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SEMI–COMPATIBILITY AND FIXED POINT THEOREM IN MENGER SPACE

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Abstract. In this paper, the concept of semi-compatibility in Menger space is introduced and it is used to prove results on the existence of a unique common fixed point of four self-maps. These results are a very wide improvement of Mishra [8], Dedeic and Sarapa [3, 4], Cain and Kasril [1], and Sehgal and Bharucha Reid [10].

1. Introduction

In [8], Mishra introduced the concept of compatible self-maps in Menger space and proved the existence of a common fixed point of a pair of compatible maps using a contractive condition. Cho, Sharma and Sahu [2] introduced the non-symmetric concept of semi-compatibility maps in a d-topological space. They defined a pair of self-maps (S, T) to be *semi-compatible of self-maps* if the conditions (i) $Sy = Ty$ implies $STy = TSy$ and (ii) $Sx_n \to x$, $Tx_n \to x$ imply $STx_n \to Tx$, as $n \to \infty$, hold. However, (ii) implies (i), taking $x_n = y$ and $x = Ty = Sy$. So we define the concept of semi-compatibility of a pair of self-maps in Menger space by the condition (ii) only.

2. Preliminaries

Throughout this paper we use all symbols and basic definitions of Mishra [8].

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Definition 2.1. A mapping $F : \mathbb{R} \to \mathbb{R}^+$ is called a *distribution* if it is non-decreasing left continuous with $\inf\{F(t) \mid t \in \mathbb{R}\} = 0$ and $\sup\{F(t) \mid t \in \mathbb{R}\} = 1$.

We shall denote by L the set of all distribution functions and denote by H the specific distribution function defined by

$$H(t) = \begin{cases} 0, & t \leq 0 \\ 1, & t > 0. \end{cases}$$

Definition 2.2. ([7]) A probabilistic metric space (PM-space) is an ordered pair (X, F) where X is an abstract set of elements and $F : X \times X \to L$ is defined by $(p, q) \to F_{p,q}$. Here the functions $F_{p,q}$ satisfy the following :

(1) $F_{p,q}(x) = 1$ for all $x > 0$ if and only if $p = q$,
(2) $F_{p,q}(0) = 0$,
(3) $F_{p,q} = F_{q,p}$,
(4) if $F_{p,q}(x) = 1$ and $F_{q,r}(y) = 1$ then $F_{p,r}(x + y) = 1$.

Definition 2.3. A mapping $t : [0, 1] \times [0, 1] \to [0, 1]$ is called a *t-norm* if

(1) $t(a, 1) = a$, $t(0, 0) = 0$,
(2) $t(a, b) = t(b, a)$,
(3) $t(c, d) \geq t(a, b)$ for $c \geq a$ and $d \geq b$,
(4) $t(t(a, b), c) = t(a, t(b, c))$.

Definition 2.4. A *Menger space* is a triplet (X, F, t), where (X, F) is a PM-space and t is a t-norm such that for all $p, q, r \in X$ and all $x, y \geq 0$

$$F_{p,r}(x + y) \geq t(F_{p,q}(x), F_{q,r}(y)).$$

Definition 2.5. A sequence $\{p_n\}$ in X is said to *converge to a point* p in X, written as $p_n \to p$, if for each $\epsilon > 0$ and each $\lambda > 0$ there is an integer $M(\epsilon,\lambda)$ such that $F_{p_n,p}(\epsilon) > 1-\lambda$ for all $n \geq M(\epsilon,\lambda)$. A sequence $\{p_n\}$ in X is said to be a *Cauchy sequence* if for each $\epsilon > 0$ and each $\lambda > 0$ there is an integer $M(\epsilon,\lambda)$ such that $F_{p_n,p_m}(\epsilon) \geq 1-\lambda$ for all $n, m \geq M(\epsilon,\lambda)$. A Menger space (X,F,t) is said to be *complete* if every Cauchy sequence in X converges to a point in X.

Definition 2.6. Self-maps A and S of a Menger space (X,F,t) are said to be *weakly compatible* or *coincidentally commuting* if they commute at their coincidence points, i.e., if $Ap = Sp$ for some $p \in X$ then $ASp = SAp$.

Definition 2.7. Self-maps A and S of a Menger space (X,F,t) are said to be *compatible* if $F_{ASp_n,SAp_n}(x) \to 1$ for all $x > 0$ whenever $\{p_n\}$ is a sequence in X such that $Ap_n, Sp_n \to u$ as $n \to \infty$ for some $u \in X$.

Here we introduce the notion of semi-compatible mappings in Menger space.

Definition 2.8. Self-maps A and S of a Menger space (X,F,t) are said to be *semi-compatible* if $F_{ASp_n,Su}(x) \to 1$ for all $x > 0$ whenever $\{p_n\}$ is a sequence in X such that $Ap_n, Sp_n \to u$ as $n \to \infty$ for some $u \in X$.

Definition 2.9. Let S and T be multivalued maps on a Menger space. A sequence $\{x_0, x_1, x_2, \cdots\}$ such that $x_1 \in Tx_0$, $x_2 \in STx_0$, $x_3 \in TSTx_0$, $\cdots$, is said to be an *orbit* of S and T at x_0 and is denoted by $O(T,S,x_0)$. The orbit is said to be *complete* if every Cauchy sequence in X converges to a point in X.

Proposition 2.1. *Let A, B, S, T be self-maps of a Menger space (X,F,t) such that $A(X) \subset T(X)$ and $B(X) \subset S(X)$. For $x_0 \in X$, define sequences $\{x_n\}$ and $\{y_n\}$ as follows:*

$$Ax_{2n} = Tx_{2n+1} = y_{2n+1} \quad \& \quad Bx_{2n+1} = Sx_{2n+2} = y_{2n+2}$$

for $n = 0, 1, 2 \cdots$. *Then*

(1) $\{x_0, x_1, x_2, \cdots\} = O(T^{-1}A, S^{-1}B, x_0)$,
(2) $\{y_1, y_2, y_3, \cdots\} = O(BT^{-1}, AS^{-1}, Ax_0)$.

Proof. It is easy to show that $Ax_0 = Tx_1$ implies $x_1 \in T^{-1}Ax_0$ and $Bx_1 = Sx_2$ gives $x_2 \in S^{-1}Bx_1 \subset (S^{-1}B)(T^{-1}A)x_0$, and so on. Hence (1) is true. Again, $y_1 = Ax_0$, $y_2 = Bx_1 \in BT^{-1}(Ax_0)$.

Similarly, $y_3 = Ax_2 \subset A(S^{-1}B)(T^{-1}A)x_0 = (AS^{-1})(BT^{-1})Ax_0$, and so on. This gives (2). □

PROPOSITION 2.2. *In a Menger space* (X, F, t), *if* $t(x, x) \geq x$ *for all* $x \in [0, 1]$ *then* $t(a, b) = \min\{a, b\}$ *for all* $a, b \in [0, 1]$.

PROPOSITION 2.3. *If* (A, S) *is a semi-compatible pair of self-maps of a Menger space* (X, F, t), *where* $t(x, x) \geq x$ *for all* $x \in [0, 1]$ *and* S *is continuous, then* (A, S) *is compatible.*

Proof. Consider a sequence $\{x_n\}$ in X such that $\{Ax_n\} \to u$ and $\{Sx_n\} \to u$. Since S is continuous, $SAx_n \to Su$ and $S^2x_n \to Su$. Since (A, S) is semi-compatible, $ASx_n \to Su$. Hence for a given pair (ϵ, λ) there is an integer $N_0(\epsilon, \lambda)$ such that

$$F_{ASx_n,Su}(\frac{\epsilon}{2}) \geq 1 - \lambda \quad \& \quad F_{SAx_n,Su}(\frac{\epsilon}{2}) \geq 1 - \lambda$$

for all $n \geq N_0$. Now

$$\begin{aligned} F_{ASx_n,SAx_n}(\epsilon) &\geq t(F_{ASx_n,Su}(\frac{\epsilon}{2}), F_{SAx_n,Su}(\frac{\epsilon}{2})) \\ &\geq t(1 - \lambda, 1 - \lambda) \\ &\geq 1 - \lambda \end{aligned}$$

for all $n \geq N_0$. Hence the pair (A, S) is compatible. □

PROPOSITION 2.4. *If (A, S) is a compatible pair of self-maps of a Menger space (X, F, t), where $t(x, x) \geq x$ for all $x \in [0, 1]$ and S is continuous, then (A, S) is semi-compatible.*

Proof. Consider a sequence $\{x_n\}$ in X such that $\{Ax_n\} \to u$ and $\{Sx_n\} \to u$. Since S is continuous, $SAx_n \to Su$. Since (A, S) is compatible, for a given pair (ϵ, λ) there is an integer $N_0(\epsilon, \lambda)$ such that

$$F_{ASx_n,SAx_n}(\frac{\epsilon}{2}) \geq 1 - \lambda \quad \& \quad F_{SAx_n,Su}(\frac{\epsilon}{2}) \geq 1 - \lambda$$

for all $n \geq N_0$. Now

$$\begin{aligned} F_{ASx_n,Su}(\epsilon) &\geq t(F_{ASx_n,SAx_n}(\frac{\epsilon}{2}), F_{SAx_n,Su}(\frac{\epsilon}{2})) \\ &\geq t(1 - \lambda, 1 - \lambda) \\ &\geq 1 - \lambda \end{aligned}$$

for all $n \geq N_0$. Hence $ASx_n \to Su$, i.e., the pair (A, S) is semi-compatible. □

PROPOSITION 2.5. ([10]) *If (X, d) is a metric space, then the metric d induces a mapping $F : X \times X \to L$, defined by $F_{p,q}(x) = H(x - d(p, q))$, $p, q \in X$ and $x \in \mathbb{R}$. Further, if $t : [0, 1] \times [0, 1] \to [0, 1]$ is defined by $t(a, b) = \min\{a, b\}$, then (X, F, t) is a Menger space. It is complete if (X, d) is complete.*

The space (X, F, t) is called an *induced Menger space.*

Now we give an example of a pair of self-maps (S, T) which is semi-compatible but not compatible. Further, we show that the semi-compatibility of the pair (S, T) need not imply the semi-compatibility of (T, S).

EXAMPLE 2.1. Let (X, d) be a metric space and (X, F, t) the induced Menger space with $F_{p,q}(\epsilon) = H(\epsilon - d(p, q))$ for all $p, q \in X$ and all $\epsilon > 0$, where $X = [0, 1]$. Define self-maps S and T as follows;

$$S(x) = \begin{cases} x, & \text{if } 0 \le x < \dfrac{1}{2} \\ 1, & \text{if } \dfrac{1}{2} \le x \le 1 \end{cases} \qquad T(x) = \begin{cases} 1 - x, & \text{if } 0 \le x < \dfrac{1}{2} \\ 1, & \text{if } \dfrac{1}{2} \le x \le 1. \end{cases}$$

And $x_n = \frac{1}{2} - \frac{1}{n}$. Then

$$F_{Sx_n, \frac{1}{2}}(\epsilon) = H(\epsilon - \frac{1}{n}).$$

So

$$\lim_{n\to\infty} F_{Sx_n, \frac{1}{2}}(\epsilon) = H(\epsilon) = 1.$$

Hence $Sx_n \to \frac{1}{2}$ as $n \to \infty$. Similarly, one can show that $Tx_n \to \frac{1}{2}$ as $n \to \infty$.

Now

$$F_{STx_n, TSx_n}(\epsilon) = H(\epsilon - (\frac{1}{2} - \frac{1}{n})).$$

So

$$\lim_{n\to\infty} F_{STx_n, TSx_n}(\epsilon) = H(\epsilon - \frac{1}{2}) \ne 1$$

for each $\epsilon > 0$. Hence (S, T) is not compatible.

On the other hand,

$$\lim_{n\to\infty} F_{STx_n, Tx}(\epsilon) = \lim_{n\to\infty} F_{STx_n, 1}(\epsilon) = H(\epsilon - (1 - 1)) = 1.$$

Thus (S, T) is not semi-compatible.

EXAMPLE 2.2. Let (X, F, t), S and $\{x_n\}$ be given as Example 2.1, and I the identity map on X. Then $\{Ix_n\} = \{x_n\} \to \frac{1}{2}$ and $\{Sx_n\} \to \frac{1}{2}$.

Now $\{ISx_n\} = \{Sx_n\} \to \frac{1}{2} \ne S(\frac{1}{2})$. Thus (I, S) is not semi-compatible.

On the other hand, for a sequence $\{x_n\}$ in X such that $\{x_n\} \to x$ and $\{Sx_n\} \to x$, we have $\{SIx_n\} = \{Sx_n\} \to x = Ix$. Thus (S, I) is semi-compatible.

The above example gives an important aspect of semi-compatibility.

LEMMA 2.6. ([11]) *Let* $\{p_n\}$ *be a sequence in a Menger space* (X, F, t) *with continuous* t*-norm and* $t(x,x) \geq x$ *for all* $x \in [0,1]$. *Assume that there is a* $k \in (0,1)$ *such that for all* $x > 0$ *and all* $n \in \mathbb{N}$

$$F_{p_n,p_{n+1}}(kx) \geq F_{p_{n-1},p_n}(x).$$

Then the sequence $\{p_n\}$ *is a Cauchy sequence in* X.

LEMMA 2.7. *Let* A, B, S *and* T *be self-maps of a Menger space* (X, F, t), *where* $t(a,b) = \min\{a,b\}$, $\forall a, b \in [0,1]$, *satisfying*

(1) $A(X) \subset T(X), \quad B(X) \subset S(X),$

(2) *there exists* $k \in (0,1)$ *and* $x_0 \in X$ *such that*

$$F_{Ap,Bq}(kx) \geq \min\{F_{Ap,Sp}(x), F_{Sq,Tq}(x), F_{Sp,Tq}(x), F_{Ap,Tq}(\alpha x), F_{Bq,Sp}((2-\alpha)x)\}$$

for all $p, q \in O(T^{-1}A, S^{-1}B, x_0)$, *all* $\alpha \in (0,1]$ *and all* $x > 0$. *Then the sequence* $\{y_n\}$, *defined by*

$$Ax_{2n} = Tx_{2n+1} = y_{2n+1} \text{ and } Bx_{2n+1} = Sx_{2n+2} = y_{2n+2} \tag{2.1}$$

for all $n = 0, 1, 2, \cdots$, *is a Cauchy sequence in* X.

Proof. Putting $p = x_{2n}$ and $q = x_{2n+1}$ in the condition (ii) and using the equation (2.1) and the properties of t-norm, we have

$$\begin{aligned}
&F_{y_{2n+1},y_{2n+2}}(kx) = F_{Ax_{2n},Bx_{2n+1}}(kx)\\
&\geq \min\{F_{Ax_{2n},Sx_{2n}}(x), F_{Bx_{2n+1},Tx_{2n+1}}(x), F_{Sx_{2n},Tx_{2n+1}}(x),\\
&\qquad F_{Ax_{2n},Tx_{2n+1}}((1-\lambda)x), F_{Sx_{2n},Bx_{2n+1}}((1+\lambda)x)\}\\
&= \min\{F_{y_{2n+1},y_{2n}}(x), F_{y_{2n+1},y_{2n+2}}(x), F_{y_{2n},y_{2n+1}}(x),\\
&\qquad F_{y_{2n+1},y_{2n+1}}((1-\lambda)x), F_{y_{2n+2},y_{2n}}((1+\lambda)x)\}\\
&= \min\{F_{y_{2n+1},y_{2n}}(x), F_{y_{2n+1},y_{2n+2}}(x), 1, F_{y_{2n},y_{2n+2}}((1+\lambda)x)\}\\
&\geq \min\{F_{y_{2n},y_{2n+1}}(x), F_{y_{2n+1},y_{2n+2}}(\lambda x)\}
\end{aligned}$$

for $\lambda \in (0,1)$, since

$$F_{y_{2n},y_{2n+2}}((1+\lambda)x) \geq \min\{F_{y_{2n},y_{2n+2}}(x), F_{y_{2n+1},y_{2n+2}}(\lambda x)\}.$$

Since t is continuous and a distribution function is left continuous, we have

$$F_{y_{2n+1},y_{2n+2}}(kx) \geq \min\{F_{y_{2n},y_{2n+1}}(x), F_{y_{2n+1},y_{2n+2}}(x)\}$$

as $\lambda \to 1$.

Similarly, one can show that

$$F_{y_{2n+2},y_{2n+3}}(kx) \geq \min\{F_{y_{2n+1},y_{2n+2}}(x), F_{y_{2n+2},y_{2n+3}}(x)\}.$$

Hence we can write

$$F_{y_n,y_{n+1}}(kx) \geq \min\{F_{y_{n-1},y_n}(x), F_{y_n,y_{n+1}}(x)\}$$

for $n = 2, 3, \cdots$. Consequently,

$$F_{y_n,y_{n+1}}(x) \geq \min\{F_{y_{n-1},y_n}(k^{-1}x), F_{y_n,y_{n+1}}(k^{-1}x)\}.$$

By repeating the above argument, we get

$$F_{y_n,y_{n+1}}(x) \geq \min\{F_{y_{n-1},y_n}(k^{-i}x), F_{y_n,y_{n+1}}(k^{-i}x)\}.$$

Since $F_{y_n,y_{n+1}}(k^{-i}x) \to 1$ as $i \to \infty$, it follows that

$$F_{y_n,y_{n+1}}(kx) \geq F_{y_{n-1},y_n}(x)$$

for all $n \in \mathbb{N}$ and all $x > 0$.

Therefore, by Lemma 2.6, $\{y_n\}$ is a Cauchy sequence in (X, F, t), as desired. □

3. Main results

THEOREM 3.1. *Let A, B, S and T be self-maps of a Menger space (X, F, t), where $t(a, b) = \min\{a, b\}$ for all $a, b \in [0, 1]$, satisfying* (i) *and* (ii) *for all $q \in X$ and all $p \in O(BT^{-1}, AS^{-1}, Ax_0) \cup O(T^{-1}A, S^{-1}B, x_0)$, and*

1. *the pair (A, S) is semi-compatible and the pair (B, T) is weakly compatible,*
2. *one of A and B is continuous,*
3. *for some $x_0 \in X$, the orbit $O(BT^{-1}, AS^{-1}, Ax_0)$ is complete.*

Then A, B, S and T have a unique common fixed point.

Proof. Define sequences $\{x_n\}$ and $\{y_n\}$ in X such that $y_{2n+1} = Ax_{2n} = Tx_{2n+1}$ and $y_{2n+2} = Bx_{2n+1} = Sx_{2n+2}$ for $n = 0, 1, 2, \cdots$. By Lemma 2.7, $\{y_n\}$ is a Cauchy sequence. Since $O(BT^1, AS^{-1}, Ax_0)$ is complete, the sequence $\{y_n\}$ converges to $u \in X$. The subsequences $\{Ax_{2n}\}$, $\{Bx_{2n+1}\}$, $\{Sx_{2n}\}$ and $\{Tx_{2n+1}\}$ converge to u, i.e.,

$$\{Ax_{2n}\} \to u, \qquad \{Bx_{2n+1}\} \to u, \tag{3.1}$$

$$\{Sx_{2n}\} \to u, \qquad \{Tx_{2n+1}\} \to u. \tag{3.2}$$

Case 1. When S is continuous:

Since S is continuous,

$$SAx_{2n} \to Su, \qquad S^2x_{2n} \to Su. \tag{3.3}$$

And the semi-compatibility of (A, S) gives

$$ASx_{2n} \to Su. \tag{3.4}$$

Since the condition (ii) holds for all α, we can take $\alpha = 1$.

Step I: Put $p = Sx_{2n}$ and $q = x_{2n+1}$. Then we get

$$F_{ASx_{2n},Bx_{2n+1}}(kx) \geq \min\{F_{ASx_{2n},SSx_{2n}}(x), F_{Bx_{2n+1},Tx_{2n+1}}(x), F_{SSx_{2n},Tx_{2n+1}}(x), F_{ASx_{2n},Tx_{2n+1}}(x), F_{Bx_{2n+1},SSx_{2n}}(x)\}.$$

Letting $n \to \infty$, and using (3.3) and (3.4) and the properties of t-norm, we get

$$F_{Su,u}(kx) \geq \min\{F_{Su,Su}(x), F_{u,u}(x), F_{Su,u}(x), F_{Su,u}(x), F_{u,Su}(x)\}.$$

Thus

$$F_{Su,u}(x) \geq F_{Su,u}(k^{-1}x).$$

By repeating the above process, we get

$$F_{Su,u}(x) \geq F_{Su,u}(k^{-1}x) \geq F_{Su,u}(k^{-2}x) \geq \cdots \geq F_{Su,u}(k^{-i}x) \geq \cdots ,$$

which tends to 1 as $i \to \infty$. Hence $F_{Su,u}(x) \geq 1$ implies that $F_{Su,u}(x) = 1$ for all x greater than zero. This gives $Su = u$.

Step II: Put $p = u$ and $q = x_{2n+1}$. Then we get

$$F_{Au,Bx_{2n+1}}(kx) \geq \min\{F_{Au,Su}(x), F_{Bx_{2n+1},Tx_{2n+1}}(x), F_{Su,Tx_{2n+1}}(x), F_{Au,Tx_{2n+1}}(x), F_{Bx_{2n+1},Su}(x)\}.$$

Letting $n \to \infty$, and using (3.1) and (3.4) and the properties of t-norm, we get

$$F_{Au,u}(kx) \geq \min\{F_{Au,u}(x), F_{u,u}(x), F_{u,u}(x), F_{Au,u}(x), F_{u,Au}(x)\}$$

for all $x > 0$. Thus

$$F_{Au,u}(kx) \geq F_{Au,u}(x)$$

for all $x > 0$. By the same reasoning as in Step I, we can show that $u = Au$. Hence

$$u = Au = Su. \tag{3.5}$$

Since $A(X) \subset T(X)$, there is a $w \in X$ such that $Au = Tw$. Thus

$$u = Au = Su = Tw. \tag{3.6}$$

Step III: Put $p = x_{2n}$ and $q = w$. Using the properties of t-norm and (3.6), we get

$$F_{Ax_{2n},Bw}(kx) \geq \min\{F_{Ax_{2n},Sx_{2n}}(x), F_{Bw,Tw}(x), F_{Sx_{2n},Tw}(x), F_{Ax_{2n},Tw}(x), F_{Bw,Sx_{2n}}(x)\}.$$

Letting $n \to \infty$, and using (3.1) and (3.6), we get

$$F_{u,Bw}(kx) \geq \min\{F_{u,u}(x), F_{u,u}(x), F_{Bw,u}(x), F_{u,Bu}(x), F_{Bw,u}(x)\}$$

for all $x > 0$. Thus

$$F_{u,Bw}(kx) \geq F_{Bw,u}(x)$$

for all $x > 0$. By the same reasoning as in Step I, we can show that $u = Bw$. Hence

$$Bw = Tw = u.$$

Since (B, T) is weak-compatible, $TBw = BTw$ gives

$$Bu = Tu. \tag{3.7}$$

Step IV: Put $p = u$ and $q = u$. Using the properties of t-norm, (3.5) and (3.7), we get

$$F_{Au,Bu}(kx) \geq \min\{F_{Au,Su}(x), F_{Bu,Su}(x), F_{Su,Tu}(x), F_{Au,Tu}(x), F_{Bu,Su}(x)\} \geq F_{Bu,u}(x).$$

So

$$F_{Bu,u}(kx) \geq F_{Bu,u}(x).$$

By the same reasoning as in Step I, we can show that $u = Bu = Tu$. Hence by (3.5)

$$u = Au = Su = Tu.$$

That is, u is a common fixed point of A, B, S and T.

Case 2. When A is continuous:

Since A is continuous, we have $ASx_{2n} \to Au$. The semi-compatibility of (A, S) gives $ASx_{2n} \to Su$. By the uniqueness of limit in a Menger space, we get $Au = Su$ and the rest of the proof is similar to the proof of Case 1.

Uniqueness : Let z be another common fixed point of A, B, S and T. Then $z = Az = Bz = Sz = Tz$. Put $p = u$ and $q = z$. Letting $\alpha = 1$ in (ii), we get

$$F_{Au,Bz}(kx) \geq \min\{F_{Au,Su}(x), F_{Bz,Tz}(x), \\ F_{Su,Tz}(x), F_{Au,Tz}(x), F_{Bz,Su}(x)\}.$$

That is,

$$F_{u,z}(kx) \geq \min\{F_{u,u}(x), F_{z,z}(x), F_{u,z}(x), F_{u,z}(x), F_{u,z}(x)\} \\ \min\{F_{u,z}(x), F_{u,z}(x)\} \geq F_{u,z}(x)$$

for all $x > 0$. By the same reasoning as in Step I, we can show that $u = z$. Therefore, u is a unique common fixed point of A, B, S and T, as desired. □

COROLLARY 3.2. *Let $k \in (0, 1)$ be fixed. Let A, B, S and T be self-maps of a complete Menger space (X, F, t), where $t(a, b) = \min\{a, b\}$, satisfying* (i) *and* (ii) *for all $p \in O(BT^{-1}, AS^{-1}, Ax_0) \cup \overline{O(T^{-1}A, S^{-1}B, x_0)}$ and all $q \in X$,* (3) *in Theorem 3.1, and*

(1) *the pairs (A, S) and (B, T) are semi-compatible,*
(2) *one of A, B, S and T is continuous.*

Then A, B, S and T have a unique common fixed point.

Proof. The semi-compatibility implies the weak-compatibility. The proof follows from Theorem 3.1. □

In [8], Mishra proved the following.

THEOREM 3.3. ([8]) *Let A, B, S and T be self-maps of a Menger space (X, F, t), with a continuous t-norm with $t(x, x) \geq x$ for all $x \in_0, 1)$, satisfying* (i) *and*

(1) *for all $p, q \in X$, all $x > 0$ and some $k \in (0, 1)$,*

$$F_{Ap,Bq}(kx) \geq t(F_{Ap,Sp}(x), t(F_{Bq,Tq}(x), t(F_{Ap,Tq}(x), t(F_{Ap,Tq}(\alpha x), F_{Bq,Sp}(2-\alpha)(x))))),$$

(2) *the pairs (A, S) and (B, T) are compatible,*
(3) *S and T are continuous.*

Then A, B, S and T have a unique common fixed point.

REMARK 3.1. In view of Proposition 2.2, let $t(a, b) = \min\{a, b\}$. Then (1) in Theorem 3.3 becomes

$$F_{Ap,Bq}(kx) \geq \min\{F_{Ap,Sp}(x), F_{Bq,Tq}(x), F_{Sp,Tq}(x), F_{Ap,Tq}(\alpha x), F_{Bq,Sp}((2-\alpha)x)\}.$$

Theorem 3.1 is a wide generalization of the above result.

Now keeping only the first three factors in R.H.S. of the contractive condition (ii) of Theorem 3.1 and taking $B = A$, we get the following.

COROLLARY 3.4. *Three self-maps A, S and T of a Menger space (X, F, t), where $t(a, b) = \min\{a, b\}$, satisfying the conditions:*

(1) *for some $x_0 \in X$ an orbit $O(AT^{-1}, AS^{-1}, Ax_0)$ is complete,*
(2) *$A(X) \subset S(X) \cap T(X)$,*

(3) *one of A and S is continuous,*
(4) *(A, S) is semi-compatible and (A, T) is weak-compatible,*
(5) *for some $x_0 \in X$, there is a $k \in (0, 1)$ such that for all $p \in O(AT^{-1}, AS^{-1}, Ax_0) \cup \overline{O(T^{-1}A, S^{1}A, x_0)}$, all $q \in X$ and all $x > 0$,*

$$F_{Ap,Aq}(kx) \geq \min\{F_{Ap,Sp}(x), F_{Aq,Tq}(x), F_{Sp,Tq}(x)\}.$$

Then A, S and T have a unique common fixed point.

REMARK 3.2. Corollary 3.4 is a generalization of the theorem of Dedeic and Sarapa [4] in the sense that commutativity of both pairs (A, S) and (A, T) is replaced by a weaker condition of semi-compatibility of one pair and weak-compatibility of the other pair. Further, we need the continuity of one of the maps A or S and not of both A and T as found there.

If we take $S = T = I$ in the above corollary, then we get the following.

COROLLARY 3.5. *Let A be a self-map on a Menger space (X, F, t) with t-norm defined by $t(a, b) = \min\{a, b\}$ satisfying*

(1) *for some $x_0 \in X$ the orbit $O(A, x_0)$ is complete,*
(2) *for all $p \in \overline{O(A, x_0)}$ and all $q \in X$, there is a $k \in (0, 1)$ such that*

$$F_{Ap,Aq}(kx) \geq \min\{F_{Ap,p}(x), F_{Ap,q}(x), F_{p,q}(x)\}.$$

Then A has a unique fixed point.

REMARK 3.3. The above result improves the result of Sehgal and Bharucha Reid [10] in the sense that the domain of p in the contractive condition is reduced to the closure of an orbit and also the domain of completeness is an orbit only, not the whole space.

If we keep only one last factor in R.H.S. of the contractive condition in the above corollary, then we get the following.

COROLLARY 3.6. *Let A be a self-map of a Menger space with t-norm defined by $t(a,b) = \min\{a,b\}$ for all $a,b \in (0,1]$, satisfying (1) in Corollary 3.5 and for all $p \in \overline{O(A,x_0)}$ and all $q \in X$ there is a $k \in (0,1)$ such that*

$$F_{Ap,Aq}(kx) \geq F_{p,q}(x)$$

for all $x > 0$. Then A has a unique fixed point.

REMARK 3.4. This result improves a Menger version of Banach Contraction Theorem as given in Hicks [5, Theorem 1] in the sense that the domain of p in the contractive condition is the closure of the orbit $O(A,x_0)$, not the whole space X.

If we take $S = T = I$ in Corollary 3.2 and keeping only one last factor in the R.H.S. of the contractive condition, then we get the following.

COROLLARY 3.7. *Let A and B be self-maps on a Menger space (X,F,t) with t-norm defined by $t(a,b) = \min\{a,b\}$ for all $a,b \in [0,1]$, satisfying*

(1) *for some $x_0 \in X$ the orbit $O(A,B,x_0)$ is complete,*
(2) *for all $p \in \overline{O(A,B,x_0)}$ and all $q \in X$ there is a $k \in (0,1)$ such that*

$$F_{Ap,Bq}(kx) \geq F_{p,q}(x).$$

Then A and B have a unique common fixed point.

THEOREM 3.8. *Let $\{T_n\}$ be a sequence of self-maps on a complete Mneger space (X,F,t) with t-norm defined by $t(a,b) = \min\{a,b\}$ for all $a,b \in X$ and there are constants $k_{i,i+1} \in (0,1)$ such that*

$$F_{T_i^{m_i}p,T_{i+1}^{m_{i+1}}q}(k_{i,i+1}x) \geq F_{p,q}(x)$$

for all $x > 0$ and all $p, q \in X$. Then the self-maps T_n have a unique common fixed point in X.

Proof. By Corollary 3.7, each pair $(T_i^{m_i}, T_{i+1}^{m_{i+1}})$ has a unique common fixed point, say, u, for all i. Hence $u = T_i^{m_i} u = T_{i+1}^{m_{i+1}} u$. Now $T_i^{m_i}(T_i u) = T_i(T_i^{m_i} u) = T_i u$, i.e., $T_i u$ is a fixed point of $T_i^{m_i}$. Similarly, $T_{i+1}u$ is a fixed point of $T_{i+1}^{m_{i+1}}$. Put $p = T_i u$ and $q = u$ in the above condition, we get

$$F_{T_i^{m_i} T_i u, T_{i+1}^{m_{i+1}} u}(k_{i,i+1}x) \geq F_{T_i u, u}(x)$$

implies

$$F_{T_i u, u}(k_{i,i+1}x) \geq F_{T_i u, u}(x),$$

which gives $T_i u = u$. Similarly, we can show that $T_{i+1}u = u$. So $T_i u = T_{i+1} u = u$. Therefore, u is a common fixed point of T_i and T_{i+1}. If v is another common fixed point of T_i and T_{i+1}, then v is also a common fixed point of $T_i^{m_i}$ and $T_{i+1}^{m_{i+1}}$, which is unique. Hence $u = v$. Thus any two consecutive maps of the sequence $\{T_n\}$ have a unique common fixed point. Let u_1 be the common fixed point of the pair (T_1, T_2) and u_2 the common fixed point of the pair (T_2, T_3). Put $p = u_1$ and $q = u_2$ in the contractive condition. We get

$$F_{u_1,u_2}(k_{1,2}x) \geq F_{u_1,u_2}(x)$$

for all $x > 0$. This implies $u_1 = u_2$. Hence each consecutive pair of $\{T_n\}$ has the same unique common fixed point, which must be the unique common fixed point of $\{T_n\}$. □

REMARK 3.5. Theorem 3.8, which is a corollary of Theorem 3.1, is a generalization of Theorems 1 and 2 of Dedeic and Sarapa [3] since the constants $k_{i,i+1}$ are distinct for different i and the powers m_i of maps T_i are distinct for different i. Also the condition of continuity of maps is not required here.

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