JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume 17, No. 2, October 2004

CONSTANTS FOR HARMONIC MAPPINGS

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Abstract. In this paper, we obtain some coefficient estimates of harmonic, orientation-preserving, univalent mappings defined on $\Delta = \{z : |z| > 1\}$.

1. Introduction

Let Σ be the set of all complex-valued, harmonic, orientation-preserving, univalent mappings

$$f(z) = h(z) + \overline{g(z)} + Alog|z| \tag{1.1}$$

of $\Delta = \{z : |z| > 1\}$, where

$$h(z) = z + \sum_{k=1}^{\infty} a_k z^{-k} \quad \text{and} \quad g(z) = \sum_{k=1}^{\infty} b_k z^{-k}$$

are analytic in Δ and $A \in \mathbb{C}$.

Hengartner and Schober [2] used the representation (1.1) to obtain some coefficient estimates and distortion theorems. Some coefficient bounds for $f \in \Sigma$ are also obtained by Jun [3].

Our purpose is to continue the investigation of this class Σ. Thus we obtain some coefficient bounds of $f \in \Sigma$ by using properties of the analytic function $h - g$.

This work was supported by the research fund of Nat. Sci. Inst., Seoul Women's University, 2004.

Received by the editors on October 05, 2004.

2000 *Mathematics Subject Classifications*: Primary 30C45, 30C50.

Key words and phrases: harmonic function, univalent function.

2. Some coefficient bounds

Let $\mathcal{S}$ be the class of functions $\tilde{f}(z) = z + \sum_{n=0}^{\infty} c_n z^{-n}$ that are analytic and univalent for $|z| > 1$.

LEMMA 2.1. ([4]) *Let $\mathcal{F}$ be a subset of $\mathcal{S}$, and let F be subordinate to a function $G \in \mathcal{S}$. If a functional $\lambda : \mathcal{F} \to \mathbb{R}$ assumes its maximum at $F \in \mathcal{F}$, then the associated functional $\psi : \mathcal{H} \to \mathbb{R}$ defined by $\psi(\tilde{f}) = \lambda(\frac{\tilde{f}}{\omega'(\infty)} \circ \omega)$ on $\mathcal{H} = \{\tilde{f} \in \mathcal{S} : \frac{\tilde{f}}{\omega'(\infty)} \circ \omega \in \mathcal{F}\}$ assumes its maximum at $K = \omega'(\infty)G$, where $\omega = G^{-1} \circ F$. In addition, the two maxima are equal.*

Proof. F is subordinate to a function $G \in \mathcal{S}$; that is, $F(\Delta) \subset G(\Delta)$ and $F(\infty) = G(\infty) = \infty$. Thus ω is univalent in Δ, maps Δ into Δ, and $\omega(\infty) = \infty$. The function K belongs to $\mathcal{H}$ since $G \circ \omega = F \in \mathcal{F}$. Furthermore, since λ assumes its maximum at F, it follows that $\psi(\tilde{f}) \leq \lambda(F) = \lambda(G \circ \omega) = \psi(\omega'(\infty)G)$. That is, ψ assumes its maximum over $\mathcal{H}$ at $K = \omega'(\infty)G$, and two maxima are equal. □

LEMMA 2.2. *Assume that $\tilde{f}(z) = z + \sum_{n=1}^{\infty} c_n z^{-n}$, $F(z) = z + \sum_{n=1}^{\infty} B_n z^{-n}$, and $K(z) = z + \sum_{n=1}^{\infty} C_n z^{-n}$ belong to $\mathcal{S}$ and that $F = \frac{K}{\omega'(\infty)} \circ \omega$. Then*

$$\frac{\tilde{f}}{\omega'(\infty)} \circ \omega(z) = z + \sum_{n=1}^{\infty} \beta_n z^{-n}$$

where

$$\beta_1 = \frac{c_1}{\omega'(\infty)^2} + B_1 - \frac{C_1}{\omega'(\infty)^2},$$

$$\beta_4 = \frac{c_4}{\omega'(\infty)^5} - 2(B_1 - \frac{C_1}{\omega'(\infty)^2})\frac{c_2}{\omega'(\infty)^3} - (B_2 - \frac{C_2}{\omega'(\infty)^3})\frac{c_1}{\omega'(\infty)^2} + B_4 + \frac{B_2 C_1}{\omega'(\infty)^2} + \frac{2B_1 C_2}{\omega'(\infty)^3} - \frac{C_4 + 3C_1 C_2}{\omega'(\infty)^5}.$$

Proof. The result is obtained by the straightforward calculation, as desired. □

THEOREM 2.3. *Let $f \in \Sigma$. If $h - g$ with $a_1 - b_1$ real is univalent and if $t \geq 1$, then*

$$Re\{(a_4 - b_4) - (3t^2 - 2)(a_2 - b_2) + 2t^3(a_1 - b_1)\} \leq 2t^3. \tag{2.1}$$

Proof. Let $\tilde{f}(z) = h(z) - g(z) = z + \sum_{n=1}^{\infty} c_n z^{-n}$, where $c_n = a_n - b_n$. Then $\tilde{f}(z) \in \mathcal{S}$. If $c_1 \geq 0$, then Kubota [5] proved that $Re\{c_4\} \leq \frac{2}{5} + \frac{729}{163840}$. Equality occurs for a function $F(z) = z + \sum_{n=1}^{\infty} B_n z^{-n}$ where $B_1 = \frac{27}{128}$, $B_2 = -\frac{27}{256}$, $B_3 = -\frac{243}{65536}$, $B_4 = \frac{2}{5} + \frac{729}{163840}$. More specifically, $w = F(z)$ satisfies the differential equation

$$\left(w - \frac{3}{8}\right)\left(w + \frac{3}{4}\right)^{1/2} \frac{dw}{dz} = z^{-7/2}\left(z^5 - \frac{27}{256}z^3 + \frac{27}{256}z^2 - 1\right).$$

In order to determine $F(1)$, we integrate this differential equation over a path from $z = -1$ to $z = 1$. A corresponding path runs from $w = -3/4$ to $w = F(1)$. The result is the equation $\frac{2}{5}X^5 - \frac{3}{4}X^3 = \frac{121}{320}$ for $X = \sqrt{F(1) + \frac{3}{4}}$. It follows from elementary calculus that this equation has only one real solution, and it is located at $X = 1.47329\cdots$. This implies that $F(1) = 1.42060\cdots$. For our purposes it is important only that $F(1) > 3/4$ so that the function F is subordinate to $G(z) = c(z + 1/z)$ for $0 < c \leq 3/8$. It will be convenient to set $t = 3/(8c)$ and restrict $t \geq 1$. Apply Lemma 2.1 and use the notation of Lemma 2.2, then we have $Re\{\beta_4\} \leq B_4$ if $\beta_1 \geq 0$. Since $C_1 = 1$ and $C_2 = C_3 = C_4 = 0$, this inequality reduces to (2.1). The constraint $\beta_1 \geq 0$ becomes $c_1 \geq 1 - \frac{3}{2}t^2$. However, if $c_1 < 1 - \frac{3}{2}t^2$, then the trivial estimates $|c_4| \leq 1/2$ and $|c_2| \leq 1/\sqrt{2}$ from the area theorem imply

$$Re\{c_4 - (3t^2 - 2)c_2 + 2t^3 c_1\} \leq \frac{1}{2} + \sqrt{2}(\frac{3}{2}t^2 - 1) - 3t^5 + 2t^3.$$

The polynomial $\frac{1}{2} + \sqrt{2}(\frac{3}{2}t^2 - 1) - 3t^5$ is negative at $t = 1$ and decreasing for $t \geq 1$. Therefore

$$Re\{(a_4 - b_4) - (3t^2 - 2)(a_2 - b_2) + 2t^3(a_1 - b_1)\} \leq 2t^3$$

holds for all real $a_1 - b_1$. □

Corollary 2.4. *Let $f \in \Sigma$. If $h-g$ with a_1-b_1 real is univalent, then*

$$Re\{(a_4 - b_4) + 4(a_1 - b_1)\} \leq 4.$$

Proof. The inequality (2.1) in Theorem 2.3 is sharp for $K(z) = z + 1/z$. When $t = 1$, it reduces to $Re\{(a_4-b_4)-(a_2-b_2)+2(a_1-b_1)\} \leq 2$ whenever $a_1 - b_1$ is real. Add this to the inequality $Re\{(a_2 - b_2) + 2(a_1-b_1)\} \leq 2$ [1] which is obtained by Garabedian and Schiffer, then we get

$$Re\{(a_4 - b_4) + 4(a_1 - b_1)\} \leq 4,$$

as desired. □

A set $E \subset \mathbb{C}$ is said to be starlike with respect to a point $w_0 \in E$ if the linear segment joining w_0 to every other point $w \in E$ lies entirely in E.

The following result was proved by Pommerenke [6].

Theorem 2.5. ([6]) *Let $\mathcal{S}^* = \{\tilde{f} \in \mathcal{S} : \tilde{f}(\Delta)^c$ is starlike with respect to the origin$\}$. Then the nth coefficient of every function in $\mathcal{S}^*$ satisfies $|c_n| \leq 2/(n+1)$, with equality only for the function*

$$\tilde{f}(z) = \{k(z^{-n-1})\}^{-1/(n+1)} = z - \frac{2}{n+1} z^{-n} + \cdots ,$$

and its rotations, $n = 1, 2, \dots$, where k is the Koebe function.

Theorem 2.6. *For each $f \in \Sigma$ with univalent starlike $h - g$, we have*

$$|a_n - b_n| \leq \frac{2}{(n+1)}.$$

Proof. $\tilde{f}(z) = h(z) - g(z) \in \mathcal{S}^*$. Thus we have estimates by Theorem 2.5. □

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