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SOME RESULTS OF ADDITIVE ENDOMORPHISMS IN RINGS

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1. Introduction

In this paper, we initiate the investigation of rings in which all the additive endomorphisms or only the left multiplication endomorphisms are generated by ring endomorphisms. This study was motivated by the Sullivan's Problem (i.e., characterize those rings in which every additive endomorphism is a ring endomorphism, these rings are called *AE rings*) [9] and the current investigation of LSD-generated algebras [1].

Throughout this paper, R denotes an associative ring not necessarily with unity, $End(R,+)$ the ring of additive endomorphisms of R, and $End(R,+,\cdot)$ the monoid of ring endomorphisms of R. For $X \subseteq R$, we use $gp(X)$ for the subgroup of $(R,+)$ generated by X. For $x \in R$, ${}_x\tau$ denotes the left multiplication map (i.e., $a \longmapsto xa$, for all $a \in R$). Observe ${}_x\tau \in End(R,+)$. $\mathcal{LGE}(R)$ is the set

$$\{x \in R \mid {}_x\tau \in gp(End(R,+,\cdot))\}.$$

Note that $\mathcal{LGE}(R)$ is a subring of R. $\mathcal{L}(R)$ is the set $\{x \in R \mid xab = xaxb\}$. $(\mathcal{L}(R),\cdot)$ is a subsemigroup of $(R,\cdot)$, and $x \in \mathcal{L}(R)$ if and only if ${}_x\tau \in End(R,+,\cdot)$. Also $\mathcal{L}(R) \subseteq \mathcal{LGE}(R)$ and $\mathcal{L}(R)$ contains all one-sided unities of R, the left annihilators of R^2 and all

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central idempotents. We use $\mathcal{RGE}(R)$ and $\mathcal{R}(R)$ for the right sided analogs of $\mathcal{LGE}(R)$ and $\mathcal{L}(R)$, respectively.

We call that a ring R is an *AGE ring* if $End(R,+) = gp(End(R,+,\cdot)$, *LGE ring* if $R = \mathcal{LGE}(R)$. Note if the left regular representation of R into $End(R,+)$ is surjective, then R is an AGE ring.

R is *LSD* (*LSD-generated*) if $R = \mathcal{L}(R)$ ($R = gp(\mathcal{L}(R))$) [1]. The classes of LSD, LSD-generated rings are closed with respect to homomorphisms and direct sums. Observe that the class of LGE rings contains both the class of AGE rings and the class of LSD-generated rings. The class of AGE rings is contained in the class of RGE rings (i.e., $R = \mathcal{RGE}(R)$). In the sequel, examples are provided to show that the classes of LGE, AGE and LSD-generated rings are distinct. Although the class of AE rings is a proper subclass of the class of SD rings, the class of AGE rings is not contained in the class of LSD-generated rings.

2. Examples and some results

Example 2.1. Rings $\mathbb{Z}$ and $\mathbb{Z}_n$ are additively generated by 1, so they are both AGE and LSD-generated rings. Since $x \in \mathcal{L}(R)$ implies $x^3 = x^n$ for $n > 3$, then $\mathcal{L}(S) = \{0\}$ for any nonzero proper subring S of $\mathbb{Z}$. Hence any nonzero proper subring of $\mathbb{Z}$ is an AGE ring which is not LSD and LSD-generated.

Example 2.2. Let S be an LSD semigroup (i.e, $xab = xaxb$, for all $x, a, b \in S$). Then the semigroup ring $K[S]$, where K is $\mathbb{Z}$ or $\mathbb{Z}_n$, is an LSD-generated ring. In particular, let S be a nonempty set and define multiplication on S by $st = t$, for each $s, t \in S$. Then $\mathbb{Z}[S]$ and $\mathbb{Z}_n[S]$ are LSD-generated rings. Furthermore if $|S| = 2$, then $\mathbb{Z}_2[S]$ is an LSD ring which is not an AGE ring.

Example 2.3. Let R be a ring and $X \subseteq R$ such that $R = gp(X)$. Let I be the ideal generated by $\{bxby - bxy \mid b, x, y \in X\}$. Then R/I

is an LSD-generated ring.

Proposition 2.1. *Let $Y \subseteq End(R,+,\cdot)$ and $S \subseteq R$ such that $f(S)$, for each $f \in Y$.*

(1) *If R is an LGE ring and for each $x \in R$, ${}_x\tau = \sum_{i\in I} \pm f_i$, where each $f_i \in Y$, then $gp(S)$ is a left ideal of R.*
(2) *If R is an AGE ring and $Y = End(R,+,\cdot)$, then $h(S) \subseteq gp(S)$, for each $h \in End(R,+)$.*

Proof. (1) Let $x \in R$ and $w \in gp(S)$. Then $w = \sum_{j\in J} k_j s_j$, where each $k_j \in \mathbb{Z}$ and each $s_j \in S$. Also there exist $f_i \in Y$ such that

$$ {}_x\tau = \sum_{i\in I} \pm f_i. $$

Hence

$$ xw = {}_x\tau(w) = \sum_{i\in I} \pm f_i(w) = \sum_{i\in I} \pm f_i(\sum_{j\in J} k_j s_j) = \sum_{i\in I}\sum_{j\in J} \pm k_j f_i(s_j) \in gp(S). $$

Thus $gp(S)$ is a left ideal of R.
(2) The proof of this part is similar to that of part (1). □

It is immediate that the classes of LGE rings and LSD-generated rings are closed with respect to direct sums. Our next result shows that the class of AGE rings is closed with respect to finite direct sums. We will use $End_{\mathbb{Z}}(R)$ to denote $End(R,+)$.

Proposition 2.2. *Let $R = \bigoplus_{i=1}^{n} A_i$ be a direct sum of rings. Then R is an AGE ring if and only if for each pair (i,j) and every $f_{jk} \in Hom_{\mathbb{Z}}(A_k, A_j)$ we have*

$$ f_{jk} = \sum_{\alpha\in\Lambda} k_\alpha h_\alpha, $$

where each $k_\alpha \in \mathbb{Z}$ *and each* $h_\alpha : A_k \longrightarrow A_j$ *is a ring homomorphism.*

Proof. For convenience, we prove the case for $n = 2$. The case for $n > 2$ is similar. We see that $End_{\mathbb{Z}}(R) = End_{\mathbb{Z}}(A_1 \oplus A_2) \simeq M$, where

$$M = \begin{bmatrix} End_{\mathbb{Z}}(A_1) & Hom_{\mathbb{Z}}(A_2, A_1) \\ Hom_{\mathbb{Z}}(A_1, A_2) & End_{\mathbb{Z}}(A_2) \end{bmatrix}.$$

So we can represent $f \in End_{\mathbb{Z}}(R)$ by the matrix

$$\begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix},$$

where $f_{jk} \in Hom_{\mathbb{Z}}(A_k, A_j)$.

($\Longrightarrow$). Assume that R is an AGE ring and $f_{jk} \in Hom_{\mathbb{Z}}(A_k, A_j)$. Consider $j = 2$ and $k = 1$. Then $\begin{bmatrix} 0 & 0 \\ f_{21} & 0 \end{bmatrix} \in M$. So $\begin{bmatrix} 0 & 0 \\ f_{21} & 0 \end{bmatrix} = \sum_{\alpha\in\Lambda} k_\alpha h_\alpha$, where each $k_\alpha \in \mathbb{Z}$ and each $h_\alpha \in M$ is a ring endomorphism. Thus

$$h_\alpha = \begin{bmatrix} h_{\alpha 11} & h_{\alpha 12} \\ h_{\alpha 21} & h_{\alpha 22} \end{bmatrix}.$$

By definition, each $h_{\alpha jk}$ is additive. Let $x, y \in A_1$. Then

$$\begin{bmatrix} h_{\alpha 11}(xy) \\ h_{\alpha 21}(xy) \end{bmatrix} = h_\alpha(\begin{bmatrix} x \\ 0 \end{bmatrix}\begin{bmatrix} y \\ 0 \end{bmatrix}) = h_\alpha(\begin{bmatrix} x \\ 0 \end{bmatrix})h_\alpha(\begin{bmatrix} y \\ 0 \end{bmatrix})$$

$$= \begin{bmatrix} h_{\alpha 11} & h_{\alpha 12} \\ h_{\alpha 21} & h_{\alpha 22} \end{bmatrix}\begin{bmatrix} x \\ 0 \end{bmatrix}\begin{bmatrix} h_{\alpha 11} & h_{\alpha 12} \\ h_{\alpha 21} & h_{\alpha 22} \end{bmatrix}\begin{bmatrix} y \\ 0 \end{bmatrix} = \begin{bmatrix} h_{\alpha 11}(x) \\ h_{\alpha 21}(x) \end{bmatrix}\begin{bmatrix} h_{\alpha 11}(y) \\ h_{\alpha 21}(y) \end{bmatrix}.$$

Thus $f_{21} = \sum_{\alpha\in\Lambda} k_\alpha h_{\alpha 21}$, where each $h_{\alpha 21} : A_1 \longrightarrow A_2$ is a ring endomorphism.

Similarly, f_{11}, f_{12} and f_{22} are shown to have the desired properties.

($\Longleftarrow$) Let $f \in End_{\mathbb{Z}}(R)$ with matrix representation $\begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} \in M$. Then

$$\begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} = \begin{bmatrix} \sum_{\alpha\in\Lambda} k_{\alpha 11} h_{\alpha 11} & \sum_{\alpha\in\Lambda} k_{\alpha 12} h_{\alpha 12} \\ \sum_{\alpha\in\Lambda} k_{\alpha 21} h_{\alpha 21} & \sum_{\alpha\in\Lambda} k_{\alpha 22} h_{\alpha 22} \end{bmatrix},$$

where each $k_{\alpha jk} \in \mathbb{Z}$ and each $h_{\alpha jk} : A_k \longrightarrow A_j$ is a ring homomorphism.

Let $x, y \in A_1$ and $\alpha \in \Lambda$. Consider

$$\begin{bmatrix} 0 & 0 \\ h_{\alpha 21} & 0 \end{bmatrix} \begin{bmatrix} xy \\ 0 \end{bmatrix} = \begin{bmatrix} h_{\alpha 21}(xy) \\ 0 \end{bmatrix} = \begin{bmatrix} h_{\alpha 21}(x)h_{\alpha 21}(y) \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ h_{\alpha 21} & 0 \end{bmatrix} \begin{bmatrix} x \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ h_{\alpha 21} & 0 \end{bmatrix} \begin{bmatrix} y \\ 0 \end{bmatrix}.$$

Clearly, $\begin{bmatrix} 0 & 0 \\ h_{\alpha 21} & 0 \end{bmatrix}$ is additive. Hence $\begin{bmatrix} 0 & 0 \\ h_{\alpha 21} & 0 \end{bmatrix}$ represents a ring endomorphism on R.

Similarly, $\begin{bmatrix} h_{\alpha 11} & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & h_{\alpha 12} \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & h_{\alpha 22} \end{bmatrix}$ are all ring endomorphisms on R. Thus

$$\begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} = \sum_{\delta \in \Delta} k_\delta h_\delta,$$

where each $k_\delta \in \mathbb{Z}$ and each h_δ represents a ring endomorphism on R. Therefore R is an AGE ring. $\square$

Let $Y \subseteq End(R, +, \cdot)$ and let R^Y denote

$$\{x \in R | f(x) = x, \text{ for each } f \in Y\}.$$

Observe that R^Y is a subring of R (when Y is a group acting as automorphisms on R, then R^Y is called the *fixed ring* under Y).

Proposition 2.3. *Let $X \subseteq \mathcal{LGE}(R)$. For each $x \in X$, pick a representation of ${}_x\tau = \sum_{i \in I} k_i f_i$ such that $k_i \in \mathbb{Z}$ and $f_i \in End(R, +, \cdot)$. Let Y_x be the set of f_i in this representation. Let $Y = \cup_{x \in X} Y_x$. If $< X >$ is the subring generated by X, then R^Y is a left $< X >$-module. If $< X >= R$, then R^Y is a left ideal of R and R is an LGE ring.*

Proof. Let $x \in X$ and $s \in R^Y$. Then there exists a representation of ${}_x\tau = \sum_{i \in I} k_i f_i$ such that $k_i \in \mathbb{Z}$ and $f_i \in Y$. Hence

$$xs = {}_x\tau(s) = \sum_{i \in I} k_i f_i(s) = (\sum_{i \in I} k_i) s \in R^Y.$$

Since X generates R, then R^Y is a left ideal of R. □

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