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THE MCSHANE INTEGRAL OF BANACH SPACE-VALUED FUNCTIONS

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Abstract. In this paper, we investigate the relations between the McShane and Pettis integrals.

1. Introduction

A general integration theory based on the concept of Riemann type integral sums was initiated around 1960 by Jaroslav Kurzweil and independently by Ralph Henstock.

The main virtue of the presentation of the Henstock–Kurzweil integral of real-valued functions is that no measure theory is required and that even sophisticated convergence results can be derived using merely elementary tools from the calculus without advanced topology.

The relatively new concepts of the Henstock–Kurzweil and McShane integral based on Riemann type sums are an interesting challenge also in the study of integration of Banach space-valued functions. The advantage of a relatively transparent and easy definition is undoubtedly an invitation to do so.

The investigations started around 1990 by the work of R. A. Gordon and since then attention has been paid to this field.

In this paper, we introduce the McShane and Pettis integral and investigate its relations for bilinear triples.

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2. Preliminaries

Assume that X, Y and Z are Banach spaces and that there is a bilinear mapping $B : X \times Y \longrightarrow Z$. We use the short notation $xy = B(x, y)$ for the values of the bilinear form B for $x \in X, y \in Y$ and assume that

$$\|xy\| \leq \|x\| \cdot \|y\|.$$

Triples of Banach spaces X, Y, Z with these properties are called *bilinear triples* and they are denoted by $\mathcal{B} = (X, Y, Z)$. For the case $\mathcal{B} = (\mathbb{R}, \mathbb{R}, \mathbb{R})$, we always assume $B(x, y) = xy$ (product).

The sets $[c, d], (c, d), (c, d], [c, d) \subset [a, b]$ are called *intervals in* $[a, b]$. Let a compact interval $[a, b] \subset \mathbb{R}$ be given.

A finite collection $\{(t_i, I_i) : i = 1, \cdots, p\}$ of nonoverlapping tagged intervals is called an *M-system in* $[a, b]$ if $I_i \subset [a, b]$ for $j = 1, \cdots, p$.

An M-system $\{(t_i, I_i) : i = 1, \cdots, p\}$ in $[a, b]$ is called an *M-partition of the interval* $[a, b]$ if

$$\bigcup_{i=1}^{p} I_i = [a, b].$$

Given a positive function δ called a *gauge* on $[a, b]$, a tagged interval (t, I) is said to be *δ-fine* if

$$I \subset (t - \delta(t), t + \delta(t)).$$

M-systems are called $\delta-$fine if all the tagged intervals $\{(t_i, I_i), i = 1, \cdots, p\}$ are δ-fine.

A figure is a finite union of intervals. Let $\mathcal{F}$ be the collection of all figures on $[a, b]$.

A function g defined on $\mathcal{F}$ with Banach space-values is *additive* if for nonoverlapping figures A and B, $g(A \cup B) = g(A) + g(B)$.

Throughout this paper we always assume that g is additive and

$$g([c, d]) = g((c, d)) = g((c, d])) = g([c, d)).$$

DEFINITION 2.1. Assume that $\mathcal{B} = (X, X, X)$ is a bilinear triple and that $f : [a, b] \longrightarrow X$ and $g : \mathcal{F} \longrightarrow X$ are given. We say that fdg is McShane(or M)-integrable on $[a, b]$ and $v \in X$ is its integral if for every $\varepsilon > 0$ there is a gauge δ on $[a, b]$ such that for

$$S(fdg, P) \equiv \sum_{i=1}^{p} f(t_i)g(I_i)$$

we have

$$\|S(fdg, P) - v\| < \varepsilon$$

provided $P = \{(t_i, I_i) : i = 1, \cdots, p\}$ is a δ-fine M-partition of $[a, b]$. In this case we denote $v = (M)\int_a^b f(s)dg(s)$. If there is no confusion we omit the prefix (M).

The following theorems describe some of the basic properties of the M-integral.

THEOREM 2.1. *Assume that $\mathcal{B} = (X, X, X)$ is a bilinear triple and that $f : [a, b] \longrightarrow X$ and $g : \mathcal{F} \longrightarrow X$ are given.*

(a) If fdg is M-integrable on $[a, b]$, then fdg is M-integrable on every subinterval of $[c, d] \subset [a, b]$.

(b) If fdg is M-integrable on each of the interval I_1 and I_2, where I_1 and I_2 are nonoverlapping and $I_1 \cup I_2 = I$ is an interval, then fdg is M-integrable on I and

$$\int_I fdg = \int_{I_1} fdg + \int_{I_2} fdg.$$

(c) If f_1dg and f_2dg are M-integrable on $[a, b]$ and α and β are real numbers, then $(\alpha f_1 + \beta f_2)dg$ is M-integrable on $[a, b]$ and

$$\int_a^b (\alpha f_1 + \beta f_2)dg = \alpha \int_a^b f_1dg + \beta \int_a^b f_2dg.$$

Proof. The proof of this theorem is essentially identical to that of [4]. $\square$

THEOREM 2.2. *Assume that* $\mathcal{B} = (X, X, X)$ *is a bilinear triple and that* $f : [a, b] \longrightarrow X$ *and* $g : \mathcal{F} \longrightarrow X$ *are given. For given* $\varepsilon > 0$ *assume that the gauge* δ *on* $[a, b]$ *is such that*

$$\left\| \sum_{i=1}^{p} f(t_i)g(I_i) - \int_a^b f dg \right\| < \varepsilon$$

for every δ*-fine partition* $P = \{(t_i, I_i) : i = 1, \cdots, p\}$ *of* $[a, b]$.

If $P^* = \{(t_i^*, I_i^*) : i = 1, \cdots, m\}$ *is a* δ*-fine* M*-system in* $[a, b]$*, then we have*

$$\left\| \sum_{i=1}^{m} \left[f(t_i)g(I_i^*) - \int_{I_i^*} f dg \right] \right\| \leq \varepsilon.$$

Proof. The proof is essentially identical to that of [4]. □

Theorem 2.2 is called the Saks-Henstock lemma for the M-integral.

DEFINITION 2.2. Assume that $g : \mathcal{F} \longrightarrow X$ is given. We define for a figure E in $[a, b]$,

$$SV(g, P, E) = \sup \left\| \sum_{i=1}^{p} x_i g(I_i) \right\|,$$

where $P = \{(t_i, I_i)\}$ is a partition with $\cup_{i=1}^{p} I_i = E$ and the supremum is taken over all possible choice of $x_i \in X, i = 1, ..., p$ with $\|x_i\| \leq 1$. We denote

$$SV(g, E) \equiv \sup_P SV(g, P, E),$$

where the supremum is taken over all $P = \{(t_i, I_i)\}$ with $\cup_{i=1}^{p} I_i = E$. If $SV(g, E) < \infty$, we say that g is of semi-bounded variation on E.

For a measurable set E, we define $|E|$ as the Lebesgue measure of E.

We say that a function $g : \mathcal{F} \longrightarrow X$ satisfies condition(P_1) if given $\varepsilon > 0$ there exists a $\eta > 0$ such that

$$SV(g, E) < \varepsilon$$

whenever E is a figure with $|E| < \eta$.

The integrals also behave like the Lebesgue integrals. Let $\|f\|_\infty \equiv \sup_{s\in[a,b]} \|f(s)\|$.

THEOREM 2.3. *Assume that $\mathcal{B} = (X, X, X)$ is a bilinear triple and that $f : [a, b] \longrightarrow X$ and $g : \mathcal{F} \longrightarrow X$ are given. Suppose that $\|f\|_\infty < \infty$ and g satisfies the condition(P_1).*

(a) If fdg is M-integrable on $[a, b]$, then for every measurable subset $E \subset [a, b]$ the M-integral $\int_a^b f\chi_E dg \equiv \int_E fdg$ exists.

(b) For every set $E \subset [a, b]$ with $|E| = 0$, $\int_E fdg = 0$.

(c) For disjoint measurable sets $E_i, i = 1, 2, ...,$ we have

$$\int_{\cup_{i=1}^\infty E_i} fdg = \sum_{i=1}^\infty \int_{E_i} fdg.$$

Proof. The proof of this theorem is considerably analogue to that of [4]. But the condition(P_1) and $\|f\|_\infty < \infty$ are crucial to prove the theorem. □

Let X^* be the set of all linear functionals on X and let $B(X^*) = \{x^* \in X^* : \|x^*\| \leq 1\}$.

DEFINITION 2.3. Assume that $\mathcal{B} = (X, X, X)$ is a bilinear triple and that $f : [a, b] \longrightarrow X$ and $g : \mathcal{F} \longrightarrow X$ are given. Let x^* be an element of X^*. Then we say that $x^*(fdg)$ is M-integrable on $[a, b]$, and $r \in \mathbb{R}$ is its integral if for every $\varepsilon > 0$ there is a gauge δ on $[a, b]$ such that for

$$S(x^*(fdg), P) \equiv \sum_{i=1}^p x^*(f(t_i)g(I_i))$$

we have

$$\|S(x^*(fdg), P) - r\| < \varepsilon$$

provided $P = \{(t_i, I_i) : i = 1, \cdot, p\}$ is a δ-fine M-partition of $[a, b]$. In this case we denote $r = \int_a^b x^*(fdg)$.

DEFINITION 2.4. Assume that $\mathcal{B} = (X, X, X)$ is a bilinear triple and that $f : [a, b] \longrightarrow X$ and $g : \mathcal{F} \longrightarrow X$ are given. Let I be an index set. Then $\{x_i^*(fdg) : x_i^* \in B(X^*), i \in I\}$ is called M-equiintegrable if for every $\varepsilon > 0$ there is a gauge δ on $[a, b]$ such that

$$\left\| \sum_{k=1}^{p} x_i^*(f(t_k)g(I_k)) - (M) \int_a^b x_i^*(fdg) \right\| < \varepsilon$$

for every δ-fine M-partition $P = \{(t_k, I_k) : k = 1, \cdots, p\}$ of $[a, b]$ and all $i \in I$.

Using the concept of M-equiintegrability we have the following convergence result for the M-integral.

THEOREM 2.4. *Assume that $\mathcal{B} = (X, X, X)$ is a bilinear triple and that $f : [a, b] \longrightarrow X$ and $g : \mathcal{F} \longrightarrow X$ are given. If the sequence $\{x_n^*(fdg) : x_n^* \in B(X^*), n = 1, 2, \cdots\}$ is M-equiintegrable and there is a $x^* \in B(X^*)$ such that*

$$\lim_{n\to\infty} x_n^*(f(t)g(I)) = x^*(f(t)g(I))$$

for all $t \in [a, b]$ and all intervals $I \subset [a, b]$, then $x^*(fdg)$ is M-integrable and

$$\lim_{n\to\infty} \int_a^b x_n^*(fdg) = \int_a^b x^*(fdg).$$

Proof. The proof is essentially same as the proof of [2]. □

Now we introduce a concept of Pettis for our bilinear forms.

DEFINITION 2.5. Assume that $\mathcal{B} = (X, X, X)$ is a bilinear triple and that $f : [a, b] \longrightarrow X$ and $g : \mathcal{F} \longrightarrow X$ are given. We say that fdg is Pettis(or P-)integrable on $[a, b]$ if for every measurable $E \subset [a, b]$ there is an element $x_E \in X$ that satisfies

$$x^*(x_E) = (M) \int_E x^*(f(s)dg(s))$$

for every $x^* \in X^*$. We denote $x_E = (P) \int_E fdg$.

3. M-integrable functions are P-integrable

We first prove some auxiliary results.

LEMMA 3.1. *Assume that $\mathcal{B} = (X, X, X)$ is a bilinear triple and that $f : [a, b] \longrightarrow X$ and $g : \mathcal{F} \longrightarrow X$ are given. Suppose that $\|f\|_\infty < \infty$ and g satisfies the condition(P_1). If fdg is M-integrable on $[a, b]$, then there is a $\eta > 0$ such that for any finite collection $\{J_j : 1 \le j \le p\}$ of nonoverlapping intervals in $[a, b]$ with $\sum_{j=1}^{p} |J_j| < \eta$ we have*

$$\left\| \sum_{j=1}^{p} \int_{J_j} fdg \right\| < \varepsilon.$$

Proof. Let $\varepsilon > 0$ be given. Since fdg is M-integrable on $[a, b]$, there exists a gauge δ on $[a, b]$ such that

$$\left\| S(fdg, P) - \int_a^b fdg \right\| < \varepsilon$$

whenever P is a δ-fine M-partition of $[a, b]$. Fix a δ-fine M-partition of $[a, b]$

$$P_0 = \{(t_i, I_i) : 1 \le i \le q\}.$$

Suppose that $\{J_j : 1 \le j \le p\}$ is a finite collection of nonoverlapping intervals in $[a, b]$ such that $\sum_{j=1}^{p} |J_j| \le \eta$ which also satisfies that $SV(g, E) < \varepsilon$ whenever $|E| < \eta$. By subdividing these intervals if necessary, we may assume that for each j, $J_j \subset I_i$ for some i. For each $i, 1 \le i \le q$, let $M_i = \{j : 1 \le j \le p, J_j \subset I_i\}$ and let

$$P = \{(t_i, J_j) : j \in M_i, i = 1, \cdots, q\}.$$

Note that P is a δ-fine M-system in $[a, b]$.

Using the Saks–Henstock lemma, we have

$$\begin{aligned} \left\| \sum_{j=1}^{p} \int_{J_j} fdg \right\| &\le \left\| \sum_{j=1}^{p} \left[\int_{J_j} fdg - f(t_i)g(J_j) \right] \right\| + \left\| \sum_{j=1}^{p} f(t_i)g(J_j) \right\| \\ &< \varepsilon + \|f\|_\infty \varepsilon. \end{aligned}$$

This completes the proof. $\square$

LEMMA 3.2. ([8]) *Assume that F is an X-valued interval function defined for intervals in $[a, b]$ such that for every $\varepsilon > 0$ there is an $\eta > 0$ such that for any finite collection $\{J_j : j = 1, \cdots, p\}$ of nonoverlapping intervals in $[a, b]$ with $\sum_{j=1}^{p} |J_j| < \eta$ we have $\| \sum_{j=1}^{p} F(J_j)\| < \varepsilon$. Then:*

(a) For any sequence $\{I_i : i = 1, 2, \cdots\}$ of nonoverlapping intervals $I_i \subset [a, b], i \in \mathbb{N}$ with $\sum_{i=1}^{\infty} |I_i| \leq (b - a)$ the limit

$$\lim_{n\to\infty} \sum_{i=1}^{n} F(I_i) = \sum_{i=1}^{\infty} F(I_i) \in X$$

exists.

(b) If for the sequence $\{I_i : i = 1, 2, \cdots\}$ of nonoverlapping intervals, then $\sum_{i=1}^{\infty} |I_i| < \eta$, where $\eta > 0$ is the value of η corresponding to $\varepsilon > 0$ by the assumption, implies $\| \sum_{i=1}^{\infty} F(I_i)\| \leq \varepsilon$.

LEMMA 3.3. *Assume that $\mathcal{B} = (X, X, X)$ is a bilinear triple and that $f : [a, b] \longrightarrow X$ and $g : \mathcal{F} \longrightarrow X$ are given. Suppose that $\|f\|_\infty < \infty$ and g satisfies the condition (P_1). If fdg is M-integrable on $[a, b]$, then for every open set $G \subset [a, b]$ there is an element $x_G \in X$ such that*

$$\int_G x^*(fdg) = x^*(x_G)$$

for every $x^* \in X^*$.

Proof. In the proof of [8, Lemma 31] we can find a measurable set $E_0 \subset G$ with $|E_0| = |G|$ and a collection $\{K_n\}$ of figures contained in $[a, b]$ such that $E_0 = \cup_{n=1}^{\infty} K_n$, $K_n^o \cap K_l^o = \emptyset$ for $n \neq l$, where K^o denotes the interior of K, and K_n is a finite union of non-overlapping intervals in $[a, b]$, i.e.,

$$K_n = \bigcup_{i=1}^{p_n} I_i^n,$$

while $\{I_i^n : i = 1, \cdots, p_n, n \in \mathbb{N}\}$ forms an at most countable system of non-overlapping intervals contained in E_0. Since $\cup_{n=1}^{p} K_n \subset E_0, p \in \mathbb{N}$, we have

$$\sum_{n=1}^{p} |K_n| = \left| \bigcup_{n=1}^{p} K_n \right| \le |E_0| = |G| \le |[a,b]| < \infty.$$

This gives

$$\sum_{n=1}^{\infty} |K_n| = \sum_{n=1}^{\infty} \left| \bigcup_{i=1}^{p_n} I_i^n \right| = \sum_{n=1}^{\infty} \sum_{i=1}^{p_n} |I_i^n| < \infty,$$

and by Lemmas 3.1 and 3.2 we obtain the existence of the limit

$$\lim_{m\to\infty} \sum_{n=1}^{m} \sum_{i=1}^{p_n} F(I_i^n) = \lim_{m\to\infty} \sum_{n=1}^{m} F(K_n) \equiv x_G \in X,$$

where F is a McShane primitive of fdg.

Given $x^* \in X^*$, the real function $x^*(fdg)$ is M-integrable for every measurable subset of $[a,b]$. Since $|G - E_0| = 0$ and $E_0 \subset G$, by Theorem 2.3 we have

$$\int_G x^*(fdg) = \int_{E_0} x^*(fdg).$$

Further we have

$$\begin{aligned}
\int_{E_0} x^*(fdg) &= \int_{\cup_{n=1}^{\infty} K_n} x^*(fdg) = \int_{\cup_{n=1}^{\infty} \cup_{i=1}^{p_n} I_i^n} x^*(fdg) \\
&= \lim_{m\to\infty} \int_{\cup_{n=1}^{m} \cup_{i=1}^{p_n} I_i^n} x^*(fdg) = \lim_{m\to\infty} x^* \left(\int_{\cup_{n=1}^{m} \cup_{i=1}^{p_n} I_i^n} fdg \right) \\
&= \lim_{m\to\infty} x^* \left(\int_{\cup_{n=1}^{m} K_n} fdg \right) = \lim_{m\to\infty} x^* \left(\sum_{n=1}^{m} F(K_n) \right) \\
&= x^*(x_G),
\end{aligned}$$

and $\int_G x^*(fdg) = x^*(x_G)$ for every $x^* \in X^*$. The proof is complete. $\square$

LEMMA 3.4. *Assume that* $\mathcal{B} = (X, X, X)$ *is a bilinear triple and that* $f : [a, b] \longrightarrow X$ *and* $g : \mathcal{F} \longrightarrow X$ *are given. Suppose that* $\|f\|_\infty < \infty$ *and* g *satisfies the condition* (P_1)*. If* fdg *is* M*-integrable on* $[a, b]$*, then for every closed set* $H \subset [a, b]$ *there is an element* $x_H \in X$ *such that*

$$\int_H x^*(fdg) = x^*(x_H)$$

for every $x^* \in X^*$.

Proof. If $H \subset [a, b]$ is closed, then $[a, b] - H$ is open and for every $x^* \in X^*$ we have

$$\begin{aligned} x^*\left((M)\int_a^b fdg\right) &= \int_a^b x^*(fdg) = \int_H x^*(fdg) + \int_{[a,b]-H} x^*(fdg) \\ &= \int_H x^*(fdg) + x^*(x_{[a,b]-H}), \end{aligned}$$

where for the open set $[a, b] - H$ the element $x_{[a,b]-H} \in X$ is given by Lemma 3.3. Hence

$$\int_H x^*(fdg) = x^*\left((M)\int_{[a,b]} fdg - x_{[a,b]-H}\right),$$

and we can take

$$x_H = (M)\int_a^b fdg - x_{[a,b]-H} \in X. \quad \square$$

LEMMA 3.5. *Assume that* $\mathcal{B} = (X, X, X)$ *is a bilinear triple and that* $f : [a, b] \longrightarrow X$ *and* $g : \mathcal{F} \longrightarrow X$ *are given. Suppose that* $\|f\|_\infty < \infty$ *and* g *satisfies the condition* (P_1)*. If* fdg *is* M*-integrable on* $[a, b]$ *and* $G \subset [a, b]$ *is open, then for every* $\varepsilon > 0$ *there is an* $\eta > 0$ *such that if* $|G| < \eta$*, then* $\|x_G\| < \varepsilon$*, where* $x_G \in X$ *is such that* $\int_G x^*(fdg) = x^*(x_G)$ *for every* $x^* \in X^*$.

Proof. As in the proof of Lemma 3.3 we say that there exists a sequence of sets $K_n \subset G, n \in \mathbb{N}$, which are finite unions of nonoverlapping intervals and satisfy $K_n^o \cap K_l^o = \emptyset$ for $n \neq l$, such that for every $x^* \in X^*$ we have

$$\int_G x^*(fdg) = \lim_{m\to\infty} x^* \left(\sum_{n=1}^m F(K_n) \right) = x^*(x_G).$$

By Lemmas 3.1 and 3.2, for every $\varepsilon > 0$ there is an $\eta > 0$ such that if $\sum_{n=1}^\infty |K_n| < \eta$ then $\| \sum_{n=1}^\infty F(K_n)\| = \|x_G\| < \varepsilon$. Hence the lemma is proved. □

THEOREM 3.6. *Assume that $\mathcal{B} = (X, X, X)$ is a bilinear triple and that $f : [a, b] \longrightarrow X$ and $g : \mathcal{F} \longrightarrow X$ are given. Suppose that $\|f\|_\infty < \infty$ and g satisfies the condition (P_1). If fdg is M-integrable on $[a, b]$, then fdg is P-integrable.*

Proof. We need to prove only that for every measurable subset E of $[a, b]$ there is an element $x_E \in X$ which satisfies $x^*(x_E) = \int_E x^*(fdg)$ for every $x^* \in X^*$.

Suppose that E is a measurable subset of $[a, b]$. Then there exists a sequence of open sets $G_n \subset [a, b], n \in \mathbb{N}$, such that

$$E \subset \cdots \subset G_{n+1} \subset G_n \subset \cdots \text{ and } |G_n - E| < \frac{1}{2n}$$

and a sequence of closed sets $H_n \subset [a, b], n \in \mathbb{N}$, such that

$$\cdots \subset H_n \subset H_{n+1} \subset \cdots \subset E \text{ and } |E - H_n| < \frac{1}{2n}.$$

Since $G_n \subset [a, b], n \in \mathbb{N}$, are open sets, by Lemma 3.3 there exists $x_{G_n}, n \in \mathbb{N}$, such that

$$\int_{G_n} x^*(fdg) = x^*(x_{G_n})$$

for every $x^* \in X^*$. Let $\varepsilon > 0$ be given and let $\eta > 0$ be the value corresponding to $\varepsilon/2$ by Lemma 3.5. Then there exists $N \in \mathbb{N}$, such that $1/n < \eta$ if $n \geq N$ and therefore $|G_n - H_n| < \eta$ for $n \geq N$.

Assume that $m_1, m_2 > N$. Then

$$\begin{aligned}
|x^*(x_{G_{m_1}} - x_{G_{m_2}})| &= \left| \int_{G_{m_1}} x^*(fdg) - \int_{G_{m_2}} x^*(fdg) \right| \\
&= \left| \int_{G_{m_1}} x^*(fdg) - \int_{H_N} x^*(fdg) - \int_{G_{m_2}} x^*(fdg) + \int_{H_N} x^*(fdg) \right| \\
&= \left| \int_{G_{m_1} - H_N} x^*(fdg) - \int_{G_{m_2} - H_N} x^*(fdg) \right| \\
&\leq \left| \int_{G_{m_1} - H_N} x^*(fdg) \right| + \left| \int_{G_{m_2} - H_N} x^*(fdg) \right| \\
&\leq \|x^*\|(\|x_{G_{m_1} - H_N}\| + \|x_{G_{m_2} - H_N}\|) \leq \|x^*\|\varepsilon
\end{aligned}$$

because $G_{m_1} - H_N \subset G_N - H_N$ and $|G_{m_1} - H_N| \leq |G_N - H_N| < \eta$, and similarly $|G_{m_2} - H_N| < \eta$.

Hence for every $x^* \in B(X^*)$ we have $|x^*(x_{G_{m_1}} - x_{G_{m_2}})| < \varepsilon$ provided $m_1, m_2 > N$ and therefore $\|x_{G_{m_1}} - x_{G_{m_2}}\| < \varepsilon$ for $m_1, m_2 > N$, the sequence $x_{G_n} \in X, n \in \mathbb{N}$, is therefore Cauchy, and consequently the limit $\lim_{m\to\infty} x_{G_m} = x_E \in X$ exists.

Moreover, we have $E \subset \cap_{m=1}^{\infty} G_m$ and $\cap_{m=1}^{\infty} G_m - E \subset G_n - E$ for every $n \in \mathbb{N}$ and therefore $| \cap_{m=1}^{\infty} G_m - E| = 0$. Hence

$$\begin{aligned}
x^*(x_E) &= \lim_{m\to\infty} x^*(x_{G_m}) = \lim_{m\to\infty} \int_{G_m} x^*(fdg) \\
&= \int_{\cap_{m=1}^{\infty} G_m} x^*(fdg) = \int_E x^*(fdg)
\end{aligned}$$

for all $x^* \in X^*$.

This holds for every measurable set $E \subset [a,b]$. Therefore, by definition, fdg is P-integrable, $(P)\int_E x^*(fdg = x^*(x_E)$, and the theorem is proved. $\square$

4. The Pettis integral and its relation to the McShane integral

LEMMA 4.1. *Assume that $\mathcal{B} = (X, X, X)$ is a bilinear triple and that $f : [a,b] \longrightarrow X$ and $g : \mathcal{F} \longrightarrow X$ are given. Suppose that fdg is P-integrable. Then fdg is M-integrable if and only if the set $\{x^*(fdg) : x^* \in B(X^*)\}$ is M-equiintegrable.*

Proof. It is obvious by definition that the set $\{x^*(fdg) : x^* \in B(X^*)\}$ is M-equiintegrable provided fdg is M-integrable. Assume that $\{x^*(fdg) : x^* \in B(X^*)\}$ is M-equiintegrable. Then, by definition, for every $\varepsilon > 0$ there exists a gauge δ on $[a,b]$, such that for every $\delta-$fine M-partition $P = \{(t_i, I_i) : i = 1, ..., q\}$ of $[a,b]$ and $x^* \in B(X^*)$ we have

$$\left| \sum_{i=1}^{q} x^*(f(t_i)g(I_i)) - \int_a^b x^*(fdg) \right| < \varepsilon.$$

Since fdg is P-integrable, we have $\int_a^b x^*(fdg) = x^*\left((P)\int_a^b fdg\right)$, and

$$\sum_{i=1}^{q} x^*(f(t_i)g(I_i) = x^*\left(\sum_{i=1}^{q} f(t_i)g(I_i)\right)$$

holds evidently. Hence for every $\delta-$fine M-partition $P = \{(t_i, I_i) : i = 1, ..., q\}$ of $[a,b]$ and $x^* \in B(X^*)$ we have

$$\left| x^*\left(S(fdg, P) - \int_a^b fdg\right) \right| < \varepsilon,$$

and this yields immediately

$$\left\| S(fdg, P) - \int_a^b fdg \right\| < \varepsilon$$

for every $\delta-$fine M-partition $P = \{(t_i, I_i) : i = 1, ..., q\}$. So we obtain that fdg is M-integrable on $[a, b]$(and $(M)\int_a^b fdg = (P)\int_a^b fdg$). □

For the next theorem we need to assume that the ball $B(X^*)$ has the following properties.

Property(P_2): There exists a sequence $\{x_n^*\} \subset B(X^*)$ such that the closure of $\{x_n^*\}$ contains $B(X^*)$ and $\{x_n^*(fdg)\}$ is M-equiintegrable on $[a, b]$.

THEOREM 4.2. *Assume that $\mathcal{B} = (X, X, X)$ is a bilinear triple and that $f : [a, b] \longrightarrow X$ and $g : \mathcal{F} \longrightarrow X$ are given. Suppose that X satisfies the condition(P_2) and g is semi-bounded variation on $[a, b]$. Then fdg is M-integrable on $[a, b]$.*

Proof. Assume that $x^* \in B(X^*)$ is given. Since X satisfies the condition(P_2), there exists a sequence $\{x_k^*\} \subset \{x_n^*\}$ such that

$$\lim_{k\to\infty} \|x_k^* - x^*\| = 0, \tag{4.1}$$

and $\{x_k^*(fdg)\}$ is M-equiintegrable. So by definition there is a gauge δ on $[a, b]$ such that

$$\left|\sum_{i=1}^{p} x_k^*(f(t_i)g(I_i)) - \int_a^b x_k^*(fdg)\right| < \varepsilon \tag{4.2}$$

Since $\{x_k^*(fdg)\}$ is M-equiintegrable and for every $t \in [a, b]$ and every interval I in $[a, b]$ $\lim_{k\to\infty} x_k^*(f(t)g(I)) = x^*(f(t)g(I))$, by Theorem 2.4, $x^*(fdg)$ is M-integrable and $\lim_{k\to\infty} \int_a^b x_k^*(fdg) = \int_a^b x^*(fdg)$. So there is a $k_0 > \mathbb{N}$ such that

$$\left|\int_a^b x_k^*(fdg) - \int_a^b x^*(fdg)\right| < \varepsilon \tag{4.3}$$

Let $P = \{(t_i, I_i) : i = 1, ..., p\}$ is a $\delta-$fine M-partition of $[a, b]$ and let $k > k_0$. Then we get

$$\begin{aligned}
&\left|\sum_{i=1}^{q} x^*(f(t_i)g(I_i)) - \int_a^b x^*(fdg)\right| \\
\leq\ & \left|\sum_{i=1}^{q}[x^*(f(t_i)g(I_i)) - x_k^*(f(t_i)g(I_i))\right| + \left|x_k^*(f(t_i)g(I_i)) - \int_a^b x_k^*(fdg)\right| \\
& + \left|\int_a^b x_k^*(fdg) - \int_a^b x^*(fdg)\right| \\
<\ & \|f\|_\infty \cdot \|x_k^* - x^*\| \cdot SV(g,[a,b]) + \varepsilon + \varepsilon \\
<\ & [2 + \|f\|_\infty \cdot SV(g,[a,b])]\varepsilon.
\end{aligned}$$

Since $x^* \in B(X^*)$ was arbitrary, we see that $\{x^*(fdg) : x^* \in B(X^*)\}$ is M-equiintegrable. By Lemma 4.1, fdg is M-integrable on $[a,b]$. $\square$.

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