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ON THE STOCHASTIC PROCESS $X(t,\omega) \in L^2_{s.a.p.}$

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Abstract. We find some properties of a stochastic process $X(t,\omega) \in L^2_{s.a.p.}$ which is of bounded variation.

1. Introduction

Throughout this paper, $(\Omega, \mathcal{F}, P)$ is the underlying probability space and , without otherwise mentioned,$X(t,\omega)$, $t \in \mathbf{R}$, is a complex valued stochastic process of the second order,where ω is an element of Ω, that is,

$$E|X(t,\omega)|^2 = ||X(t,\omega)||^2 < \infty \qquad \text{for every } t.$$

Suppose that $X(t,\omega)$ is measurable on $\mathbf{R} \times \Omega$ and also suppose that

$$\int_a^b ||X(t,\omega)||^2 \, dt < \infty, \qquad \text{for every finite } a < b.$$

In this case, $X(t,\omega)$ is of $L^2(a,b)$ as a function of t almost surely.

Definition 1.1. $X(t,\omega) \in L^2_{s.a.p.}$ if and only if the set

$$S^2(\epsilon, X) \equiv \{\tau; \sup_{u \in \mathbf{R}} \int_u^{u+1} ||X(t+\tau,\omega) - X(t,\omega)||^2 \, dt < \epsilon\}$$

is relatively dense for every $\epsilon > 0$.

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Proposition 1.1. *Let* $X(t,\omega) \in L^2_{s.a.p.}$. *For*

$$\alpha(\lambda) = l.i.m._{T\to\infty}\frac{1}{T}\int_0^T X(t,\omega)e^{-i\lambda t}dt,$$

there exists $\Lambda = \{\lambda_n\} \subset \mathbf{R}$ *such that* $\alpha(\lambda) \neq 0$ *for* $\lambda \in \Lambda$ *and* $\alpha(\lambda) = 0$ *for* $\lambda \notin \Lambda$. *Let* $\alpha(\lambda) \equiv \alpha_n, n = 1, 2, ...$ *And then Parseval relation*

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T ||X(t,\omega)||^2\,dt = \sum_{n=1}^{\infty}||\alpha_n||^2$$

holds.(We call the numbers $\lambda_1, \lambda_2, ...,$ *Fourier exponents and the numbers* $\alpha_1, \alpha_2, ...,$ *Fourier coefficients of* $X(t,\omega) \in L^2_{s.a.p.}$ *)*

Proof. We know[1] there exist $\Lambda = \{\lambda_n\}$ and $\{\lambda_n\} \subset \mathbf{R}^+$ such that $\sum_{n=1}^{\infty}\gamma_n < \infty, \phi(u) = \sum_{n=1}^{\infty}\gamma_n e^{i\lambda_n u}$ where

$$\phi(u) = \lim_{T\to\infty}\frac{1}{T}\int_{\gamma}^{\gamma+T} < X(t+u), X(t) > dt.$$

(Convergence of the above limit is uniform for γ and u.)

Therefore,

$$\begin{aligned}\lim_{T\to\infty}\frac{1}{T}\int_0^T \phi(u)e^{-i\lambda u}du &= \lim_{T\to\infty}\frac{1}{T}\int_0^T\sum_{n=1}^{\infty}\gamma_n e^{i(\lambda_n-\lambda)u}du\\ &= \sum_{n=1}^{\infty}\gamma_n(\lim_{T\to\infty}\frac{1}{T}\int_0^T e^{i(\lambda_n-\lambda)u}du)\end{aligned}$$

The above value is γ_n if $\lambda = \lambda_n \in \Lambda, n = 1, 2, ...$ and 0 if $\lambda \in \Lambda^c$.

Otherwise,

$$\begin{aligned}
&\lim_{T\to\infty} \frac{1}{T}\int_0^T \phi(u)e^{-i\lambda u}du\\
&= \lim_{T\to\infty} \frac{1}{T}\int_0^T e^{-i\lambda u}\{\lim_{S\to\infty} \frac{1}{S}\int_0^S < X(u+t), X(t) > dt\}du\\
&= \lim_{T\to\infty}\lim_{S\to\infty} \frac{1}{TS}\int_0^S\int_0^T e^{-i\lambda u} < X(u+t), X(t) > dtdu\\
&= \lim_{S\to\infty} < l.i.m._{T\to\infty}\frac{1}{T}\int_{-t}^{T-t} e^{-i\lambda u}X(u)du, \frac{1}{S}\int_0^S e^{-i\lambda t}X(t)dt >\\
&= \lim_{S\to\infty} < \alpha(\lambda), \frac{1}{S}\int_0^S e^{-i\lambda t}X(t)dt >\\
&=< \alpha(\lambda), l.i.m._{S\to\infty}\frac{1}{S}\int_0^S e^{-i\lambda t}X(t)dt >\\
&=< \alpha(\lambda), \alpha(\lambda) >\\
&= ||\alpha(\lambda)||^2.
\end{aligned}$$

Hence

$$||\alpha(\lambda_n)||^2 = ||\alpha_n||^2 = \gamma_n, n = 1, 2, ...$$

$$||\alpha(\lambda)||^2 = 0, \lambda \in \Lambda^c.$$

Therefore, we have

$$\begin{aligned}
\phi(0) &= \lim_{T\to\infty}\frac{1}{T}\int_0^T ||X(t)||^2\, dt\\
&= \sum_{n=1}^{\infty}\gamma_n = \sum_{n=1}^{\infty}||\alpha(\lambda_n)||^2 = \sum_{n=1}^{\infty}||\alpha_n||^2,
\end{aligned}$$

as desired. □

We, throughout this paper, make the following assumption:

There exists some $\delta > 0$ such that $|\lambda_m - \lambda_n| > \delta$, for $m \neq n$, where $\lambda_n, n = 1, 2, ...$, are Fourier exponents.

Without otherwise mentioned,$X(t,\omega) \in L^2_{s.a.p.}$ means $X(t,\omega) \in L^2_{s.a.p.}$ which admits the above condition.

DEFINITION 1.2. Let $X(t,\omega), t \in \mathbf{R}$ be of $L^2(a,b)$. If

$$\sup_D \sum_{j=1}^{n} ||X(t_j,\omega) - X(t_{j-1},\omega)|| = V < \infty,$$

where sup is taken for all divisions $D; a \leq t_0 < t_1 < ... < t_n \leq b$, for every finite $[a,b] \subset \mathbf{R}$, then we say that $X(t,\omega)$ is of bounded variation and write $X(t,\omega) \in BV$.

In this paper, we find some propositions of a stochastic process $X(t,\omega) \in L^2_{s.a.p.}$ which is of bounded variation.

2. Bounded variation

PROPOSITION 2.1. *If $X(t,\omega) \in BV$ then*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T ||X(t+h,\omega) - X(t,\omega)||dt \leq c|h|,$$

for some constant c.

Proof. In the case of $h > 0$,

$$\begin{aligned}
&\lim_{T\to\infty} \frac{1}{T} \int_0^T ||X(t+h,\omega) - X(t,\omega)||dt \\
&= \lim_{T\to\infty} \frac{1}{T} \int_0^T dt \int_t^{t+h} d||X(u,\omega)|| \\
&= \lim_{T\to\infty} \frac{1}{T} [\int_0^h d||X(u,\omega)|| \int_0^u dt \\
&+ \int_h^T d||X(u,\omega)|| \int_{u-h}^u dt + \int_T^{T+h} d||X(u,\omega)|| \int_{u-h}^T dt] \\
&\leq \lim_{T\to\infty} \frac{1}{T} h 3 c_1 T \\
&\leq c_2 h,
\end{aligned}$$

for some constants c_1, c_2.

Similarly, in the case of $h > 0$, we can also have some constant c_3 such that

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T ||X(t+h,\omega) - X(t,\omega)||dt \le c_3|h|,$$

as desired. □

PROPOSITION 2.2. *If* $X(t,\omega) \in L^2_{s.a.p.}$ *and* $X(t,\omega) \in BV$*, then* $||\alpha_n(\omega)|| \le \frac{c}{|\lambda_n|}$ *for some constant* c*.*

Proof. For each T,

$$\int_0^T X(t,\omega)e^{-i\lambda_n t}dt = [\frac{e^{-i\lambda_n t}X(t,\omega)}{-i\lambda_n}]_0^T + \frac{1}{i\lambda_n}\int_0^T e^{-i\lambda_n t}dX(t,\omega).$$

Therefore,

$$\begin{aligned} E|\alpha_n(\omega)|^2 &= \lim_{T\to\infty}\frac{1}{T^2}E|\int_0^T X(t,\omega)e^{-i\lambda_n t}dt|^2 \\ &\le \lim_{T\to\infty}[E|\frac{e^{-i\lambda_n T}X(T,\omega) - X(0,\omega)}{\lambda_n T}|^2] \\ &+ \lim_{T\to\infty}[\frac{1}{|\lambda_n|^2}\frac{1}{T^2}E|\int_0^T |dX(t,\omega)||^2]. \end{aligned}$$

And

$$\begin{aligned} &E|\frac{e^{-i\lambda_n T}X(T,\omega) - X(0,\omega)}{\lambda_n T}|^2 \\ &\le \frac{1}{|\lambda_n T|^2}[E|X(T,\omega) - X(0,\omega)|^2 + 2E|X(0,\omega)|^2] \\ &\le \frac{c_1}{|\lambda_n|^2}, \end{aligned}$$

for large T and some constant c_1.

Also we have

$$E|\int_0^T |dX(t,\omega)||^2 \leq (\int_0^T d||X(t,\omega)||^2)^2 \leq c_2^2 T^2,$$

for some constant c_2.

Therefore, by the following relation,

$$\begin{aligned}\lim_{T\to\infty} \frac{1}{|\lambda_n|^2}\frac{1}{T^2}E|\int_0^T |dX(t,\omega)||^2 &\leq \lim_{T\to\infty}\frac{1}{|\lambda_n|^2}\frac{1}{T^2}c_2^2T^2 \\ &\leq \frac{c_3}{|\lambda_n|^2},\end{aligned}$$

for some constant c_3. We have the conclusion. □

PROPOSITION 2.3. *Let $X(t,\omega) \in L^2_{s.a.p.}$ and $X(t,\omega) \in BV$. If $0 < \nu < \frac{1}{2}$ then $\alpha_n(\omega) = o(|\lambda_n|^{-\nu}), a.s.$*

Proof. For any $A > 0$,

$$P(\{\omega : |\alpha_n(\omega)| > A|\lambda_n|^{-\nu}\}) \leq (A|\lambda_n|^{-\nu})^{-2}E|\alpha_n(\omega)|^2.$$

By Proposition 2.2., for some constant c_1, the last term is not greater than $c_1 A^{-2}|\lambda_n|^{2(\nu-1)}$.

Since there exists some constant c_2 such that $|\lambda_n| > c_2 n$, we have, for $0 < \nu < \frac{1}{2}$,

$$c_1|\lambda_n|^{2(\nu-1)} < c_3 n^{2(\nu-1)},$$

for some constant c_3.

Since $\sum_{n=1}^{\infty} n^{2(\nu-1)} < \infty$, we have

$$\sum_{n=1}^{\infty} P(\{\omega : |\alpha_n(\omega)| > A|\lambda_n|^{-\nu}\}) < \infty.$$

By the Borel-Cantelli lemma, we have

$$\lim_{n\to\infty}\frac{|\alpha_n(\omega)|}{|\lambda_n|^{-\nu}} = 0, a.s.,$$

as desired. □

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