JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume 17, No. 2, October 2004

SEMI–COMPATIBILITY AND FIXED POINTS OF EXPANSION MAPPINGS IN 2–METRIC SPACES

Bijendra Singh* and Shobha Jain**

Abstract. This paper introduces the notion of semi-compatible self-maps in 2-metric spaces and establishes a fixed point theorem for four self-maps, satisfying an implicit relation through semi-compatibility of a pair of self-maps. This results in another fixed point theorem for four expansion maps which generalizes and improves many results of Kang et. al. [5] with an application.

1. Introduction

The concept of 2-metric space was initially given by Gahler [3] whose abstract properties were suggested by the area of function in Euclidean space. Iseki [4] set out the tradition of proving fixed point theorem in 2-metric spaces employing various contractive conditions. Later on, Naidu and Prasad [7] introduced the concept of weak commutativity while Murthy et. al. [6] introduced the concept of compatible maps in 2-metric spaces. In [2], Cho, Sharma and Sahu introduced the concept of semi-compatibility maps in d-topological spaces. They defined a pair of self-maps (S,T) to be semi-compatible if the conditions (i) $Sy = Ty$ implies $STy = TSy$ (ii) $\{Sx_n\} \to x$, $\{Tx_n\} \to x$ implies $STx_n \to Tx$, as $n \to \infty$, hold. However, (ii) implies (i), taking $x_n = y$ and $x = Ty = Sy$. So in a 2-metric space, we define semi-compatibility by the condition (ii) only.

Received by the editors on June 11, 2004.
2000 *Mathematics Subject Classifications*: Primary 54H25, 47H10.
Key words and phrases: 2-metric space, fixed point, semi-compatible map, weak-compatible map.

The study of common fixed points of expansion type mappings has been an active field of research activity during two decades. In [5], Kang et. al. have established some remarkable results for two surjective self-maps satisfying an expansive condition for fixed point theory in 2-metric spaces.

In the beginning we prove some properties of semi-compatible maps in 2-metric spaces and establish a fixed point theorem for four self-maps, two of which are semi-compatible and the remaining two are weak-compatible. For this, we employ a new class F_4 of functions from $(\mathbb{R}^+)^4$ to $\mathbb{R}$ satisfying some conditions.

As an application a linear characterization of $F \in F_4$ has to be established. It has been applied to prove another fixed point theorem for four expansive self-maps. This theorem turns out to be a generalization and improvement of many results of [5] to four self-maps. At the same time, a condition of F_4 results in the uniqueness of the fixed point unlike [5].

2. Preliminaries

Let X be a non-empty set with real-valued function d on $X \times X \times X$ satisfying the followings:

(1) $d(x, y, z) = 0$ if at least two of x, y, z are equal,

(2) $d(x, y, z) = d(p(x, y, z))$ for all $x, y, x \in X$ and each permutation $p(x, y, z)$ of x, y, z,

(3) $d(x, y, z) \le d(x, y, w) + d(x, w, z) + d(w, y, z)$ for all $x, y, z, w \in X$.

The function d is called a *2-metric* on X and the pair (X, d) is called a *2-metric space*.

A sequence $\{x_n\}$ is said to be *2-convergent* to a point $x \in X$ if $\lim_{n\to\infty} d(x_n, x, a) = 0$, and is said to be *2-Cauchy* sequence if $\lim_{n,m\to\infty} d(x_n, x_m, a) = 0$ for all $a \in X$. The 2-metric space (X, d)

is called *complete* if every Cauchy sequence in X converges to a point of X.

For a pair of self-maps (S, T) on a 2-metric space (X, d):

(I) (S, T) is said to be *compatible* if $\lim_{n\to\infty} d(STx_n, TSx_n, a) = 0$ for all $a \in X$, whenever the sequence $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = x$.

(II) (S, T) is said to be *semi-compatible* if $\lim_{n\to\infty} d(STx_n, Tx, a) = 0$ for all $a \in X$, whenever the sequence $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = x$.

(III) (S, T) is said to be *weak-compatible* or *coincidently commuting* if $Sy = Ty$ for some $y \in X$ then $TSy = STy$.

Proposition 2.1. *If S and T are semi-compatible self-maps on a 2-metric space (X, d) then the pair (S, T) is weak-compatible.*

Proof. Since (S, T) is semi-compatible, $\lim_{n\to\infty} d(STx_n, Tx, a) = 0$ for all $a \in X$, whenever the sequence $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = x$. Take $x_n = y$ and $x = Ty = Sy$. Since (S, T) is semi-compatible, $STy = Tx = TSy$, i.e., $STy = TSy$. □

However, weak-compatibility does not imply semi-compatibility. It is clear from Example 2.1 that the pair of self-maps (I, S) is weak-compatible but it is not semi-compatible.

Proposition 2.2. *If S and T are compatible self-maps on a 2-metric space (X, d) and T is continuous then the pair (S, T) is semi-compatible.*

Proof. Let $\{Sx_n\} \to x$ and let $\{Tx_n\} \to x$. Since T is continuous, $TSx_n \to Tx$. Since the pair (S, T) is compatible, we get $\lim_{n\to\infty} d(STx_n, TSx_n, a) = 0$, i.e., $\lim_{n\to\infty} d(STx_n, Tx, a) = 0$. So (S, T) is semi-compatible. □

Here we give an example of pair of self-maps (I, S) on a 2-metric space, which is compatible but not semi-compatible. Further, we see that the semi-compatibility of a pair (S, I) need not imply the semi-compatibility of (I, S).

Example 2.1. Let $X = \{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots, \frac{1}{n}, \cdots\}$. Define $d : X \times X \times X \to (0, \infty)$ by $d(x, y, z) = 0$ if x, y, z are distinct and $\{\frac{1}{n}, \frac{1}{n+1}\} \subset \{x, y, z\}$, $\quad = 1$ otherwise. Then (X, d) is a 2-metric space as shown in [7]. Let I be the identity on X and define a self-map S as follows: $S(\frac{1}{n}) = \frac{1}{n+2}, S(0) = 1$ and $x_n = \frac{1}{n}$. Then

$$\lim_{n\to\infty} d(Ix_n, 0, a) = \lim_{n\to\infty} d(x_n, 0, a) = 0,$$
$$\lim_{n\to\infty} d(Sx_n, 0, a) = \lim_{n\to\infty} d(\frac{1}{n+2}, 0, a) = 0$$

for all $a \in X$. Thus $\{x_n\}$ and $\{Sx_n\}$ converge to $x = 0$. Now, the pair (I, S) is commuting. Hence it is compatible. But $\{ISx_n\} = \{Sx_n\} \to 0 \neq S(0)$ as $\{Sx_n\} \to 0$, and we get that (I, S) is not semi-compatible. Also for any sequence $\{x_n\} \to x$

$$\lim_{n\to\infty} d(SIx_n, Ix, a) = \lim_{n\to\infty} d(Sx_n, x, a) = 0.$$

Thus (S, I) is semi-compatible.

The above example gives an important aspect of semi-compatibility since the pair (I, S) is commuting hence it is weakly commuting, compatible and weak-compatible but it is not semi-compatible.

Definition 2.1. ([8]) Let F_4 be the class of upper semi-continuous functions on the right from $(\mathbb{R}^+)^4 \to \mathbb{R}$ such that for some $h \in (0, 1)$

(1) $F(u, v, u, v) \geq 0$ implies $v \leq hu$.
(2) $F(u, v, v, u) \geq 0$ implies $v \leq hu$.
(3) $F(u, u, 0, 0) \geq 0$ implies $u = 0$.

REMARK 2.1. It follows from the first two conditions of F_4 that for $F \in F_4$,

(1) $F(0, L, 0, L) \geq 0$ implies $L = 0$.

(2) $F(0, L, L, 0) \geq 0$ implies $L = 0$.

S. L. Singh [9] proved the following.

LEMMA 2.3. ([9]) *Let $\{x_n\}$ be a sequence in a complete 2-metric space X. If there exists a $h \in (0, 1)$ such that*

$$d(x_n, x_{n+1}, a) \leq h \cdot d(x_{n-1}, x_n, a)$$

for all $a \in X$ and all n, then $\{x_n\}$ converges to a point in X.

Before proving the main result we need the following lemma.

LEMMA 2.4. *Let A, B, S and T be four self-maps of a complete 2-metric space (X, d) such that*

(i) *$A(X) \subset T(X)$ and $B(X) \subset S(X)$,*

(ii) *for some $F \in F_4$,*

$$F[d(Sx, Ty, z), d(Ax, By, z), d(Ax, Sx, z), d(By, Ty, z)] \geq 0$$

for all $x, y, z \in X$.

Then the sequence $\{y_n\}$ converges to a point in X, where the sequences $\{x_n\}$ and $\{y_n\}$ are defined by $Ax_{2n} = Tx_{2n+1} = y_{2n+1}$ and $Bx_{2n+1} = Sx_{2n+2} = y_{2n+2}$ for $n = 0, 1, 2, \cdots$.

Proof. For some $x_0 \in X$, define sequences $\{x_n\}$ and $\{y_n\}$ in X as follows: $Ax_{2n} = Tx_{2n+1} = y_{2n+1}$ and $Bx_{2n+1} = Sx_{2n+2} = y_{2n+2}$ for $n = 0, 1, 2, \cdots$.

Put $x = x_{2n}$ and $y = x_{2n+1}$ in (ii), we get

$$F[d(Sx_{2n}, Tx_{2n+1}, z), d(Ax_{2n}, Bx_{2n+1}, z), d(Ax_{2n}, Sx_{2n}, z), d(Bx_{2n+1}, Tx_{2n+1}, z)] \geq 0$$

implies that

$$F[d(y_{2n}, y_{2n+1}, z), d(y_{2n+1}, y_{2n+2}, z), d(y_{2n}, y_{2n+1}, z), \\ d(y_{2n+1}, y_{2n+2}, z)] \geq 0$$

for all $z \in X$. That is, $F[U, V, U, V] \geq 0$ implies that $V \leq h \cdot U$, where $U = d(y_{2n}, y_{2n+1}, z), V = d(y_{2n+1}, y_{2n+2}, z)$, and $h \in (0, 1)$. So

$$d(y_{2n+1}, y_{2n+2}, z) \leq h \cdot d(y_{2n}, y_{2n+1}, z)$$

for $h \in (0, 1)$.

Similarly, if we take $x = x_{2n}$ and $y = x_{2n-1}$ in (ii), then we get

$$F[d(Sx_{2n}, Tx_{2n-1}, z), d(Ax_{2n}, Bx_{2n-1}, z), d(Ax_{2n}, Sx_{2n}, z), \\ d(Bx_{2n-1}, Tx_{2n-1}, z)] \geq 0$$

implies that

$$F[d(y_{2n}, y_{2n-1}, z), d(y_{2n+1}, y_{2n}, z), d(y_{2n+1}, y_{2n}, z), \\ d(y_{2n}, y_{2n+1}, z)] \geq 0$$

for all $z \in X$. That is, $F[U, V, V, U] \geq 0$ implies that $V \leq h \cdot U$, where $U = d(y_{2n}, y_{2n-1}, z), V = d(y_{2n+1}, y_{2n}, z)$, and $h \in (0, 1)$. So

$$d(y_{2n}, y_{2n+1}, z) \leq h \cdot d(y_{2n-1}, y_{2n}, z)$$

for $h \in (0, 1)$. Therefore, for all n even or odd $d(y_n, y_{n+1}, z) \leq h \cdot d(y_{n-1}, y_n, z)$ for $h \in (0, 1)$. By Lemma 2.3, $\{y_n\}$ converges to some $u \in X$. □

It has been shown in [3] that, although d is a continuous function of any of its three augments, it need not to be continuous in two augments. If it is continuous in two augments then it is continuous in all three augments. For brevity, a d, which is continuous in all of its augments, will be called *continuous*.

From now on, the 2-metric d is assumed to be continuous.

3. Main results

THEOREM 3.1. *Let A, B, S and T be four self-maps of a complete 2-metric space (X, d) satisfying* (i), (ii) *and*

(iii) (S, A) *is semi-compatible and* (T, B) *is weak-compatible,*

(iv) A *is continuous.*

Then the four self-maps A, B, S and T have a unique common fixed point.

Proof. Let $x_0 \in X$. From the condition (i) we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that $Ax_{2n} = Tx_{2n+1} = y_{2n+1}$ and $Bx_{2n+1} = Sx_{2n+2} = y_{2n+2}$ for all n. By Lemma 2.4, $\{y_n\} \to u \in X$. Also its subsequences converge to u. That is,

$$\{Ax_{2n}\} \to u \qquad \& \qquad \{Sx_{2n}\} \to u, \tag{1}$$

$$\{Bx_{2n+1}\} tou \qquad \& \qquad \{Tx_{2n+1}\} tou. \tag{2}$$

Since (S, A) is semi-compatible, $SAx_{2n} \to Au$. Since A is continuous, $A^2x_{2n} \to Au$.

Step 1: Put $x = Ax_{2n}$ and $y = x_{2n+1}$ in (ii), we get

$$F[d(SAx_{2n}, Tx_{2n+1}, z), d(A^2x_{2n}, Bx_{2n+1}, z), d(A^2x_{2n}, SAx_{2n}, z), d(Bx_{2n+1}, Tx_{2n+1}, z)] \geq 0$$

for all $z \in X$. Letting $n \to \infty$, we get

$$F[d(Au, u, z), d(Au, u, z), d(Au, Au, z), d(u, u, z)] \geq 0$$

for all $z \in X$. That is,

$$F[d(Au, u, z), d(Au, u, z), 0, 0] \geq 0$$

for all $z \in X$, which gives $d(Au, u, z) = 0$. Hence $Au = u$. Since $A(X) \subset T(X)$, there exists a $v \in X$ such that $u = Au = Tv$.

Step 2: Put $x = x_{2n}$ and $y = v$ in (ii), we get

$$F[d(Sx_{2n}, Tv, z), d(Ax_{2n}, Bv, z), d(Ax_{2n}, Sx_{2n}, z), \\ d(Bv, Tv, z)] \geq 0$$

for all $z \in X$. Letting $n \to \infty$ and using $u = Tv$, we get

$$F[d(u, u, z), d(u, Bv, z), d(u, u, z), d(u, Bv, z)] \geq 0$$

for all $z \in X$. That is,

$$F[0, d(Bv, u, z), 0, d(u, Bv, z)] \geq 0$$

for all $z \in X$, which gives $d(u, Bv, z) = 0$ for all $z \in X$. So $u = Bv$. Hence $u = Bv = Tv$. Since (T, B) is weak-compatible, we have $Bu = Tu$.

Step 3: Put $x = x_{2n}$ and $y = u$ in (ii), we get

$$F[d(Sx_{2n}, Tu, z), d(Ax_{2n}, Bu, z), d(Ax_{2n}, Sx_{2n}, z), d(Bu, Tu, z)] \geq 0$$

for all $z \in X$. Letting $n \to \infty$ and using $Tu = Bu$, we get

$$F[d(u, Bu, z), d(u, Bu, z), d(u, u, z), d(Bu, Bu, z)] \geq 0$$

for all $z \in X$. Let $U = d(u, Bu, z)$. Then $F[U, U, 0, 0] \geq 0$ implies $U = 0$. That is, $d(u, Bu, z) = 0$ for all $z \in X$, which gives $u = Bu$. Hence $u = Bu = Tu$. Since $B(X) \subset S(X)$, there exists a $w \in X$ such that $u = Bu = Sw$. Therefore, $u = Bu = Tu = Sw$.

Step 4: Put $x = w$ and $y = u$ in (ii), we get

$$F[d(Sw, Tu, z), d(Aw, Bu, z), d(Aw, Sw, z), d(Bu, Tu, z)] \geq 0$$

foe all $z \in X$.

$$F[d(u, u, z), d(Aw, u, z), d(Aw, u, z), d(u, u, z)] \geq 0$$

implies that

$$F[0, d(Aw, u, z), d(Aw, u, z), 0] \geq 0$$

for all $z \in X$. Let $U = d(Aw, u, z)$. Then $F[0, U, U, 0] \geq 0$, which gives $U = 0$, i.e., $d(Aw, u, z) = 0$ for all $z \in X$. So $u = Aw$. Hence $u = Aw = Sw$. Since (S, A) is semi-compatible, it is weak-compatible. So we have $Au = Su$. Therefore, $u = Au = Su$. Hence $u = Au = Su = Bu = Tu$.

Step 5 (Uniqueness): Let u_1 be another common fixed point of A, B, S and T. Then $u_1 = Au_1 = Bu_1 = Su_1 = Tu_1$. Put $x = u$ and $y = u_1$ in (ii), we get

$$F[d(u, u_1, z), d(u, u_1, z), d(u, u, z), d(u_1, u_1, z)] \geq 0,$$
$$F[d(u, u_1, z), d(u, u_1, z), 0, 0] \geq 0$$

for all $z \in X$. By the same reasoning as given in Step 3, we have $d(u, u_1, z) = 0$ for all $z \in X$, which gives $u = u_1$. Hence u is a unique common fixed point A, B, S and T. □

PROPOSITION 3.2. *Let F be a function from $(\mathbb{R}^+)^4 \to \mathbb{R}$ such that $F(t_1, t_2, t_3, t_4) = t_1 - at_2 - bt_3 - ct_4$, where $a, b, c \in \mathbb{R}^+$ with $b < 1, c < 1$ and $a > 1$. Then $F \in F_4$.*

Proof. For $u, v \in \mathbb{R}^+$, $F(u, v, u, v) \geq 0$ implies that $v \leq h_1 u$, where $h_1 = \frac{1-b}{a+c} < 1$, as $b < 1$ and $a + b + c > 1$. Again, $F(u, v, v, u) \geq 0$ implies that $v \leq h_2 u$, where $h_2 = \frac{1-c}{a+b} < 1$, as $c < 1$. $F(u, u, 0, 0) \geq 0$ implies that $(1 - a)u \geq 0$, which implies that $(a - 1)u \leq 0$, $\quad (a > 1)$. So $u \leq 0$, which gives $u = 0$. Now, take $h = \max\{h_1, h_2\}$. Hence $F \in F_4$. □

In what follows we proceed to arrive at a generalization of Kang et. al. [5].

COROLLARY 3.3. *Let A, B, S and T be four self-maps of a complete 2-metric space (X, d) satisfying* (i), (iii), (iv) *and*

(v) *for some $a, b, c \in \mathbb{R}^+, a > 1, b < 1, c < 1$,*

$$d(Sx, Ty, z) \geq a \cdot d(Ax, By, z) + b \cdot d(Ax, Sx, z) + c \cdot d(By, Ty, z)$$

for all $x, y, z \in X$, or

(vi) *$d(Sx, Ty, z) \geq ad(Ax, By, z)$ for all $x, y, z \in X$ and some $a > 1$.*

Then the four self-maps A, B, S and T have a unique common fixed point.

In [5], Kang , Chang and Ryu proved the following two results:

THEOREM 3.4. ([5]) *Let S and T be surjective mappings from a complete 2-metric space (X, d) into itself. Suppose that there exist non-negative real numbers $a < 1, b < 1, c$ $(a + b + c > 1)$ such that*

$$d(Sx, Ty, z) \geq a \cdot d(x, Sx, z) + b \cdot d(x, Ty, z) + c \cdot d(x, y, z)$$

for all $x, y, z \in X$. Then S and T have a unique common fixed point.

COROLLARY 3.5. ([5]) *Let S and T be surjective mappings from a complete 2-metric space (X, d) into itself. Suppose that there exists a non-negative real number $c > 1$ such that*

$$d(Sx, Ty, z) \geq c \cdot d(x, y, z)$$

for all $x, y, z \in X$. Then S and T have a unique common fixed point.

Note that in Theorem 3.4 [5] the uniqueness was not proved there and it is not even true.

Corollary 3.5 generalizes the results from two self-maps to four self-maps and it also improves the first result in establishing the uniqueness too.

THEOREM 3.6. *Let A, S and T be three self-maps of a complete 2-metric space (X, d) satisfying* (iv) *and*

(vii) $A(X) \subset T(X) \cap S(X)$,

(viii) *for some $F \in F_4$,*

$$F[d(Sx, Ty, z), d(Ax, Ay, z), d(Ax, Sx, z), d(Ay, Ty, z)] \geq 0$$

for all $x, y, z \in X$.

(ix) *(S, A) is semi-compatible and the pair (T, A) is weak-compatible.*

Then A, S and T have a unique common fixed point.

Proof. The result follows from Theorem 3.1, by taking $B = A$. □

COROLLARY 3.7. *Let A, S and T be three self-maps of a complete 2-metric space (X, d) satisfying* (iv), (vii), (ix) *and*

(x) *for some $a, b, c \in \mathbb{R}^+$, $a > 1, b < 1, c < 1$,*

$$d(Sx, Ty, z) \geq a \cdot d(Ax, Ay, z) + b \cdot d(Ax, Sx, z) + c \cdot d(Ay, Ty, z)$$

or (xi) $d(Sx, Ty, z) \geq a \cdot d(Ax, Ay, z)$ *for all $x, y, z \in X$ and some $a > 1$.*

Then A, S and T have a unique common fixed point.

COROLLARY 3.8. *Let A and S be two self-maps of a complete 2-metric space (X, d). If A is continuous and*

(1) $A(X) \subset S(X)$,

(2) *for some $F \in F_4$,*

$$F[d(Sx, Sy, z), d(Ax, Ay, z), d(Ax, Sx, z), d(Ay, Sy, z)] \geq 0$$

for all $x, y, z \in X$.

(3) *(S, A) is semi-compatible.*

Then A and S have a unique common fixed point.

Proof. The result follows from Theorem 3.6, by taking $T = S$. □

Corollary 3.9. *Let S and T be two surjective self-maps of a complete 2-metric space (X, d) such that*

(xii) for some $a, b, c \in \mathbb{R}^+, a > 1$ and $b, c \in [0, 1)$,

$$d(Sx, Ty, z) \geq a \cdot d(x, y, z) + b \cdot d(x, Sx, z) + c \cdot d(y, Ty, z)$$

or (xiii) $d(Sx, Ty, z) \geq a \cdot d(x, y, z)$ for all $x, y, z \in X$ and some $a > 1$.

Then S and T have a unique common fixed point.

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School of Studies in Mathematics
Vikram University
Ujjain-456010(M. P.), India

**
M. B. Khalsa College
Rajmohalla, Indore (M. P.)
India
E-mail: shobajain1@yahoo.com