JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **17**, No.2, October 2004

WEAK INVERSE SHADOWING AND Ω-STABILITY

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Abstract. We give characterization of Ω-stable diffeomorphisms via the notions of weak inverse shadowing. More precisely, it is proved that the C^1 interior of the set of diffeomorphisms with the weak inverse shadowing property with respect to the class $\mathcal{T}_h$ coincides with the set of Ω-stable diffeomorphisms.

1. Introduction

Ω-stable systems have been the main objects of interests in the global qualitative theory of dynamical systems over recent 30 years and various attempts have been made to characterize the systems via the notions of hyperbolicity, shadowing, Axiom A, no-cycle condition, etc.

The weak shadowing property which is really weaker than the shadowing property was introduced by Corless and Pilyugin ([3]), and they proved that the weak shadowing property is generic in the set of homoemorphisms on a C^∞ closed manifold endowed with the C^0 topology. Moreover Sakai ([13]) showed that every element in the C^1 interior of the set of diffeomorphisms on a C^∞ closed surface having the weak shadowing property is Ω-stable, but the converse does not hold in higher dimension.

The inverse shadowing property which is a "dual" notion of shadowing property was introduced by Corless and Pilyugin in [3], and the qualitative theory of dynamical systems with the property was developed by various authors (see [1], [3],[4],[6],[7],[11],[12], for example).

Received by the editors on June 16, 2004.
2000 *Mathematics Subject Classifications* : Primary 37C50, 37MM05.
Key words and phrases: Ω-stable, weak inverse shadowing, topologically Ω-stable.

In the paper [2], Choi, Lee and Zhang introduced the notion of weak [resp. orbital] inverse shadowing which is a "dual" notion of weak [resp. orbital] shadowing. In that paper, they proved that the C^1 interior of the set of diffeomorphisms with the weak [resp. orbital] inverse shadowing property with respect to the class $\mathcal{T}_d$ coincides with the set of Ω-stable [resp. structurally stable] diffeomorphisms. In this paper, we will prove that the C^1 interior of the set of diffeomorphisms with the weak inverse shadowing property with respect to the class $\mathcal{T}_h$ coincides with the set of Ω-stable diffeomorphisms which is a generalization of one of the results in the paper [2].

2. Preliminaries

Consider a dynamical system generated by a homeomorphism f of a metric space X with a metric d. For a point $x \in X$, we denote by $O(x, f)$ its orbit in the system f : i.e., the set

$$O(x, f) = \{f^n(x) : n \in \mathbb{Z}\} \ .$$

We say that a sequence $\xi = \{x_n \in X : n \in \mathbb{Z}\}$ is a δ-*pseudo orbit* of f if the inequalities

$$d(f(x_n), x_{n+1}) < \delta, \ n \in \mathbb{Z}$$

hold. A δ-pseudo orbit is a natural model of computer output in a process of numerical investigation of the system f. In this case, the value δ measures errors of the method, round-off errors, etc.

Recall that f has the *shadowing property* if given $\varepsilon > 0$ there exists $\delta > 0$ such that for any δ-pseudo orbit $\xi = \{x_n : n \in \mathbb{Z}\}$ we can find a point $y \in X$ with the property

$$d(f^n(y), x_n) < \varepsilon, \ n \in \mathbb{Z} \ .$$

Of course, if f has the shadowing property formulated above, then results of its numerical study with a proper accuracy reflect its qualitative structure.

It is said that f has the *weak shadowing property* if given $\varepsilon > 0$ there exists $\delta > 0$ such that for any δ-pseudo orbit $\xi = \{x_n\}$ of f we can find a point $y \in X$ with the property

$$\xi \subset N(\varepsilon, O(y, f)).$$

The weak shadowing property was introduced in [3].

Let $X^{\mathbb{Z}}$ be the space of all two sided sequences $\xi = \{x_n : n \in \mathbb{Z}\}$ with elements $x_n \in X$, endowed with the product topology. For $\delta > 0$, let $\Phi_f(\delta)$ denote the set of all δ-pseudo orbits of f. A mapping $\varphi : X \to \Phi_f(\delta) \subset X^{\mathbb{Z}}$ is said to be a *δ-method* for f if $\varphi(x)_0 = x$, where $\varphi(x)_0$ is the 0-component of $\varphi(x)$. Then each $\varphi(x)$ is a δ-pseudo orbit of f through x. For convenience, write $\varphi(x)$ for $\{\varphi(x)_k\}_{k\in\mathbb{Z}}$. Say that φ is a *continuous δ-method* for f if the map φ is continuous. The set of all δ-methods [resp. continuous δ-methods] for f will be denoted by $\mathcal{T}_0(f, \delta)$ [resp. $\mathcal{T}_c(f, \delta)$].

Let $Z(X)$ denote the set of all homeomorphisms on X with the C^0 metric d_0 where $d_0(f, g) = \sup_{x\in X}\{d(f(x), g(x)), d(f^{-1}(x), g^{-1}(x))\}$, $f, g \in Z(X)$. If $g : X \to X$ is a homeomorphism with $d_0(f, g) < \delta$, then g induces a continuous δ-method φ_g for f by defining

$$\varphi_g(x) = \{g^n(x) : n \in \mathbb{Z}\} .$$

Let $\mathcal{T}_h(f, \delta)$ denote the set of all continuous δ-methods φ_g for f which are induced by $g \in Z(X)$ with $d_0(f, g) < \delta$. We define $\mathcal{T}_\alpha(f)$ by

$$\mathcal{T}_\alpha(f) = \bigcup_{\delta>0} \mathcal{T}_\alpha(f, \delta) ,$$

where $\alpha = 0, c, h$. Clearly,

$$\mathcal{T}_h(f) \ \subset \ \mathcal{T}_c(f) \ \subset \ \mathcal{T}_0(f) .$$

Note that a method in $\mathcal{T}_c(f)$ need not be generated by a single mapping.

We say that f has the *shadowing property* [resp. *inverse shadowing property*] with respect to the class $\mathcal{T}_\alpha$, $\alpha = 0, c, h$, if for any $\varepsilon > 0$ there is $\delta > 0$ such that for any δ-method φ in $\mathcal{T}_\alpha(f,\delta)$ and any point $x \in X$ there exists a point $y \in X$ for which

$$d(f^n(y), \varphi(x)_n) < \varepsilon \text{ [resp. } d(f^n(x), \varphi(y)_n) < \varepsilon],\ n \in \mathbb{Z}.$$

For our purpose, let M be a C^∞ closed n-dimensional manifold with a metric d induced by a Riemannian metric $||\cdot||$ on TM, and let $\mathrm{Diff}(M)$ denote the space of C^1 diffeomorphisms on M with the C^1 metric d_1. As before, for any $\delta > 0$ and $f \in \mathrm{Diff}(M)$, every $g \in \mathrm{Diff}(M)$ with $d_1(f,g) < \delta$ induces a continuous δ-method $\varphi_g : M \to M^{\mathbb{Z}}$ for f by defining

$$\varphi_g(x) = \{g^k(x) : k \in \mathbb{Z}\}.$$

Let $\mathcal{T}_d(f,\delta)$ denote the set of all continuous δ-methods φ_g for f which are induced by $g \in \mathrm{Diff}(M)$ with $d_1(f,g) < \delta$. Put

$$\mathcal{T}_d(f) = \bigcup_{\delta>0} \mathcal{T}_d(f,\delta)\ .$$

Similarly we say that f has the *shadowing property* [resp. *inverse shadowing property*] with respect to the class $\mathcal{T}_d$ if for any $\varepsilon > 0$ there is $\delta > 0$ such that for any δ-method φ in $\mathcal{T}_d(f,\delta)$ and any point $x \in M$ there exists a point $y \in M$ for which

$$d(f^n(y), \varphi(x)_n) < \varepsilon \text{ [resp. } d(f^n(x), \varphi(y)_n) < \varepsilon],\ n \in \mathbb{Z}.$$

Now let us recall the notion of weak inverse shadowing which is a "dual" notion of weak shadowing (see [1]).

Definition 2.1. We say that f has the *weak inverse shadowing property* with respect to the class $\mathcal{T}_\alpha,\ \alpha = 0, c, h, d$ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any δ-method $\varphi \in \mathcal{T}_\alpha(f)$ and any point $x \in M$ there is a point $y \in M$ for which

$$\varphi(y) \subset N(\varepsilon, O(x,f)).$$

Remark 1. An appropriate choice of the class of admissible pseudo orbits is crucial when we consider the notions of inverse shadowing (see

[4],[7],[11]). Suppose that $\mathcal{T}_a(f) \subset \mathcal{T}_b(f)$ for some $f \in \mathrm{Diff}(M)$ and $a, b \in \{0, c, h, \}$. If f has the weak [resp. orbital] inverse shadowing property with respect to the class $\mathcal{T}_b$ then it clearly has the weak [resp. orbital] inverse shadowing property with respect to the class $\mathcal{T}_a$. We can easily show that every irrational rotation f on the unit circle S^1 has the weak inverse shadowing with respect to the class $\mathcal{T}_c$ (or $\mathcal{T}_h$), but it does not have the inverse shadowing property with respect to the class $\mathcal{T}_c$ (or $\mathcal{T}_h$). Recently Choi, Kim and Lee [1] showed that the weak inverse shadowing property is generic in the space of homeomorphisms on a compact metric space with the C^0 topology.

3. Weak inverse shadowing and Ω-stability

The aim of this paper is to investigate the dynamics of diffeomorphisms belonging to the C^1 interior of the set of diffeomorphisms having the weak inverse shadowing property with respect to the class $\mathcal{T}_h$.

Let $\Omega(f)$ denote the set of nonwandering points of f. Recall that f is Ω-*stable* if there is a C^1 neighborhood $\mathcal{V}$ of f such that for any $g \in \mathcal{V}$ the restriction of f to $\Omega(f)$ and g to $\Omega(g)$ are topologically conjugate.

We say that $f \in \mathrm{Diff}(M)$ is *hyperbolic* on a closed invariant set $\Lambda \subset M$ if there is a continuous splitting of the tangent bundle, $TM|_\Lambda = E^s \oplus E^u$, and there are constants $C > 0, 0 < \lambda < 1$, such that

$$\|Df^n|_{E^s_x}\| < C\lambda^n \quad \text{and} \quad \|Df^{-n}|_{E^u_x}\| < C\lambda^n$$

for any $n > 0$ and $x \in \Lambda$. The *stable* [resp. *unstable*] *manifold* of $x \in \Lambda$ is defined by $W^s(x, f)$ [resp. $W^u(x, f)$], which is the set of points $p \in M$ such that $d(f^k(x), f^k(p))$ tends to 0 as k tends to ∞ [resp. $-\infty$]. We say that f satisfies *Axiom A* if its periodic points are dense in the set of nonwandering points $\Omega(f)$, and f is hyperbolic on $\Omega(f)$.

Let $\mathcal{F}(M)$ be the set of $f \in \mathrm{Diff}(M)$ having a C^1 neighborhood $\mathcal{U} \subset \mathrm{Diff}(M)$ such that for $g \in \mathcal{U}$, every periodic point of g is hyperbolic. Hayashi [5] proved that the set $\mathcal{F}(M)$ is contained the set of diffeomorphisms satisfying Axiom A with no-cycle condition. It is well known that f is Ω-stable if and only if it satisfies Axiom A and no-cycle condition.

We denote by $WISP_h(M)$ the set of $f \in \mathrm{Diff}(M)$ having the weak inverse shadowing property with respect to the class $\mathcal{T}_h$. At first, we begin with the following definition.

DEFINITION 3.1. ([9]) We say that a homeomorphism f of M is *topologically Ω-stable* if given $\varepsilon > 0$ there exists a neighborhood W of f in $Z(M)$ such that for any system $g \in W$ there is a continuous map h mapping $\Omega(g)$ onto $\Omega(f)$ and such that

(a) $d(x, h(x)) \leq \varepsilon$ for all $x \in \Omega(g)$;
(b) the following diagram is commutative:

$$\begin{array}{ccc} \Omega(g) & \xrightarrow{g} & \Omega(g) \\ \downarrow h & & \downarrow h \\ \Omega(f) & \xrightarrow{f} & \Omega(f) \end{array}$$

LEMMA 3.1. *If $f \in Z(M)$ is topologically Ω-stable, then it has the weak inverse shadowing property with respect to the class $\mathcal{T}_h$.*

Proof. For any $\varepsilon > 0$, let $W = B_{d_0}(f, \delta)$ be the set in the definition of topological Ω-stability.

For any $g \in B_{d_0}(f, \delta)$, there exists a semi-conjugacy h between $g|_{\Omega(g)}$ and $f|_{\Omega(f)}$ in the definition of topologically Ω-stability which satisfies Definition 3.1. (a) and (b).

For any point $x \in M$, $\omega(x, f) \subset \Omega(f)$. Take a point $z \in \omega(x, f)$. Let $y \in \Omega(g)$ with $h(y) = z$. Because h is surjective, so y can be obtained.

Then we have

$$O(y, g) \subset N(\varepsilon, O(z, f)) \subset N(\varepsilon, \omega(x, f)) \subset N(\varepsilon, O(x, f)).$$

This means that f has the weak inverse shadowing property with respect to the class $\mathcal{T}_h$, and so completes the proof. □

THEOREM 3.2. ([9]) *If f is a C^1 diffeomorphism satisfying Axiom A and the no-cycle condition, then f is topologically Ω-stable.*

THEOREM 3.3. *The C^1 interior of $WISP_h(M)$ in Diff(M) coincides with the set of Ω-stable diffeomorphisms on M.*

Proof. First we show that every element f in the C^1 interior of $WISP_h(M)$ in Diff(M), $WISP_h(M)^0$, belongs to the set $\mathcal{F}(M)$. This is clear by the [2, Theorem 3.1.].

Next we show that every Ω-stable diffeomorphism has the weak inverse shadowing property with respect to the class $\mathcal{T}_h$. This follows from Lemma 3.1. and Theorem 3.2. □

Remark 2. It was proved in [13] that every element in the C^1 interior of the set of diffeomorphisms on a C^∞ closed surface having the weak shadowing property is Ω-stable, but the converse does not hold in general. We can see that the above result does not generalize to higher dimensions. In fact, it was proved in [8] that there is a C^1 open set $\mathcal{U}$ of the set of diffeomorphisms on the 3-torus such that every $f \in \mathcal{U}$ is topologically transitive and it is not Anosov. This means that every $f \in \mathcal{U}$ is not Ω-stable but it has the weak shadowing property.

Remark 3. Let's also mention a diffeomorphism f of the two-dimensional torus T^2 studied in [10]. The nonwandering set $\Omega(f)$ consists of 4 hyperbolic fixed points,

$$\Omega(f) = \{p_1, p_2, p_3, p_4\},$$

where p_1 is a sink, p_4 is a source, and p_2, p_3 are saddles such that

$$W^s(p_2) \cup \{p_3\} \ = \ W^u(p_3) \cup \{p_2\}$$

(i.e., f has the so-called saddle connection). It is assumed that the eigenvalues of $Df(p_2)$ are $-\mu, \nu$ with $\mu > 1, 0 < \nu < 1$, and the eigenvalues of $Df(p_3)$ are $-\lambda, \kappa$ with $\kappa > 1, 0 < \lambda < 1$ (in addition, it is assumed that f satisfies some local linearity conditions). Plamenevskaya shows that f has the weak shadowing property if and only if the value

$\log(\lambda)/\log(\mu)$ is irrational. But It is easily checked that f is Ω-stable. So f has the weak inverse shadowing property with respect to the class $\mathcal{T}_h$.

Remark 4. Choi, Lee and Zhang in [2] introduced the notion of weak inverse shadowing which is a "dual" notion of weak shadowing . They proved that the C^1 interior of the set of diffeomorphisms with the weak inverse shadowing property with respect to the class $\mathcal{T}_d$ coincides with the set of Ω-stable diffeomorphisms. This theorem improves the results in [2] which say that the C^1 interior of the set of diffeomorphisms with the weak inverse shadowing property with respect to the class $\mathcal{T}_h$ coincides with the set of Ω-stable diffeomorphisms.

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