m) \tag{xii}$$

for all $x_i\in {}_B\mathcal{A}_i$, $i=1,2,\cdots,m$.

THEOREM 4. *Let* $f:\prod_{i=1}^m {}_B\mathcal{A}_i\to {}_B\mathcal{D}$ *be a mapping for which there exists a function* $\varphi:\prod_{i=1}^m {}_B\mathcal{A}_i^n\to[0,\infty)$ *for some integer* $n>1$ *such that*

$$\widetilde{\varphi}(x_{11},\cdots,x_{1n},\cdots,x_{m1},\cdots,x_{mn}) \\ =\sum_{k=1}^{\infty} n^{mk}\varphi(\frac{x_{11}}{n^k},\cdots,\frac{x_{1n}}{n^k},\cdots,\frac{x_{m1}}{n^k},\cdots,\frac{x_{mn}}{n^k})<\infty$$

$$\|f(\sum_{i=1}^n x_{1i},\cdots,\sum_{i=1}^n x_{mi})-\sum_{i_1,\cdots,i_m=1}^n f(x_{1i_1},\cdots,x_{mi_m})\| \\ \leq \varphi(x_{11},\cdots,x_{1n},\cdots,x_{m1},\cdots,x_{mn})$$

$$\|f(a_1x_1,\cdots,a_mx_m)-a_1\cdots a_m\cdot f(x_1,\cdots,x_m)\| \\ \leq \varphi(x_1,\cdots,x_1,\cdots,x_m,\cdots,x_m)$$

for all $a_1,\cdots,a_m\in B$ *and all* $x_i,x_{i1},\cdots,x_{in}\in {}_B\mathcal{A}_i$, $i=1,\cdots,m$. *Then there exists a unique linear* (n,m)*-mapping* $T:\prod_{i=1}^m {}_B\mathcal{A}_i\to {}_B\mathcal{D}$ *such that*

$$\|f(x_1,\cdots,x_m)-T(x_1,\cdots,x_m)\|\leq\frac{1}{n^m}\widetilde{\varphi}(x_1,\cdots,x_1,\cdots,x_m,\cdots,x_m)$$

for all $x_i\in {}_B\mathcal{A}_i$, $i=1,\cdots,m$.

DEFINITION 3. Let B be a unital C^*-algebra, $\mathcal{U}(B)$ the unitary group of B, and ${}_B\mathcal{A}_i$ left B-modules, $i = 1, \cdots, m$. Let ${}_B\mathcal{D}$ be a left Banach B-module with norm $\|\cdot\|$. A *unitary linear* (n,m)*-mapping* $f : \prod_{i=1}^m {}_B\mathcal{A}_i \to {}_B\mathcal{D}$ is a mapping such that

(a) $f(\sum_{i=1}^n x_{1i}, \cdots, \sum_{i=1}^n x_{mi}) = \sum_{i_1,\cdots,i_m=1}^n f(x_{1i_1}, \cdots, x_{mi_m})$ for all $x_{ji_j} \in {}_B\mathcal{A}_j$ $(1 \le i_j \le n)$, $j = 1, 2, \cdots, m$,

(b) $f(u_1x_1, \cdots, u_mx_m) = u_1 \cdots u_m \cdot f(x_1, \cdots, x_m)$ for all $u_1, \cdots, u_m \in \mathcal{U}(B)$ and all $x_i \in {}_B\mathcal{A}_i$, $i = 1, 2, \cdots, m$.

THEOREM 5. *Let B be a unital C^*-algebra with norm $|\cdot|$. Let $f : \prod_{i=1}^m {}_B\mathcal{A}_i \to {}_B\mathcal{D}$ be a mapping for which there exists a function $\varphi : \prod_{i=1}^m {}_B\mathcal{A}_i^n \to [0,\infty)$ for some integer $n > 1$ satisfying* (xi) *such that*

$$\|f(\sum_{i=1}^n x_{1i}, \cdots, \sum_{i=1}^n x_{mi}) - \sum_{i_1,\cdots,i_m=1}^n f(x_{1i_1}, \cdots, x_{mi_m})\| \le \varphi(x_{11}, \cdots, x_{1n}, \cdots, x_{m1}, \cdots, x_{mn})$$

$$\|f(u_1x_1, \cdots, u_mx_m) - u_1 \cdots u_m \cdot f(x_1, \cdots, x_m)\| \le \varphi(x_1, \cdots, x_1, \cdots, x_m, \cdots, x_m)$$

for all $u_1, \cdots, u_m \in \mathcal{U}(B)$ and all $x_i, x_{i1}, \cdots, x_{in} \in {}_B\mathcal{A}_i$, $i = 1, 2, \cdots, m$. Then there exists a unique B-multilinear unitary linear (n,m)-mapping $T : \prod_{i=1}^m {}_B\mathcal{A}_i \to {}_B\mathcal{D}$ satisfying (xii).

Proof. By the same method as the proof of Theorem 2, one can show that there exists a unique unitary linear (n,m)-mapping $T : \prod_{i=1}^m {}_B\mathcal{A}_i \to {}_B\mathcal{D}$ satisfying (xii).

It is obvious that the unique unitary linear (n,m)-mapping $T : \prod_{i=1}^m {}_B\mathcal{A}_i \to {}_B\mathcal{D}$ is a multi-additive mapping.

Now let $a \in B$ $(a \ne 0)$ and K an integer greater than $4|a|$. Then

$$|\frac{a}{K}| = \frac{1}{K}|a| < \frac{|a|}{4|a|} = \frac{1}{4} < 1 - \frac{2}{3} = \frac{1}{3}.$$

By [2, Theorem 1], there exist three elements $u_1, u_2, u_3 \in \mathcal{U}(B)$ such that $3\frac{a}{K} = u_1 + u_2 + u_3$. And

$$\begin{aligned} T(x_1, \cdots, &x_{i-1}, x_i, x_{i+1}, \cdots, x_m) \\ &= T(x_1, \cdots, x_{i-1}, 3 \cdot \frac{1}{3}x_i, x_{i+1}, \cdots, x_m) \\ &= 3T(x_1, \cdots, x_{i-1}, \frac{1}{3}x_i, x_{i+1}, \cdots, x_m) \end{aligned}$$

for all $(x_1, \cdots, x_m) \in \prod_{i=1}^m {}_B\mathcal{A}_i$. So

$$\begin{aligned} T(x_1, \cdots, x_{i-1}, \frac{1}{3}x_i, x_{i+1}, \cdots, x_m) \\ = \frac{1}{3}T(x_1, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots, x_m) \end{aligned}$$

for all $(x_1, \cdots, x_m) \in \prod_{i=1}^m {}_B\mathcal{A}_i$. Thus

$$\begin{aligned} T(x_1, \cdots, ax_i, \cdots, x_m) &= T(x_1, \cdots, \frac{K}{3} \cdot 3\frac{a}{K}x_i, \cdots, x_m) \\ &= \frac{K}{3}T(x_1, \cdots, 3\frac{a}{K}x_i, \cdots, x_m) \\ &= \frac{K}{3}T(x_1, \cdots, u_1x_i + u_2x_i + u_3x_i, \cdots, x_m) \\ &= \frac{K}{3}(u_1 + u_2 + u_3)T(x_1, \cdots, x_i, \cdots, x_m) \\ &= \frac{K}{3} \cdot 3\frac{a}{K}T(x_1, \cdots, x_i, \cdots, x_m) \\ &= aT(x_1, \cdots, x_i, \cdots, x_m) \end{aligned}$$

for all $(x_1, \cdots, x_m) \in \prod_{i=1}^m {}_B\mathcal{A}_i$. Obviously,

$$T(x_1, \cdots, 0x_i, \cdots, x_m) = 0T(x_1, \cdots, x_i, \cdots, x_m)$$

for all $(x_1, \cdots, x_m) \in \prod_{i=1}^m {}_B\mathcal{A}_i$. So the unique unitary linear (n, m)-mapping $T : \prod_{i=1}^m {}_B\mathcal{A}_i \to {}_B\mathcal{D}$ is a B-multilinear mapping. □

Similarly, one can obtain a similar result to Theorem 4.

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