JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume 17, No. 2, October 2004

FUNCTIONAL EQUATIONS IN THREE VARIABLES

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Abstract. Let r, s be nonzero real numbers. Let X, Y be vector spaces. It is shown that if a mapping $f : X \to Y$ satisfies $f(0) = 0$, and

$$sf(\frac{x+y\pm z}{r})+f(x)+f(y)\pm f(z) = sf(\frac{x+y}{r})+sf(\frac{y\pm z}{r})+sf(\frac{x\pm z}{r}),$$

or

$$sf(\frac{x+y\pm z}{r}) + f(x) + f(y) \pm f(z) = f(x+y) + f(y\pm z) + f(x\pm z)$$

for all $x, y, z \in X$, then there exist an additive mapping $A : X \to Y$ and a quadratic mapping $Q : X \to Y$ such that $f(x) = A(x) + Q(x)$ for all $x \in X$.

Furthermore, we prove the Cauchy–Rassias stability of the functional equations as given above.

1. Introduction

In 1940, S.M. Ulam [7] raised the following question: Under what conditions does there exist an additive mapping near an approximately additive mapping?

Let X and Y be Banach spaces with norms $||\cdot||$ and $\|\cdot\|$, respectively. Hyers [4] showed that if $\epsilon > 0$ and $f : X \to Y$ such that

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon$$

Received by the editors on October 08, 2004.
2000 *Mathematics Subject Classifications*: Primary 39B52.
Key words and phrases: functional equation, Cauchy–Rassias stability.

for all $x, y \in X$, then there exists a unique additive mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \leq \epsilon$$

for all $x \in X$.

Consider $f : X \to Y$ to be a mapping such that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$. Assume that there exist constants $\epsilon \geq 0$ and $p \in [0, 1)$ such that

$$\|f(x + y) - f(x) - f(y)\| \leq \epsilon(||x||^p + ||y||^p)$$

for all $x, y \in X$. Th.M. Rassias [5] showed that there exists a unique $\mathbb{R}$-linear mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \leq \frac{2\epsilon}{2 - 2^p} ||x||^p$$

for all $x \in X$. Găvruta [3] generalized the Rassias' result.

A square norm on an inner product space satisfies the important parallelogram equality

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

The functional equation

$$f(x + y) + f(x - y) = 2f(x) + 2f(y)$$

is called a *quadratic functional equation*. In particular, every solution of the quadratic functional equation is said to be a *quadratic function*. A Hyers–Ulam stability problem for the quadratic functional equation was proved by Skof [6] for mappings $f : E_1 \to E_2$, where E_1 is a normed space and E_2 is a Banach space. Cholewa [1] noticed that the theorem of Skof is still true if the relevant domain E_1 is replaced by an Abelian group. In [2], Czerwik proved the Hyers–Ulam–Rassias stability of the quadratic functional equation.

Throughout this paper, assume that r and s are nonzero real numbers.

In this paper, we are going to investigate functional equations of sum type of an additive mapping and a quadratic mapping between vector spaces, and prove the Cauchy–Rassias stability of the functional equations between Banach spaces.

2. Functional equations in three variables

Throughout this section, assume that X and Y are vector spaces.

THEOREM 2.1. *If a mapping $f : X \to Y$ satisfies $f(0) = 0$ and*

$$\begin{aligned} sf(\frac{x+y+z}{r}) + f(x) + f(y) + f(z) = sf(\frac{x+y}{r}) + sf(\frac{y+z}{r}) \\ + sf(\frac{x+z}{r}) \end{aligned} \tag{2.i}$$

for all $x, y, z \in X$, then there exist an additive mapping $A : X \to Y$ and a quadratic mapping $Q : X \to Y$ such that $f(x) = A(x) + Q(x)$ for all $x \in X$.

Proof. Let $A : X \to Y$ and $Q : X \to Y$ be the mappings defined by

$$A(x) := \frac{f(x) - f(-x)}{2},$$
$$Q(x) := \frac{f(x) + f(-x)}{2}$$

for all $x \in X$. It is obvious that $A : X \to Y$ is an odd mapping and $Q : X \to Y$ is an even mapping, and that $f(x) = A(x) + Q(x)$ for all $x \in X$.

It follows from (2.i) that

$$sA(\frac{x+y+z}{r}) + A(x) + A(y) + A(z) = sA(\frac{x+y}{r}) + sA(\frac{y+z}{r}) + sA(\frac{x+z}{r}) \tag{2.1}$$

for all $x, y, z \in X$. Put $y = z = 0$ in (2.1). Then one can obtain that

$$sA(\frac{x}{r}) + A(x) = 2sA(\frac{x}{r}),$$
$$A(\frac{x}{r}) = \frac{1}{s}A(x)$$

for all $x \in X$. So

$$A(x+y+z)+A(x)+A(y)+A(z) = A(x+y)+A(y+z)+A(x+z) \tag{2.2}$$

for all $x, y, z \in X$. Replacing z by $-y$ in (2.2), one can get

$$2A(x) = A(x+y) + A(x-y)$$

for all $x, y \in X$. Let $\frac{x+y}{2} = z$ and $\frac{x-y}{2} = w$. Then

$$2A(z+w) = A(2z) + A(2w) \tag{2.3}$$

for all $z, w \in X$. Let $w = 0$ in (2.3). $2A(z) = A(2z)$, and so

$$A(z+w) = A(z) + A(w)$$

for all $z, w \in X$. Thus the mapping $A : X \to Y$ is additive.

It follows from (2.i) that

$$sQ(\frac{x+y+z}{r}) + Q(x) + Q(y) + Q(z) = sQ(\frac{x+y}{r}) + sQ(\frac{y+z}{r}) + sQ(\frac{x+z}{r}) \tag{2.4}$$

for all $x, y, z \in X$. Put $y = z = 0$ in (2.4). Then one can obtain that

$$sQ(\frac{x}{r}) + Q(x) = 2sQ(\frac{x}{r}),$$
$$Q(\frac{x}{r}) = \frac{1}{s}Q(x)$$

for all $x \in X$. So

$$Q(x+y+z)+Q(x)+Q(y)+Q(z) = Q(x+y)+Q(y+z)+Q(x+z) \tag{2.5}$$

for all $x, y, z \in X$. Replacing z by $-y$ in (2.5), one can get

$$2Q(x) + 2Q(y) = Q(x + y) + Q(x - y)$$

for all $x, y \in X$. Thus the mapping $Q : X \to Y$ is quadratic.

Therefore, there exist an additive mapping $A : X \to Y$ and a quadratic mapping $Q : X \to Y$ such that $f(x) = A(x) + Q(x)$ for all $x \in X$. □

THEOREM 2.2. *If a mapping $f : X \to Y$ satisfies $f(0) = 0$ and*

$$sf(\frac{x+y+z}{r}) + f(x) + f(y) + f(z) = f(x + y) + f(y + z) + f(x + z)$$

for all $x, y, z \in X$, then there exist an additive mapping $A : X \to Y$ and a quadratic mapping $Q : X \to Y$ such that $f(x) = A(x) + Q(x)$ for all $x \in X$.

Proof. The proof is similar to the proof of Theorem 2.1. □

THEOREM 2.3. *If a mapping $f : X \to Y$ satisfies $f(0) = 0$ and*

$$\begin{aligned} sf(\frac{x+y-z}{r}) + f(x) + f(y) + f(z) = sf(\frac{x+y}{r}) + sf(\frac{y-z}{r}) \\ + sf(\frac{x-z}{r}) \end{aligned} \tag{2.ii}$$

for all $x, y, z \in X$, then the mapping $f : X \to Y$ is quadratic.

Proof. Put $y = z = 0$ in (2.ii). Then one can obtain that

$$sf(\frac{x}{r}) + f(x) = 2sf(\frac{x}{r}),$$
$$f(\frac{x}{r}) = \frac{1}{s}f(x)$$

for all $x \in X$. It follows from (2.ii) that

$$f(x+y-z)+f(x)+f(y)+f(z) = f(x+y)+f(y-z)+f(x-z) \tag{2.6}$$

for all $x, y, z \in X$. Replacing z by y in (2.6), one can get

$$2f(x) + 2f(y) = f(x + y) + f(x - y)$$

for all $x, y \in X$. Thus the mapping $f : X \to Y$ is quadratic. □

THEOREM 2.4. *If a mapping $f : X \to Y$ satisfies $f(0) = 0$ and*

$$sf(\frac{x+y-z}{r}) + f(x) + f(y) + f(z) = f(x + y) + f(y - z) + f(x - z)$$

for all $x, y, z \in X$, then the mapping $f : X \to Y$ is quadratic.

Proof. The proof is similar to the proof of Theorem 2.3. □

THEOREM 2.5. *If a mapping $f : X \to Y$ satisfies $f(0) = 0$ and*

$$\begin{aligned} sf(\frac{x+y-z}{r}) + f(x) + f(y) - f(z) &= sf(\frac{x+y}{r}) + sf(\frac{y-z}{r}) \\ &\quad + sf(\frac{x-z}{r}) \end{aligned} \tag{2.iii}$$

for all $x, y, z \in X$, then the mapping $f : X \to Y$ is additive.

Proof. By a similar method to the proof of Theorem 2.3, one can obtain that

$$f(x+y-z)+f(x)+f(y)-f(z) = f(x+y)+f(y-z)+f(x-z) \tag{2.7}$$

for all $x, y, z \in X$. Replacing z by y in (2.7), one can get

$$2f(x) = f(x+y) + f(x-y)$$

for all $x, y \in X$. Let $\frac{x+y}{2} = z$ and $\frac{x-y}{2} = w$. Then

$$2f(z+w) = f(2z) + f(2w) \tag{2.8}$$

for all $z, w \in X$. Let $w = 0$ in (2.8). $2f(z) = f(2z)$, and so

$$f(z+w) = f(z) + f(w)$$

for all $z, w \in X$. Thus the mapping $f : X \to Y$ is additive. □

THEOREM 2.6. *If a mapping $f : X \to Y$ satisfies $f(0) = 0$ and*

$$sf(\frac{x+y-z}{r}) + f(x) + f(y) - f(z) = f(x+y) + f(y-z) + f(x-z)$$

for all $x, y, z \in X$, then the mapping $f : X \to Y$ is additive.

Proof. The proof is similar to the proofs of Theorems 2.3 and 2.5. □

3. Stability of functional equations in three variables

Throughout this section, assume that X is a normed vector space with norm $||\cdot||$ and that Y is a Banach space with norm $\|\cdot\|$.

THEOREM 3.1. *Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ for which there exists a function $\varphi : X \times X \times X \to [0, \infty)$ satisfying $\varphi(x, y, z) = \varphi(-x, -y, -z)$ such that*

$$\widetilde{\varphi}(x, y, z) := \sum_{j=0}^{\infty} |s|^j \varphi(\frac{x}{r^j}, \frac{y}{r^j}, \frac{z}{r^j}) < \infty, \tag{3.i}$$

$$\|sf(\frac{x+y+z}{r}) + f(x) + f(y) + f(z) - sf(\frac{x+y}{r}) - sf(\frac{y+z}{r})$$
$$-sf(\frac{x+z}{r})\| \le \varphi(x, y, z) \tag{3.ii}$$

for all $x, y, z \in X$. Then there exist an additive mapping $A : X \to Y$ and a quadratic mapping $Q : X \to Y$ such that

$$\|\frac{f(x) - f(-x)}{2} - A(x)\| \le \widetilde{\varphi}(x, 0, 0), \tag{3.iii}$$

$$\|\frac{f(x) + f(-x)}{2} - Q(x)\| \le \widetilde{\varphi}(x, 0, 0) \tag{3.iv}$$

for all $x \in X$.

Proof. Let $f_1 : X \to Y$ be the mapping defined by $f_1(x) := \frac{f(x)-f(-x)}{2}$. It follows from (3.ii) that

$$\|f_1(x) - sf_1(\frac{x}{r})\| \le \frac{\varphi(x,0,0) + \varphi(-x,0,0)}{2} = \varphi(x, 0, 0),$$

$$\|sf_1(\frac{x+y+z}{r}) + f_1(x) + f_1(y) + f_1(z) - sf_1(\frac{x+y}{r}) - sf_1(\frac{y+z}{r}) \\ -sf_1(\frac{x+z}{r})\| \le \varphi(x, y, z) \tag{3.1}$$

for all $x, y, z \in X$. Then

$$\|s^n f_1(\frac{x}{r^n}) - s^{n+1} f_1(\frac{x}{r^{n+1}})\| \le |s|^n \varphi(\frac{x}{r^n}, 0, 0)$$

for all $x \in X$ and all $n = 1, 2, \cdots$. So

$$\|s^l f_1(\frac{x}{r^l}) - s^m f_1(\frac{x}{r^m})\| \le \sum_{j=l}^{m-1} |s|^j \varphi(\frac{x}{r^j}, 0, 0) \tag{3.2}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.i) and (3.2) that the sequence $\{s^n f_1(\frac{x}{r^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{s^n f_1(\frac{x}{r^n})\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{n\to\infty} s^n f_1(\frac{x}{r^n}) \tag{3.3}$$

for all $x \in X$.

By (3.i), (3.1) and (3.3),

$$sA(\frac{x+y+z}{r}) + A(x) + A(y) + A(z) - sA(\frac{x+y}{r}) - sA(\frac{y+z}{r}) - sA(\frac{x+z}{r}) = 0$$

for all $x, y, z \in X$. It is obvious that $A(-x) = -A(x)$ for all $x \in X$. By the same reasoning as in the proof of Theorem 2.1, the mapping $A : X \to Y$ is additive. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.2), one can obtain that

$$\|f_1(x) - A(x)\| \le \sum_{j=0}^{\infty} |s|^j \varphi(\frac{x}{r^j}, 0, 0) = \widetilde{\varphi}(x, 0, 0)$$

for all $x \in X$. That is, the inequality (3.iii) holds for all $x \in X$.

Now let $f_2 : X \to Y$ be the mapping defined by $f_2(x) := \frac{f(x)+f(-x)}{2}$. It follows from (3.ii) that

$$\|f_2(x) - sf_2(\frac{x}{r})\| \le \frac{\varphi(x,0,0) + \varphi(-x,0,0)}{2} = \varphi(x, 0, 0),$$

$$\|sf_2(\frac{x+y+z}{r}) + f_2(x) + f_2(y) + f_2(z) - sf_2(\frac{x+y}{r}) - sf_2(\frac{y+z}{r}) - sf_2(\frac{x+z}{r})\| \le \varphi(x, y, z) \tag{3.4}$$

for all $x, y, z \in X$. Then

$$\|s^n f_2(\frac{x}{r^n}) - s^{n+1} f_2(\frac{x}{r^{n+1}})\| \le |s|^n \varphi(\frac{x}{r^n}, 0, 0)$$

for all $x \in X$ and all $n = 1, 2, \cdots$. So

$$\|s^l f_2(\frac{x}{r^l}) - s^m f_2(\frac{x}{r^m})\| \le \sum_{j=l}^{m-1} |s|^j \varphi(\frac{x}{r^j}, 0, 0) \tag{3.5}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.i) and (3.5) that the sequence $\{s^n f_2(\frac{x}{r^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{s^n f_2(\frac{x}{r^n})\}$ converges. So one can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{n\to\infty} s^n f_2(\frac{x}{r^n}) \tag{3.6}$$

for all $x \in X$.

By (3.i), (3.4) and (3.6),

$$sQ(\frac{x+y+z}{r}) + Q(x) + Q(y) + Q(z) - sQ(\frac{x+y}{r}) - sQ(\frac{y+z}{r}) - sQ(\frac{x+z}{r}) = 0$$

for all $x, y, z \in X$. It is obvious that $Q(-x) = Q(x)$ for all $x \in X$. By the same reasoning as in the proof of Theorem 2.1, the mapping $Q : X \to Y$ is quadratic. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.5), one can obtain that

$$\|f_2(x) - Q(x)\| \le \sum_{j=0}^{\infty} |s|^j \varphi(\frac{x}{r^j}, 0, 0) = \widetilde{\varphi}(x, 0, 0)$$

for all $x \in X$. That is, the inequality (3.iv) holds for all $x \in X$. □

COROLLARY 3.2. *Let p and θ be positive real numbers with $|r|^p > |s|$, and $f : X \to Y$ a mapping satisfying $f(0) = 0$ and*

$$\|sf(\frac{x+y+z}{r}) + f(x) + f(y) + f(z) - sf(\frac{x+y}{r}) - sf(\frac{y+z}{r}) - sf(\frac{x+z}{r})\| \le \theta(||x||^p + ||y||^p + ||z||^p)$$

for all $x, y, z \in X$. Then there exist an additive mapping $A : X \to Y$ and a quadratic mapping $Q : X \to Y$ such that

$$\|\frac{f(x) - f(-x)}{2} - A(x)\| \le \frac{|r|^p\theta}{|r|^p - |s|}||x||^p,$$

$$\|\frac{f(x) + f(-x)}{2} - Q(x)\| \le \frac{|r|^p\theta}{|r|^p - |s|}||x||^p$$

for all $x \in X$.

Proof. Define $\varphi(x, y, z) = \theta(||x||^p + ||y||^p + ||z||^p)$, and apply Theorem 3.1. □

THEOREM 3.3. *Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ for which there exists a function $\varphi : X \times X \times X \to [0, \infty)$ satisfying $\varphi(x, y, z) = \varphi(-x, -y, -z)$ such that*

$$\widetilde{\varphi}(x, y, z) := \sum_{j=0}^{\infty} |s|^j \varphi(\frac{x}{2r^j}, \frac{y}{2r^j}, \frac{z}{2r^j}) < \infty, \tag{3.v}$$

$$\begin{aligned} &\|sf(\frac{x+y+z}{r}) + f(x) + f(y) + f(z) - f(x+y) - f(y+z) \\ &\qquad -f(x+z)\| \le \varphi(x, y, z) \end{aligned} \tag{3.vi}$$

for all $x, y, z \in X$. Then there exist an additive mapping $A : X \to Y$ and a quadratic mapping $Q : X \to Y$ such that

$$\|\frac{f(x) - f(-x)}{2} - A(x)\| \le \widetilde{\varphi}(x, x, 0), \tag{3.vii}$$

$$\|\frac{f(x) + f(-x)}{2} - Q(x)\| \le \widetilde{\varphi}(x, x, 0) \tag{3.viii}$$

for all $x \in X$.

Proof. Let $f_1 : X \to Y$ be the mapping defined by $f_1(x) := \frac{f(x)-f(-x)}{2}$. It follows from (3.vi) that

$$\|f_1(2x) - sf_1(\frac{2x}{r})\| \le \frac{\varphi(x, x, 0) + \varphi(-x, -x, 0)}{2} = \varphi(x, x, 0),$$

$$\begin{aligned} &\|sf_1(\frac{x+y+z}{r}) + f_1(x) + f_1(y) + f_1(z) - f_1(x+y) \\ &\qquad -f_1(y+z) - f_1(x+z)\| \le \varphi(x, y, z) \end{aligned} \tag{3.7}$$

for all $x, y, z \in X$. So

$$\|f_1(x) - sf_1(\frac{x}{r})\| \le \frac{\varphi(\frac{x}{2}, \frac{x}{2}, 0) + \varphi(-\frac{x}{2}, -\frac{x}{2}, 0)}{2} = \varphi(\frac{x}{2}, \frac{x}{2}, 0)$$

for all $x \in X$. Then

$$\|s^n f_1(\frac{x}{r^n}) - s^{n+1} f_1(\frac{x}{r^{n+1}})\| \le |s|^n \varphi(\frac{x}{2r^n}, \frac{x}{2r^n}, 0)$$

for all $x \in X$ and all $n = 1, 2, \cdots$. So

$$\|s^l f_1(\frac{x}{r^l}) - s^m f_1(\frac{x}{r^m})\| \leq \sum_{j=l}^{m-1} |s|^j \varphi(\frac{x}{2r^j}, \frac{x}{2r^j}, 0) \tag{3.8}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.v) and (3.8) that the sequence $\{s^n f_1(\frac{x}{r^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{s^n f_1(\frac{x}{r^n})\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{n\to\infty} s^n f_1(\frac{x}{r^n}) \tag{3.9}$$

for all $x \in X$.

By (3.v), (3.7) and (3.9),

$$sA(\frac{x+y+z}{r}) + A(x) + A(y) + A(z) - A(x+y) - A(y+z) - A(x+z) = 0$$

for all $x, y, z \in X$. It is obvious that $A(-x) = -A(x)$ for all $x \in X$. By the same reasoning as in the proof of Theorem 2.1, the mapping $A : X \to Y$ is additive. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.8), one can obtain that

$$\|f_1(x) - A(x)\| \leq \sum_{j=0}^{\infty} |s|^j \varphi(\frac{x}{2r^j}, \frac{x}{2r^j}, 0) = \widetilde{\varphi}(x, x, 0)$$

for all $x \in X$. That is, the inequality (3.vii) holds for all $x \in X$.

Now let $f_2 : X \to Y$ be the mapping defined by $f_2(x) := \frac{f(x)+f(-x)}{2}$. It follows from (3.vi) that

$$\|f_2(x) - sf_2(\frac{x}{r})\| \leq \frac{\varphi(\frac{x}{2}, \frac{x}{2}, 0) + \varphi(-\frac{x}{2}, -\frac{x}{2}, 0)}{2} = \varphi(\frac{x}{2}, \frac{x}{2}, 0),$$

$$\begin{aligned} &\|sf_2(\frac{x+y+z}{r}) + f_2(x) + f_2(y) + f_2(z) - f_2(x+y) \\ &-f_2(y+z) - f_2(x+z)\| \leq \varphi(x, y, z) \end{aligned} \tag{3.10}$$

for all $x, y, z \in X$. Then

$$\|s^n f_2(\frac{x}{r^n}) - s^{n+1} f_2(\frac{x}{r^{n+1}})\| \leq |s|^n \varphi(\frac{x}{2r^n}, \frac{x}{2r^n}, 0)$$

for all $x \in X$ and all $n = 1, 2, \cdots$. So

$$\|s^l f_2(\frac{x}{r^l}) - s^m f_2(\frac{x}{r^m})\| \leq \sum_{j=l}^{m-1} |s|^j \varphi(\frac{x}{2r^j}, \frac{x}{2r^j}, 0) \tag{3.11}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.v) and (3.11) that the sequence $\{s^n f_2(\frac{x}{r^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{s^n f_2(\frac{x}{r^n})\}$ converges. So one can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{n\to\infty} s^n f_2(\frac{x}{r^n}) \tag{3.12}$$

for all $x \in X$.

By (3.v), (3.10) and (3.12),

$$sQ(\frac{x+y+z}{r}) + Q(x) + Q(y) + Q(z) - Q(x+y) - Q(y+z) - Q(x+z) = 0$$

for all $x, y, z \in X$. It is obvious that $Q(-x) = Q(x)$ for all $x \in X$. By the same reasoning as in the proof of Theorem 2.1, the mapping $Q : X \to Y$ is quadratic. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.11), one can obtain that

$$\|f_2(x) - Q(x)\| \leq \sum_{j=0}^{\infty} |s|^j \varphi(\frac{x}{2r^j}, \frac{x}{2r^j}, 0) = \widetilde{\varphi}(x, x, 0)$$

for all $x \in X$. That is, the inequality (3.viii) holds for all $x \in X$. □

COROLLARY 3.4. *Let p and θ be positive real numbers with $|r|^p > |s|$, and $f : X \to Y$ a mapping satisfying $f(0) = 0$ and*

$$\|sf(\frac{x+y+z}{r}) + f(x) + f(y) + f(z) - f(x+y) - f(y+z) - f(x+z)\| \le \theta(||x||^p + ||y||^p + ||z||^p)$$

for all $x, y, z \in X$. Then there exist an additive mapping $A : X \to Y$ and a quadratic mapping $Q : X \to Y$ such that

$$\|\frac{f(x) - f(-x)}{2} - A(x)\| \le \frac{2|r|^p\theta}{2^p(|r|^p - |s|)}||x||^p,$$
$$\|\frac{f(x) + f(-x)}{2} - Q(x)\| \le \frac{2|r|^p\theta}{2^p(|r|^p - |s|)}||x||^p$$

for all $x \in X$.

Proof. Define $\varphi(x, y, z) = \theta(||x||^p + ||y||^p + ||z||^p)$, and apply Theorem 3.3. □

THEOREM 3.5. *Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ for which there exists a function $\varphi : X \times X \times X \to [0, \infty)$ satisfying $\varphi(x, y, z) = \varphi(-x, -y, -z)$ and (3.i) such that*

$$\|sf(\frac{x+y-z}{r}) + f(x) + f(y) + f(z) - sf(\frac{x+y}{r}) - sf(\frac{y-z}{r}) - sf(\frac{x-z}{r})\| \le \varphi(x, y, z) \tag{3.ix}$$

for all $x, y, z \in X$. Then there exists a quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \le \widetilde{\varphi}(x, 0, 0)$$

for all $x \in X$.

Proof. Putting $y = z = 0$ in (3.ix), one can obtain that

$$\|f(x) - sf(\frac{x}{r})\| \le \varphi(x, 0, 0)$$

for all $x \in X$. Then

$$\|s^n f(\frac{x}{r^n}) - s^{n+1} f(\frac{x}{r^{n+1}})\| \le |s|^n \varphi(\frac{x}{r^n}, 0, 0)$$

for all $x \in X$ and all $n = 1, 2, \cdots$. So

$$\|s^l f(\frac{x}{r^l}) - s^m f(\frac{x}{r^m})\| \le \sum_{j=l}^{m-1} |s|^j \varphi(\frac{x}{r^j}, 0, 0) \tag{3.13}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.ix) and (3.13) that the sequence $\{s^n f(\frac{x}{r^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{s^n f(\frac{x}{r^n})\}$ converges. So one can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{n\to\infty} s^n f(\frac{x}{r^n}) \tag{3.14}$$

for all $x \in X$.

By (3.i), (3.ix) and (3.14),

$$sQ(\frac{x+y-z}{r}) + Q(x) + Q(y) + Q(z) - sQ(\frac{x+y}{r}) - sQ(\frac{y-z}{r}) - sQ(\frac{x-z}{r}) = 0$$

for all $x, y, z \in X$. By the same reasoning as in the proof of Theorem 2.3, the mapping $Q : X \to Y$ is quadratic. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.13), one can obtain that

$$\|f(x) - Q(x)\| \le \sum_{j=0}^{\infty} |s|^j \varphi(\frac{x}{r^j}, 0, 0) = \widetilde{\varphi}(x, 0, 0)$$

for all $x \in X$. □

COROLLARY 3.6. *Let p and θ be positive real numbers with $|r|^p > |s|$, and $f : X \to Y$ a mapping satisfying $f(0) = 0$ and*

$$\|sf(\frac{x+y-z}{r}) + f(x) + f(y) + f(z) - sf(\frac{x+y}{r}) - sf(\frac{y-z}{r}) \\ -sf(\frac{x-z}{r})\| \leq \theta(||x||^p + ||y||^p + ||z||^p)$$

for all $x, y, z \in X$. Then there exists a quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \leq \frac{|r|^p\theta}{|r|^p - |s|}||x||^p$$

for all $x \in X$.

Proof. Define $\varphi(x, y, z) = \theta(||x||^p + ||y||^p + ||z||^p)$, and apply Theorem 3.5. □

THEOREM 3.7. *Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ for which there exists a function $\varphi : X \times X \times X \to [0,\infty)$ satisfying $\varphi(x, y, z) = \varphi(-x,-y,-z)$ and (3.v) such that*

$$\|sf(\frac{x+y-z}{r}) + f(x) + f(y) + f(z) - f(x+y) - f(y-z) \\ -f(x-z)\| \leq \varphi(x, y, z) \tag{3.x}$$

for all $x, y, z \in X$. Then there exists a quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \leq \widetilde{\varphi}(x, x, 0)$$

for all $x \in X$.

Proof. It follows from (3.x) that

$$\|f(x) - sf(\frac{x}{r})\| \leq \varphi(\frac{x}{2}, \frac{x}{2}, 0)$$

for all $x \in X$. Then

$$\|s^n f(\frac{x}{r^n}) - s^{n+1} f(\frac{x}{r^{n+1}})\| \leq |s|^n \varphi(\frac{x}{2r^n}, \frac{x}{2r^n}, 0)$$

for all $x \in X$ and all $n = 1, 2, \cdots$. So

$$\|s^l f(\frac{x}{r^l}) - s^m f(\frac{x}{r^m})\| \leq \sum_{j=l}^{m-1} |s|^j \varphi(\frac{x}{2r^j}, \frac{x}{2r^j}, 0) \tag{3.15}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.v) and (3.15) that the sequence $\{s^n f(\frac{x}{r^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{s^n f(\frac{x}{r^n})\}$ converges. So one can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{n\to\infty} s^n f(\frac{x}{r^n}) \tag{3.16}$$

for all $x \in X$.

By (3.v), (3.x) and (3.16),

$$sQ(\frac{x+y-z}{r}) + Q(x) + Q(y) + Q(z) - Q(x+y) - Q(y-z) \\ - Q(x-z) = 0$$

for all $x, y, z \in X$. It is obvious that $Q(-x) = Q(x)$ for all $x \in X$. By the same reasoning as in Theorem 2.4, the mapping $Q : X \to Y$ is quadratic. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.15), one can obtain that

$$\|f(x) - Q(x)\| \leq \sum_{j=0}^{\infty} |s|^j \varphi(\frac{x}{2r^j}, \frac{x}{2r^j}, 0) = \widetilde{\varphi}(x, x, 0)$$

for all $x \in X$. □

COROLLARY 3.8. *Let p and θ be positive real numbers with $|r|^p > |s|$, and $f : X \to Y$ a mapping satisfying $f(0) = 0$ and*

$$\|sf(\frac{x+y-z}{r}) + f(x) + f(y) + f(z) - f(x+y) \\ -f(y-z) - f(x-z)\| \leq \theta(||x||^p + ||y||^p + ||z||^p)$$

for all $x, y, z \in X$. Then there exists a quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \leq \frac{2|r|^p\theta}{2^p(|r|^p - |s|)}||x||^p$$

for all $x \in X$.

Proof. Define $\varphi(x, y, z) = \theta(||x||^p + ||y||^p + ||z||^p)$, and apply Theorem 3.7. □

THEOREM 3.9. *Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ for which there exists a function $\varphi : X \times X \times X \to [0, \infty)$ satisfying $\varphi(x, y, z) = \varphi(-x, -y, -z)$ and (3.i) such that*

$$\begin{aligned} \|sf(\frac{x+y-z}{r}) + f(x) + f(y) - f(z) - sf(\frac{x+y}{r}) - sf(\frac{y-z}{r}) \\ -sf(\frac{x-z}{r})\| \leq \varphi(x, y, z) \end{aligned} \tag{3.xi}$$

for all $x, y, z \in X$. Then there exists an additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \widetilde{\varphi}(x, 0, 0)$$

for all $x \in X$.

Proof. Putting $y = z = 0$ in (3.xi), one can obtain that

$$\|f(x) - sf(\frac{x}{r})\| \leq \varphi(x, 0, 0)$$

for all $x \in X$. Then

$$\|s^n f(\frac{x}{r^n}) - s^{n+1} f(\frac{x}{r^{n+1}})\| \leq |s|^n \varphi(\frac{x}{r^n}, 0, 0)$$

for all $x \in X$ and all $n = 1, 2, \cdots$. So

$$\|s^l f(\frac{x}{r^l}) - s^m f(\frac{x}{r^m})\| \leq \sum_{j=l}^{m-1} |s|^j \varphi(\frac{x}{r^j}, 0, 0) \tag{3.17}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.i) and (3.17) that the sequence $\{s^n f(\frac{x}{r^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{s^n f(\frac{x}{r^n})\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{n\to\infty} s^n f(\frac{x}{r^n}) \tag{3.18}$$

for all $x \in X$.

By (3.i), (3.xi) and (3.18),

$$sA(\frac{x+y-z}{r}) + A(x) + A(y) - A(z) - sA(\frac{x+y}{r}) - sA(\frac{y-z}{r}) - sA(\frac{x-z}{r}) = 0$$

for all $x, y, z \in X$. By the same reasoning as in the proof of Theorem 2.5, the mapping $A : X \to Y$ is additive. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.17), one can obtain that

$$\|f(x) - A(x)\| \le \sum_{j=0}^{\infty} |s|^j \varphi(\frac{x}{r^j}, 0, 0) = \widetilde{\varphi}(x, 0, 0)$$

for all $x \in X$. □

Corollary 3.10. *Let p and θ be positive real numbers with $|r|^p > |s|$, and $f : X \to Y$ a mapping satisfying $f(0) = 0$ and*

$$\|sf(\frac{x+y-z}{r}) + f(x) + f(y) - f(z) - sf(\frac{x+y}{r}) - sf(\frac{y-z}{r}) - sf(\frac{x-z}{r})\| \le \theta(||x||^p + ||y||^p + ||z||^p)$$

for all $x, y, z \in X$. Then there exists an additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \frac{|r|^p\theta}{|r|^p - |s|}||x||^p$$

for all $x \in X$.

Proof. Define $\varphi(x, y, z) = \theta(||x||^p + ||y||^p + ||z||^p)$, and apply Theorem 3.9. □

Theorem 3.11. *Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ for which there exists a function $\varphi : X \times X \times X \to [0, \infty)$ satisfying $\varphi(x, y, z) = \varphi(-x, -y, -z)$ and* (3.v) *such that*

$$\begin{aligned} &\|sf(\frac{x+y-z}{r}) + f(x) + f(y) - f(z) - f(x+y) - f(y-z) \\ &\qquad -f(x-z)\| \le \varphi(x, y, z) \end{aligned} \tag{3.xii}$$

for all $x, y, z \in X$. Then there exists an additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \widetilde{\varphi}(x, x, 0)$$

for all $x \in X$.

Proof. It follows from (3.xii) that

$$\|f(x) - sf(\frac{x}{r})\| \le \varphi(\frac{x}{2}, \frac{x}{2}, 0)$$

for all $x \in X$. Then

$$\|s^n f(\frac{x}{r^n}) - s^{n+1} f(\frac{x}{r^{n+1}})\| \le |s|^n \varphi(\frac{x}{2r^n}, \frac{x}{2r^n}, 0)$$

for all $x \in X$ and all $n = 1, 2, \cdots$. So

$$\|s^l f(\frac{x}{r^l}) - s^m f(\frac{x}{r^m})\| \le \sum_{j=l}^{m-1} |s|^j \varphi(\frac{x}{2r^j}, \frac{x}{2r^j}, 0) \tag{3.19}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.v) and (3.19) that the sequence $\{s^n f(\frac{x}{r^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{s^n f(\frac{x}{r^n})\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{n\to\infty} s^n f(\frac{x}{r^n}) \tag{3.20}$$

for all $x \in X$.

By (3.v), (3.xii) and (3.20),

$$sA(\frac{x+y-z}{r})+A(x)+A(y)-A(z)-A(x+y)-A(y-z)\\ -A(x-z)=0$$

for all $x,y,z\in X$. It is obvious that $A(-x)=A(x)$ for all $x\in X$. By the same reasoning as in Theorem 2.6, the mapping $A:X\to Y$ is additive. Moreover, letting $l=0$ and passing the limit $m\to\infty$ in (3.19), one can obtain that

$$\|f(x)-A(x)\|\le\sum_{j=0}^{\infty}|s|^j\varphi(\frac{x}{2r^j},\frac{x}{2r^j},0)=\widetilde{\varphi}(x,x,0)$$

for all $x\in X$. □

Corollary 3.12. *Let p and θ be positive real numbers with $|r|^p>|s|$, and $f:X\to Y$ a mapping satisfying $f(0)=0$ and*

$$\|sf(\frac{x+y-z}{r})+f(x)+f(y)+f(z)-f(x+y)\\ -f(y-z)-f(x-z)\|\le\theta(||x||^p+||y||^p+||z||^p)$$

for all $x,y,z\in X$. Then there exists an additive mapping $A:X\to Y$ such that

$$\|f(x)-A(x)\|\le\frac{2|r|^p\theta}{2^p(|r|^p-|s|)}||x||^p$$

for all $x\in X$.

Proof. Define $\varphi(x,y,z)=\theta(||x||^p+||y||^p+||z||^p)$, and apply Theorem 3.11. □

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