JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **17**, No.2, October 2004

POISSON DERIVATIONS ACTING ON MULTI-PARAMETER SYMPLECTIC POISSON ALGEBRA

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1. Introduction

A class of algebras $K_{n,\Gamma}^{P,Q}$, constructed by Horton in [2], includes the multiparameter quantized coordinate rings of symplectic and Euclidean $2n$-spaces, the graded quantized Weyl algebra, the quantized Heisenberg space, and is similar to a class of iterated skew polynomial rings constructed by Gómez-Torrecillas and Kaoutit in [1]. The prime and primitive spectra for the multiparameter quantized coordinate rings of symplectic and Euclidean $2n$-spaces were established by Gómez-Torrecillas and Kaoutit in [1], by Horton in [2] and by the author in [3]. Moreover the author constructed a class of Poisson algebras $A_{n,\Gamma}^{P,Q}$ in [5], whose quantization is the algebra $K_{n,\Gamma}^{P,Q}$. Here we consider an additive group K acting by Poisson derivations on $A_{n,\Gamma}^{P,Q}$ which gives a classification of K-prime Poisson ideals of $A_{n,\Gamma}^{P,Q}$ and we see that the additive group K is considered as a Poisson version of a multiplicative group acting by automorphisms on $K_{n,\Gamma}^{P,Q}$.

Assume throughout the paper that $\mathbf{k}$ denotes an algebraically closed field of characteristic zero and that all vector spaces are over $\mathbf{k}$. A Poisson algebra A is always a commutative $\mathbf{k}$-algebra with $\mathbf{k}$-bilinear map $\{\cdot,\cdot\}$, called a Poisson bracket, such that $(A,\{\cdot,\cdot\})$ is a Lie algebra

Received by the editors on October 28, 2004.
2000 *Mathematics Subject Classifications* : Primary 17B63; Secondary 16W30.
Key words and phrases: Poisson algebra, quantized algebra.

and $\{\cdot,\cdot\}$ satisfies the Leibniz rule, that is,

$$\{ab, c\} = a\{b, c\} + b\{a, c\}$$

for all $a, b, c \in A$. Hence, for any element $a \in A$, the map

$$h_a : A \longrightarrow A, \quad h_a(b) = \{a, b\}$$

is a derivation in A which is called a Hamiltonian defined by a.

2. Poisson polynomial ring

Let A be a Poisson algebra. A derivation δ on A is said to be a Poisson derivation if $\delta(\{a, b\}) = \{\delta(a), b\} + \{a, \delta(b)\}$ for all $a, b \in A$.

THEOREM 2.1. *For a Poisson algebra A with Poisson bracket $\{\cdot,\cdot\}_A$ and* **k***-linear maps α, δ from A into itself, the polynomial ring $A[x]$ is a Poisson algebra with Poisson bracket*

$$\{a, x\} = \alpha(a)x + \delta(a) \tag{2.1}$$

for all $a \in A$ if and only if α is a Poisson derivation and δ is a derivation such that

$$\delta(\{a, b\}_A) - \{\delta(a), b\}_A - \{a, \delta(b)\}_A = \delta(a)\alpha(b) - \alpha(a)\delta(b) \tag{2.2}$$

for all $a, b \in A$. In this case, we denote the Poisson algebra $A[x]$ by $A[x; \alpha, \delta]_p$ and if $\delta = 0$ then we simply write $A[x; \alpha]_p$ for $A[x; \alpha, 0]_p$.

Proof. [4, 1.1 Theorem] □

PROPOSITION 2.2. *Let A be a Poisson algebra. For Poisson derivations α and β on A, $c \in$* **k** *and $u \in A$ such that*

$$\alpha\beta = \beta\alpha, \quad \{a, u\} = (\alpha + \beta)(a)u$$

for all $a \in A$, the polynomial ring $A[y, x]$ has the following Poisson bracket

$$\{a, y\} = \alpha(a)y, \ \{a, x\} = \beta(a)x, \ \{y, x\} = cyx + u \tag{2.3}$$

for all $a \in A$. The Poisson algebra $A[y, x]$ with Poisson bracket (2.3) can be presented by $A[y; \alpha]_p[x; \beta', \delta]_p$, where β' is the Poisson derivation *on $A[y;\alpha]_p$ such that $\beta'|_A = \beta$ and $\beta'(y) = cy$, and δ is the derivation on $A[y;\alpha]_p$ such that $\delta|_A = 0$, $\delta(y) = u$.*

We often denote by $(A;\alpha,\beta,c,u)$ the Poisson algebra $A[y,x]$ with Poisson bracket (2.3).

Proof. By Theorem 2.1, there exists the Poisson algebra $A[y;\alpha]_p$ with Poisson bracket $\{a,y\} = \alpha(a)y$ for all $a \in A$ and the derivation β is extended to a derivation, denoted by β', to $A[y;\alpha]_p$ by setting $\beta'(y) = cy$. Note that the derivation $\delta = u\frac{d}{dy}$ on $A[y;\alpha]_p$ satisfies $\delta(y) = u$ and $\delta(a) = 0$ for all $a \in A$. Let us prove that, for all $f,g \in A[y;\alpha]_p$,

$$\begin{aligned} &\beta'(\{f,g\}) = \{\beta'(f),g\} + \{f,\beta'(g)\} \\ &\delta(\{f,g\}) = \{\delta(f),g\} + \{f,\delta(g)\} + \delta(f)\beta'(g) - \beta'(f)\delta(g). \end{aligned} \tag{2.4}$$

If $f,g \in A$ then the formulas in (2.4) hold trivially since β' is a Poisson derivation on A. Hence it is enough to prove (2.4) for the case $f = a \in A$ and $g = y$. Now we have that

$$\begin{aligned} \beta'(\{a,y\}) &= \beta'(\alpha(a)y) = \alpha(a)\beta'(y) + \beta'(\alpha(a))y \\ &= c\alpha(a)y + \alpha(\beta(a))y = \{\beta'(a),y\} + \{a,\beta'(y)\} \\ \delta(\{a,y\}) &= \delta(\alpha(a)y) = \alpha(a)u = \{a,u\} - \beta(a)u \\ &= \{\delta(a),y\} + \{a,\delta(y)\} + \delta(a)\beta'(y) - \beta'(a)\delta(y), \end{aligned}$$

as claimed.

Therefore β' is a Poisson derivation on $A[y;\alpha]_p$ such that the pair (β',δ) satisfies (2.2), and thus, by Theorem 2.1, there exists the Poisson algebra $A[y,x] = A[y;\alpha]_p[x;\beta',\delta]_p$ with the Poisson bracket (2.3). □

3. Poisson algebra $A_n = A_{n,\Gamma}^{P,Q}$

DEFINITION 3.1. ([5, Theorem 1.2]) Let $\Gamma = (\gamma_{ij})$ be a skew-symmetric $n \times n$-matrix with entries in $\mathbf{k}$, that is, $\gamma_{ij} = -\gamma_{ji}$ for all $i, j = 1, \cdots, n$. Let $P = (p_1, p_2, \cdots, p_n)$ and $Q = (q_1, q_2, \cdots, q_n)$ be elements of $\mathbf{k}^n$ such that $p_i \neq q_i$ for each $i = 1, \cdots, n$. Then the Poisson algebra $\mathbf{k}[y_1, x_1, \cdots, y_n, x_n]$ with Poisson bracket:

$$
\begin{array}{ll}
\{y_i, y_j\} = \gamma_{ij} y_i y_j & (\text{all } i, j) \\
\{x_i, y_j\} = (p_j - \gamma_{ij}) y_j x_i & (i < j) \\
\{y_i, x_j\} = -(q_i + \gamma_{ij}) y_i x_j & (i < j) \\
\{x_i, x_j\} = (q_i - p_j + \gamma_{ij}) x_i x_j & (i < j) \\
\{x_i, y_i\} = q_i y_i x_i + \sum_{k=1}^{i-1} (q_k - p_k) y_k x_k & (\text{all } i)
\end{array}
\tag{3.1}
$$

is called the multi-parameter symplectic Poisson algebra and denoted by $A_{n,\Gamma}^{P,Q}$ or by A_n unless any confusion arises.

REMARK 3.2. Set

$$A_0 = \mathbf{k}, \qquad A_j = \mathbf{k}[y_1, x_1, \cdots, y_j, x_j] \subseteq A_{n,\Gamma}^{P,Q}$$

for each $j = 0, 1, \cdots, n$. Then each A_j is a Poisson subalgebra of $A_{n,\Gamma}^{P,Q}$ and $A_j = A_{j-1}[y_j, x_j]$ for each j, and thus, by Theorem 2.1, there exist Poisson derivations α_j, β_j and a derivation δ_j such that A_j can be presented by

$$A_j = A_{j-1}[y_j; \alpha_j]_p[x_j; \beta_j, \delta_j]_p,$$

where

(3.2)

$$
\begin{array}{lll}
\alpha_j(y_i) = \gamma_{ij} y_i, & \alpha_j(x_i) = (p_j - \gamma_{ij}) x_i & (i < j) \\
\beta_j(y_i) = -(q_i + \gamma_{ij}) y_i, & \beta_j(x_i) = (q_i - p_j + \gamma_{ij}) x_i & (i < j) \\
\delta_j(y_i) = 0, & \delta_j(x_i) = 0 & (i < j) \\
\beta_j(y_j) = -q_j y_j & \delta_j(y_j) = -\sum_{k=1}^{j-1} (q_k - p_k) y_k x_k. &
\end{array}
$$

Set

$$\Omega_0 = 0, \qquad \Omega_j = \sum_{k=1}^{j} (q_k - p_k) y_k x_k$$

for all $j = 1, \cdots, n-1$, and note that

$$\alpha_j\beta_j = \beta_j\alpha_j, \quad \{a, \Omega_{j-1}\} = (\alpha_j + \beta_j)(a)\Omega_{j-1}$$

for all $a \in A_{j-1}$. Hence we have $A_j = (A_{j-1}; \alpha_j, \beta_j, -q_j, -\Omega_{j-1})$ by Proposition 2.2 and so the Poisson algebra $A_n = A_{n,\Gamma}^{P,Q}$ has the chain of Poisson subalgebras

$$A_0 = \mathbf{k} \subseteq A_1 = (A_0; \alpha_1, \beta_1, -q_1, 0) \subseteq \cdots \subseteq A_n = (A_{n-1}; \alpha_n, \beta_n, -q_n, -\Omega_{n-1}).$$

LEMMA 3.3. *As in Remark 3.2, set*

$$\Omega_i = \sum_{k=1}^{i}(q_k - p_k)y_k x_k \in A_n = A_{n,\Gamma}^{P,Q}$$

for each $i = 1, \cdots, n$ *and* $\Omega_0 = 0$.

(a) For any Ω_j,

$$\begin{array}{lll} \{y_i, \Omega_j\} = -q_i y_i \Omega_j, & \{x_i, \Omega_j\} = q_i x_i \Omega_j, & (i \le j) \\ \{y_i, \Omega_j\} = -p_i y_i \Omega_j, & \{x_i, \Omega_j\} = p_i x_i \Omega_j, & (i > j) \\ \{\Omega_i, \Omega_j\} = 0, & & (\textit{all } i, j) \end{array}$$

(b) We have the following relations:

$$\Omega_{i-1} = \{x_i, y_i\} - q_i y_i x_i, \quad \Omega_i = \{x_i, y_i\} - p_i y_i x_i \tag{3.3}$$

Hence, y_i *and* x_i *are Poisson normal modulo* $\langle \Omega_i \rangle$ *and* $\langle \Omega_{i-1} \rangle$.

Proof. The formulas of (a) follow from (3.1) and the formulas of (b) follow immediately since $\Omega_i = (q_i - p_i)y_i x_i + \Omega_{i-1}$ and $\{x_i, y_i\} = q_i y_i x_i + \Omega_{i-1}$. □

DEFINITION 3.4. ([3, Definition 1.4]) Let $\mathcal{P}_n = \{\Omega_1, y_1, x_1, \cdots, \Omega_n, y_n, x_n\} \subseteq A_n$. A subset T of $\mathcal{P}_n$ is said to be *admissible* if it satisfies the conditions:

(a) y_i or $x_i \in T \Leftrightarrow \Omega_i$ and $\Omega_{i-1} \in T \quad (2 \le i \le n)$

(b) y_1 or $x_1 \in T \Leftrightarrow \Omega_1 \in T$.

PROPOSITION 3.5. *(a) For every admissible set T, the ideal $\langle T\rangle$ is a prime Poisson ideal of A_n.*

(b) For every prime Poisson ideal P of A_n, $P\cap\mathcal{P}_n$ is an admissible set.

Proof. [5, 1.5 and 1.6] □

4. K-actions on $A_{n,\Gamma}^{P,Q}$

In this section, we will show that every K-prime Poisson ideal of $A_{n,\Gamma}^{P,Q}$ is generated by an admissible set. The statements and proofs of this section are modified from those of [2, §3].

DEFINITION 4.1. Let

$$K=\{(h_1,h_2,\cdots,h_{2n-1},h_{2n})\in\mathbf{k}^{2n}\mid h_{2i-1}+h_{2i}=h_{2j-1}+h_{2j}\text{ for all }i,j=1,\cdots,n\}.$$

The additive group K acts on A_n as follows:

$$(h_1,h_2,\cdots,h_{2n-1},h_{2n})(f)=\sum_i(h_{2i-1}y_i\frac{\partial f}{\partial y_i}+h_{2i}x_i\frac{\partial f}{\partial x_i})$$

for all elements $f\in A_n$. Note that each element of K acts on A_n by a Poisson derivation.

Let A be a Poisson algebra and let an additive group H act on A by Poisson derivations. A proper Poisson ideal Q of A is said to be H-prime Poisson ideal if Q is H-stable such that whenever I,J are H-stable Poisson ideals of A with $IJ\subseteq Q$, either $I\subseteq Q$ or $J\subseteq Q$. A Poisson algebra A is said to be H-simple if 0 and A are the only H-stable Poisson ideals of A.

LEMMA 4.2. *Let A be a Poisson algebra and let α be a Poisson derivation on A. Suppose that H acts on $A[x^{\pm1};\alpha]_p$ so that x is an H-eigenvector and A is both H-stable and H-simple, where H acts on A by restriction. If H contains a Poisson derivation g such that $g|_A=\alpha$ and $g(x)=cx$ for some $0\neq c\in\mathbf{k}$ then $A[x^{\pm1};\alpha]_p$ is H-simple.*

Proof. Let I be a nonzero proper H-Poisson ideal of $A[x^{\pm 1};\alpha]_p$. Then choose $0 \neq a \in I$, of shortest length with respect to x, say $a = a_k x^k + \cdots + a_m x^m$ for some $k \leq m$, where $a_i \in A$ for each i and $a_k, a_m \neq 0$. Since x is unit and $A \cap I = 0$, we may assume that $k = 0$ and $a = a_0 + \cdots + a_m x^m$, where $m > 0$ and $a_0, a_m \neq 0$. Set $J = \{r \in A \mid r + r_1 x + \cdots + r_m x^m \in I \text{ for some } r_1, \cdots, r_m \in A\}$ and note that J is a Poisson ideal of A. Given any $h \in H$, let λ_h be the h-eigenvalue of x. Since I is H-stable, $h(r + r_1 x + \cdots + r_m x^m) = h(r) + (h(r_1) + \lambda_h r_1)x + \cdots + (h(r_m) + m\lambda_h r_m)x^m \in I$, and so $h(r) \in J$. Hence J is an H-Poisson ideal of A, and thus either $J = 0$ or $J = A$; by our choice of a, $1 \in J$. Thus we may assume that $a = 1 + a_1 x + \cdots, a_m x^m$. Since I is H-stable, $g(a) = (g(a_1) + ca_1)x + \cdots + (g(a_m) + mca_m)x^m \in I$, which has the length less than a, hence $g(a) = 0$ and $g(a_i) + ica_i = \alpha(a_i) + ica_i = 0$ for each $i = 1, \cdots, m$. Now, $\{a, x\} = \alpha(a_1)x^2 + \cdots + \alpha(a_m)x^{m+1}$ is an element of I with the length less than a. Hence $\alpha(a_i) = 0$ and thus $a_i = 0$ for all $i = 1, \cdots, m$. It follows that $a = 1 \in I$, a contradiction. As a result, $A[x^{\pm 1};\alpha]_p$ is H-simple. $\square$

LEMMA 4.3. *Let $B = A[y;\alpha]_p[x;\beta]_p$, where A is a prime Poisson algebra and both α and β are Poisson derivations, such that $\beta(A) \subseteq A$ and $\beta(y) = cy$ for some $c \in \mathbf{k}$, and that H is a group of Poisson derivations on B such that A is H-stable and y, x are H-eigenvectors. If there exist $f, g \in H$ such that $f|_A = \alpha$ with $f(y) = ay$ and $g|_{A[y;\alpha]_p} = \beta$ with $g(x) = bx$ for some $a, b \in \mathbf{k}^\times$, and if A is H-simple, then*

(a) $B[y^{-1}][x^{-1}], B/\langle y, x\rangle, (B/\langle y\rangle)[x^{-1}]$, *and* $(B/\langle x\rangle)[y^{-1}]$ *are H-simple.*
(b) *B has only four H-prime Poisson ideals* $0, \langle y\rangle, \langle x\rangle, \langle y, x\rangle$.

Proof. (a) Note that

$$B[y^{-1}] = A[y^{\pm 1};\alpha]_p[x;\beta]_p, \quad B[y^{-1}][x^{-1}] = A[y^{\pm 1};\alpha]_p[x^{\pm 1};\beta]_p.$$

By Lemma 4.2, $A[y^{\pm 1};\alpha]_p$ is H-simple. Now apply Lemma 4.2 twice to obtain that $B[y^{-1}][x^{-1}] = A[y^{\pm 1};\alpha]_p[x^{\pm 1};\beta]_p$ is H-simple.

Since $B/\langle y,x\rangle \cong_H A$, it follows that $B/\langle y,x\rangle$ is H-simple. Next, the Poisson algebra $(B/\langle y\rangle)[x^{-1}] \cong_H A[x^{\pm 1};\beta]_p$ is H-simple by Lemma 4.2. Analogously, $(B/\langle x\rangle)[y^{-1}] \cong_H A[y^{\pm 1};\alpha]_p$ is H-simple.

(b) Clearly, $0, \langle y\rangle, \langle x\rangle, \langle y,x\rangle$ are all H-prime Poisson ideals. Suppose that P is a nonzero H-prime Poisson ideal of B. The extended ideal $P^e = PB[y^{-1}][x^{-1}]$ contains the multiplicative identity because $B[y^{-1}][x^{-1}]$ is H-simple. Thus, $y^i x^j \in P$ for some i,j and thus P contains y or x since $\langle y\rangle$ and $\langle x\rangle$ are both H-stable Poisson ideals of B. If $x \in P$ then $P/\langle x\rangle$ is an H-prime Poisson ideal of $B/\langle x\rangle$, and thus $P = \langle x\rangle$ or $P = \langle x,y\rangle$ since $(B/\langle x\rangle)[y^{-1}]$ is H-simple. Analogously, if P contains y then $P = \langle y\rangle$ or $P = \langle x,y\rangle$. As a result, B has only four H-prime Poisson ideals $0, \langle y\rangle, \langle x\rangle, \langle y,x\rangle$. □

Lemma 4.4. *Let $B = (A;\alpha,\beta,c,u) = A[y;\alpha]_p[x;\beta',\delta]_p$ be the Poisson algebra given in Proposition 2.2. Assume, in addition, that A is a prime Poisson algebra, $\alpha(u) = du$, $\beta(u) = -du$ for some $d \in \mathbf{k}$ with $c+d \neq 0$ and $0 \neq \delta(y) = u \in A$ is Poisson normal in B. Set $z = (c+d)yx + \delta(y)$. Let H be a group of Poisson derivations on B such that A is H-stable and y, x and z are H-eigenvectors. Suppose that there exist $f, g \in H$ such that $f|_A = \alpha$ with $f(y) = ay$ for some $a \in \mathbf{k}^\times$ and $g|_{A[y;\alpha]_p} = \beta'$ with $g(y^{-1}z) = by^{-1}z$ for some $b \in \mathbf{k}^\times$. If A is H-simple, then*

(a) *$\delta(y)$ is invertible in B.*

(b) *no proper H-stable Poisson ideal of B contains a power of y.*

(c) *$B[y^{-1}][z^{-1}]$, $B[z^{-1}]$ and $B/\langle z\rangle$ are H-simple.*

(d) *the only H-prime Poisson ideals of B are 0 and $\langle z\rangle$.*

Proof. (a) Since $\delta(y) = \{y,x\} - cyx$ is H-eigenvector and Poisson normal, $\langle\delta(y)\rangle$ is an H-stable Poisson ideal of B. Thus $I = \langle\delta(y)\rangle \cap A$ is a nonzero H-stable Poisson ideal of A, and hence $1 \in I$ since A is H-simple. In particular, $1 \in \langle\delta(y)\rangle$ and so $\delta(y)B = \langle\delta(y)\rangle = B$. Consequently, $\delta(y)$ is invertible in B.

(b) Suppose that P is a proper H-Poisson ideal of B such that $y^j \in P$ for some $j > 0$. Whenever $y^j \in P$ for some $j > 0$, we have that

$$jy^{j-1}\delta(y) = \delta(y^j) = \{y^j, x\} - \beta'(y^j)x = \{y^j, x\} - jcy^jx \in P,$$

and hence $y^{j-1} \in P$ since $\delta(y)$ is invertible in B by (a). The repeated applications of the above argument guarantee that $y \in P$. Therefore $\delta(y) = \{y, x\} - cyx \in P$, and thus no proper H-Poisson ideal contains a power of y since $\delta(y)$ is invertible in B by (a).

(c) Note that $B[y^{-1}] = A[y^{\pm 1}; \alpha]_p[y^{-1}z; \beta']_p$ and

$$B[y^{-1}][z^{-1}] = A[y^{\pm 1}; \alpha]_p[(y^{-1}z)^{\pm 1}; \beta']_p, \quad g|_{A[y^{\pm 1};\alpha]_p} = \beta'.$$

Applying Lemma 4.2 yields that both $A[y^{\pm 1}; \alpha]_p$ and $A[y^{\pm 1}; \alpha]_p[(y^{-1}z)^{\pm 1}; \beta']_p$ are H-simple, so $B[y^{-1}][z^{-1}]$ is H-simple.

Let P be an H-prime Poisson ideal of $B[z^{-1}]$. Then P is induced from an H-prime Poisson ideal $\check{P}$ of B disjoint from $\{z^j \mid j = 0, 1, \cdots\}$. By (b), $\check{P}$ contains no y^j. Suppose that $\check{P}$ contains some y^iz^j. Since z and y are Poisson normal and H-eigenvectors and the hypothesis, we have that $y^i \in \check{P}$ or $z^j \in \check{P}$, a contradiction. Thus $\check{P}$ is disjoint from the multiplicative set generated by y and z. Hence the extension $\check{P}^e$ to $B[y^{-1}][z^{-1}]$ is an H-prime Poisson ideal. Since $B[y^{-1}][z^{-1}]$ is H-simple, $\check{P}^e = 0$, and so $\check{P} = 0$, so $P = 0$. Thus $B[z^{-1}]$ contains no nonzero H-prime Poisson ideals.

If I is a proper H-Poisson ideal of $B[z^{-1}]$ then I is contained in a prime Poisson ideal P of $B[z^{-1}]$. Set $Q = (P : H)$ the largest H-stable Poisson ideal contained in P. If I and J are H-stable Poisson ideals such that $IJ \subseteq Q$ then either $I \subseteq P$ or $J \subseteq P$, and thus either $I \subseteq Q$ or $J \subseteq Q$. It follows that Q is an H-prime Poisson ideal such that $I \subseteq Q \subseteq P$. Since $B[z^{-1}]$ does not have a nonzero H-prime Poisson ideal, we have that $I = Q = 0$. Hence, $B[z^{-1}]$ is H-simple.

Note that $\langle z\rangle$ is a Poisson ideal of B since z is Poisson normal, and $zB[y^{-1}]$ is also a Poisson ideal of $B[y^{-1}]$. Observe that

$$\begin{aligned}(B/\langle z\rangle)[y^{-1}] &\cong_H B[y^{-1}]/(zB[y^{-1}])\\ &= A[y^{\pm 1};\alpha]_p[y^{-1}z;\beta']_p/(zA[y^{\pm 1};\alpha]_p[y^{-1}z;\beta']_p)\\ &\cong_H A[y^{\pm 1};\alpha]_p.\end{aligned}$$

Thus $(B/\langle z\rangle)[y^{-1}]$ is H-simple by Lemma 4.2. Denote by $\overline{b}$ the canonical homomorphic image of $b \in B$ in $B/\langle z\rangle$. Since $\overline{yx} = -(c+d)^{-1}\overline{\delta(y)}$ and $\delta(y)$ is invertible in A by (a), $\overline{y}$ is invertible in $B/\langle z\rangle$, and thus $B/\langle z\rangle = (B/\langle z\rangle)[y^{-1}]$ is H-simple.

(d) Clearly 0 is an H-prime Poisson ideal of B since B is a prime Poisson algebra. Further, $\langle z\rangle$ is H-stable and prime Poisson since z is an H-eigenvector and Poisson normal in B. Now, let P be an H-prime Poisson ideal of B. If P contains no z^i then P extends to an H-prime Poisson ideal $\check{P}$ of $B[z^{-1}]$. Since $B[z^{-1}]$ is H-simple by (c), $\check{P} = 0$, and so $P = 0$. Assume that P contains some z^i. Then $z \in P$ since $\langle z\rangle$ is an H-stable Poisson ideal and P is an H-prime Poisson ideal. Thus 0 and $\langle z\rangle$ are the only H-prime Poisson ideals of B since $B/\langle z\rangle$ is H-simple by (c). □

Definition 4.5. Given an admissible set T of A_n, let N_T be the subset of $\mathcal{P}_n$ defined by

(a) $y_1 \in N_T$ if and only if $y_1 \notin T$
(b) $x_1 \in N_T$ if and only if $x_1 \notin T$
(c) for $i > 1$, $\Omega_i \in N_T$ if and only if $\Omega_{i-1} \notin T$ and $\Omega_i \notin T$
(d) for $i > 1$, $y_i \in N_T$ if and only if $\Omega_{i-1} \in T$ and $y_i \notin T$
(e) for $i > 1$, $x_i \in N_T$ if and only if $\Omega_{i-1} \in T$ and $x_i \notin T$

Theorem 4.6. *For an admissible set T, let E_T be the multiplicative set generated by N_T.*

(a) $E_T \cap \langle T\rangle = \phi$.

(b) $A_n^T = (A_n/\langle T\rangle)[E_T^{-1}]$ *is H-simple.*

Proof. (a) It follows immediately from Proposition 3.5.

(b) We proceed by induction on n. Let $n = 1$ and we will apply Lemma 4.3 (a). By Remark 3.2, $A_1 = (\mathbf{k}, 0, 0, -q_1, 0) = \mathbf{k}[y_1; 0]_p[x_1, \beta_1]_p$, where $\beta_1(y_1) = -q_1y_1$, and consider $f = (1, 1), g = (-q_1, 1) \in K$. Then g acts as β_1 on A_1 and $g(x) = x$. There are four possible cases for T:

$$\phi, \{y_1, \Omega_1\}, \{x_1, \Omega_1\}, \{y_1, x_1, \Omega_1\}.$$

Hence A_1^T is one of the forms $A_1[y^{-1}][x^{-1}]$, $(A_1/\langle y_1\rangle)[x_1^{-1}]$, $(A_1/\langle x_1\rangle)[y_1^{-1}]$, $A_1/\langle y_1, x_1\rangle$. Applying Lemma 4.3 (a), A_1^T is H-simple.

Suppose that $n > 1$ and A_{n-1}^S is K-simple for any admissible set $S \subseteq \mathcal{P}_{n-1}$. Note that

$$A_n = A_{n-1}[y_n; \alpha_n]_p[x_n; \beta_n, \delta_n]_p = (A_{n-1}; \alpha_n, \beta_n, -q_n, -\Omega_{n-1})$$
$$\alpha_n(-\Omega_{n-1}) = p_n(-\Omega_{n-1}), \quad \beta_n(-\Omega_{n-1}) = -p_n(-\Omega_{n-1})$$

by Remark 3.2 and Lemma 3.3. Given an admissible set T of A_n, set $T' = T \cap \mathcal{P}_{n-1}$ and let I be the ideal of A_{n-1} generated by T'. Then, since I is $\{\alpha_n, \beta_n, \delta_n\}$-stable, we have the following K-equivalence:

$$A_n/IA_n \cong_K (A_{n-1}/I)[y_n; \overline{\alpha}_n]_p[x_n; \overline{\beta}_n, \overline{\delta}_n]_p,$$

where $\overline{\delta}_n = 0$ if $\Omega_{n-1} \in T'$, and thus we have

$$(A_n/IA_n)[E_{T'}^{-1}] \cong_K (A_{n-1}/I)[E_{T'}^{-1}][y_n; \overline{\alpha}_n]_p[x_n; \overline{\beta}_n, \overline{\delta}_n]_p.$$

Set $A = (A_{n-1}/I)[E_{T'}^{-1}]$ and $S = T\backslash T'$. Then $\langle T\rangle = IA_n + \langle S\rangle$ and

$$A_n/\langle T\rangle \cong_K (A_n/IA_n)/(\langle T\rangle/IA_n)$$
$$A_n/\langle T\rangle[E_{T'}^{-1}] \cong_K A[y_n; \overline{\alpha}_n]_p[x_n; \overline{\beta}_n, \overline{\delta}_n]_p/\langle S\rangle.$$

Let E be the multiplicative set generated by $N_T\backslash(N_{T'}\cap\mathcal{P}_{n-1})$. Then

$$\begin{aligned} A_n/\langle T\rangle[E_T^{-1}] &= A_n/\langle T\rangle[E_{T'}^{-1}][E^{-1}] \\ &\cong_K (A[y_n;\overline{\alpha}_n]_p[x_n;\overline{\beta}_n,\overline{\delta}_n]_p/\langle S\rangle)[E^{-1}]. \end{aligned}$$

In order to apply Lemma 4.3 and Lemma 4.4, we will define the necessary elements of K. Set

$$\begin{aligned} f &= (\gamma_{1n}, p_n-\gamma_{1n}, \gamma_{2n}, p_n-\gamma_{2n}, \cdots, \gamma_{n-1,n}, p_n-\gamma_{n-1,n}, 1, p_n-1) \\ g &= (-q_1-\gamma_{1n}, q_1-p_n+\gamma_{1n}, -q_2-\gamma_{2n}, q_2-p_n+\gamma_{2n}, \cdots, \\ &\qquad -q_{n-1}-\gamma_{n-1,n}, q_{n-1}-p_n+\gamma_{n-1,n}, -q_n, q_n-p_n). \end{aligned}$$

Then $f,g\in K$ and $f|_{A_{n-1}}=\alpha_n, f(y_n)=y_n, f(x_n)=(p_n-1)x_n$ and $g|_{A_{n-1}[y_n;\alpha_n]_p}=\beta_n, g(x_n)=(q_n-p_n)x_n$. Note that $(-q_n+p_n)y_nx_n-\Omega_{n-1}=-\Omega_n$ and $g(-y_n^{-1}\Omega_n)=(q_n-p_n)(-y_n^{-1}\Omega_n)$. As defined, 1 and q_n-p_n are nonzero.

There are five possible cases for S:

$$\phi,\quad \{\Omega_n\},\quad \{y_n,\Omega_n\},\quad \{x_n,\Omega_n\},\quad \{y_n,x_n,\Omega_n\}.$$

If $S=\phi$ then $\langle S\rangle=0$, and if $\Omega_{n-1}\in T'$, then E is generated by y_n and x_n, so that

$$\begin{aligned} A_n^T &\cong_K (A[y_n;\overline{\alpha}_n]_p[x_n;\overline{\beta}_n,\overline{\delta}_n]_p/\langle S\rangle)[E^{-1}] \\ &= (A[y_n;\overline{\alpha}_n]_p[x_n;\overline{\beta}_n]_p)[y_n^{-1}][x_n^{-1}] \\ &= A[y_n^{\pm1};\overline{\alpha}_n]_p[x_n^{\pm1};\overline{\beta}_n]_p. \end{aligned}$$

since $\overline{\delta}_n=0$. Applying Lemma 4.3 yields that A_n^T is K-simple. If $\Omega_{n-1}\notin T'$ then E is generated by Ω_n and $A_n^T\cong_K (A[y_n;\overline{\alpha}_n]_p[x_n;\overline{\beta}_n,\overline{\delta}_n]_p)[\Omega_n^{-1}]$ is K-simple by Lemma 4.4.

If $S=\{\Omega_n\}$ then $\Omega_{n-1}\notin T'$ and $E=\{1\}$, and so

$$A_n^T\cong_K (A[y_n;\overline{\alpha}_n]_p[x_n;\overline{\beta}_n,\overline{\delta}_n]_p)/\langle\Omega_n\rangle$$

is K-simple by Lemma 4.4.

If $S = \{y_n, \Omega_n\}$ then $\langle S\rangle = \langle y_n\rangle$ and E is generated by x_n. Further $\overline{\delta}_n = 0$ since $\Omega_{n-1} \in T'$ and

$$A_n^T \cong_K (A[y_n; \overline{\alpha}_n]_p[x_n; \overline{\beta}_n, \overline{\delta}_n]_p/\langle S\rangle)[x_n^{-1}]$$
$$= (A[y_n; \overline{\alpha}_n]_p[x_n; \overline{\beta}_n]_p/\langle y_n\rangle)[x_n^{-1}]$$

is K-simple by Lemma 4.3.

If $S = \{x_n, \Omega_n\}$ then $\langle S\rangle = \langle x_n\rangle$ and E is generated by y_n. Moreover $\overline{\delta}_n = 0$ and

$$A_n^T \cong_K (A[y_n; \overline{\alpha}_n]_p[x_n; \overline{\beta}_n, \overline{\delta}_n]_p/\langle S\rangle)[y_n^{-1}]$$
$$= (A[y_n; \overline{\alpha}_n]_p[x_n; \overline{\beta}_n]_p/\langle x_n\rangle)[y_n^{-1}]$$

is K-simple by Lemma 4.3.

Lastly, if $S = \{y_n, x_n, \Omega_n\}$ then $\Omega_{n-1} \in T'$ and $E = \{1\}$, and so

$$A_n^T \cong_K (A[y_n; \overline{\alpha}_n]_p[x_n; \overline{\beta}_n]_p)/\langle y_n, x_n\rangle$$

is K-simple by Lemma 4.3. Therefore we conclude that A_n^T is K-simple for every admissible set T. $\square$

LEMMA 4.7. *Let P be a K-prime Poisson ideal of A_n. Then $T = P \cap \mathcal{P}_n$ is an admissible set.*

Proof. For convenience, set $\Omega_0 = 0$. Suppose that $y_i \in T$, $i = 1, \cdots, n$. Then $\Omega_{i-1} = \{x_i, y_i\} - q_i y_i x_i \in P$ and $\Omega_i = (q_i - p_i)y_i x_i + \Omega_{i-1} \in P$ by Lemma 3.3. It follows that if $y_i \in T$ then $\Omega_i, \Omega_{i-1} \in T$. Similarly, if $x_i \in T$, $i = 1, \cdots, n$ then $\Omega_i, \Omega_{i-1} \in T$. Conversely, suppose that $\Omega_i, \Omega_{i-1} \in T$, $i = 1, \cdots, n$. Then $(q_i - p_i)y_i x_i = \Omega_i - \Omega_{i-1} \in P$. Since y_i and x_i are both K-eigenvectors and Poisson normal modulo $\langle \Omega_{i-1}\rangle$, we have that $\langle y_i, \Omega_{i-1}\rangle$ and $\langle x_i, \Omega_{i-1}\rangle$ are K-stable Poisson ideals and $\langle y_i, \Omega_{i-1}\rangle\langle x_i, \Omega_{i-1}\rangle \subseteq P$, and hence we have $y_i \in P$ or $x_i \in P$. Therefore, if $\Omega_i, \Omega_{i-1} \in T$, $i = 1, \cdots, n$ then $y_i \in T$ or $x_i \in T$. It follows that T is an admissible set of A_n. $\square$

Theorem 4.8. *Every K-prime Poisson ideal of A_n is generated by an admissible set.*

Proof. Let P be a K-prime Poisson ideal of A_n and let $T = P \cap \mathcal{P}_n$. Then T is an admissible set by Lemma 4.7 and $P/\langle T\rangle$ is a K-prime Poisson ideal of $A_n/\langle T\rangle$. By definition, $N_T \cap T = \phi$ and so $N_T \cap P = \phi$, and hence $\overline{N}_T \cap P/\langle T\rangle = \phi$, where each element of $\overline{N}_T$ is Poisson normal in $A_n/\langle T\rangle$. Recalling that $\overline{E}_T$ is the multiplicative set generated by $\overline{N}_T$, we have that $\overline{E}_T \cap P/\langle T\rangle = \phi$. Hence $(P/\langle T\rangle)[\overline{E}_T{}^{-1}]$ is a K-prime Poisson ideal of A_n^T, and so $(P/\langle T\rangle)[\overline{E}_T{}^{-1}] = 0$ since A_n^T is K-simple by Theorem 4.6. Therefore, $P/\langle T\rangle = 0$, so $P = \langle T\rangle$. □

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