JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume 17, No. 2, October 2004

COMMON FIXED POINTS OF WEAK–COMPATIBLE MAPS ON D–METRIC SPACE

Bijendra Singh* and Shobha Jain**

Abstract. In [4], Dhage proved a result for common fixed point of two self-maps satisfying a contractive condition in D-metric spaces. This note proves a fixed point theorem for five self-maps under weak-compatibility in D-metric space which improves and generalizes the above mentioned result.

1. Introduction

Dhage [2] introduced D-metric space and proved the existence of unique fixed point of a self-map satisfying a contractive condition. Rhoades [7] generalized Dhage's contractive condition by increasing the number of factors and proved the existence of unique fixed point of a self-map in D-metric space. Recently, Dhage [4] extended Rhoades' above contractive condition to two maps. At the same time, Dhage [3] proved the existence of a unique common fixed point of a pair of self-maps under weak-compatibility.

Sessa [8] initiated the tradition of improving the commutative condition in fixed point theory by introducing the notion of *weakly commuting* mappings. Jungck [6] introduced a more general concept known as compatible mapping in metric spaces. Recently, Cho [1] proved the existence of a unique common fixed point of five self-maps using α-compatibility.

Received by the editors on April 04, 2004.
2000 *Mathematics Subject Classifications*: Primary 54H25, 47H10.
Key words and phrases: D-metric space, common fixed point, orbit, bounded orbit, complete orbit, weak-compatible.

In this paper, we prove the existence of unique common fixed point of five self-maps in a D-metric space under weak-compatibility using the contractive condition of Dhage [4]. This result generalizes and improves the result of Dhage [4] by increasing the number of self-maps from two to five and restricting the domain and that of boundedness and completeness to some orbits only.

2. Preliminaries

Definition 2.1. Let X be a non-empty set. A generalized metric (or D-metric) on X is a function from $X \times X \times X$ to $\mathbb{R}^+$ (the set of non-negative real numbers) satisfying the followings:

(1) $D(x, y, z) = 0$ if and only if $x = y = z$,
(2) $D(x, y, z) = D(p(x, y, z))$ for all $x, y, x \in X$ and each permutation $p(x, y, z)$ of x, y, z,
(3) $D(x, y, z) \leq D(x, y, w) + D(x, w, z) + D(w, y, z)$ for all $x, y, z, w \in X$.

The pair (X, D) is called a D-metric space.

Definition 2.2. A sequence $\{x_n\}$ of points in a D-metric space (X, D) is said to be *D-convergent to a point $x \in X$ if for each $\epsilon > 0$, there exists an $n_0 \in \mathbb{N}$ such that $D(x_m, x_n, x) < \epsilon$ for all $m, n > n_0$. The sequence $\{x_n\}$ is said to be D-Cauchy if for each $\epsilon > 0$, there exists an $n_0 \in \mathbb{N}$ such that $D(x_m, x_n, x_p) < \epsilon$ for all $m, n, p > n_0$. A D-metric space (X, D) is said to be complete if every Cauchy sequence converges to some point of X.*

Definition 2.3. *Let T be a multivalued map on D-metric space (X, D). Let $x_0 \in X$ be arbitrary. A sequence $\{x_n\}$ in X is said to be an orbit of T at x_0 denoted by $O(T, x_0)$ if $x_{n-1} \in T^{n-1}(x_0)$ for all $n \in \mathbb{N}$. If T is a single-valued self-map on X then for $x_0 \in X$, let $x_1 = Tx_0$, $x_2 = Tx_1 = T^2x_0, \cdots, x_{n-1} = T^{n-1}x_0$. Then the sequence $\{x_n\}$ is called the orbit of T at the point x_0 and is denoted* *by* $O(T, x_0)$.

DEFINITION 2.4. *Let* (X, D) *be a D-metric space. Let* S *and* T *be two set valued mappings from* X *into* 2^X. *For* $x_0 \in X$, *let* $x_1 \in Sx_0$, $x_2 \in Tx_1 \subset TSx_0, x_3 \in Sx_2 \subset STSx_0$, *etc. Then the sequence* $\{x_0, x_1, x_2, \cdots\} = \{x_n\}$ *is called an orbit of* S *and* T *at the point* x_0 *and is denoted by* $O(S, T, x_0)$. *An orbit* $O(S, T, x_0)$ *is said to be complete if every Cauchy sequence converges to an element of* X. *An orbit* $O(S, T, x_0)$ *is said to be bounded if there exists* $M > 0$ *such that* $D(u, v, w) leM$ *for all* $u, v, w \in O(S, T, x_0)$ *and the number* M *is said to be a bound of the orbit* $O(S, T, x_0)$.

DEFINITION 2.5. *A pair* (S, T) *of self-maps on a D-metric space* (X, D) *is said to be weak-compatible if* $Sy = Ty$ *for some* $y \in X$ *implies* $TSy = STy$.

We require the followings to prove the main result.

PROPOSITION 2.1. *Let* P, Q *and* R *be three self-maps on a D-metric space* (X, D) *such that* $P(X) \subset R(X)$ *and* $Q(X) \subset R(X)$. *For some* $x_0 \in X$, *define sequences* $\{x_n\}$ *and* $\{y_n\}$ *in* X *as follows:* $Px_{2n} = Rx_{2n+1} = y_{2n+1}$ and $Qx_{2n+1} = Rx_{2n+2} = y_{2n+2}$ *for* $n = 0, 1, 2, \cdots$. *Then*

$$\{x_0, x_1, x_2, \cdots\} = \{x_n\} = O(R^{-1}P, R^{-1}Q, x_0)$$
$$\{y_1, y_2, \cdots\} = \{y_n\} = O(QR^{-1}, PR^{-1}, Px_0).$$

Proof. We have

$$y_1 = Px_0 = Rx_1 \quad \& \quad y_2 = Qx_1 = Rx_2$$
$$y_3 = Px_2 = Rx_3 \quad \& \quad y_4 = Qx_3 = Rx_4$$
$$\vdots \quad \& \quad \vdots$$

Therefore, $x_1 \in R^{-1}Px_0$, $x_2 \in R^{-1}Qx_1 \subset (R^{-1}Q)(R^{-1}P)x_0$, and $x_3 \in R^{-1}Px_2 \subset (R^{-1}P)(R^{-1}Q)(R^{-1}P)x_0$. Hence $\{x_0, x_1, x_2, \cdots\} =$

$\{x_n\} = O(R^{-1}P, R^{-1}Q, x_0)$. Also $y_1 = Px_0, y_2 = Qx_1Q(R^{-1}P)x_0 = (QR^{-1})Px_0, y_3 = Px_2(PR^{-1})(QR^{-1})Px_0$. Hence $\{y_1, y_2, y_3, \cdots\} = \{y_n\} = O(QR^{-1}, PR^{-1}, Px_0)$. □

Note that if $AB(X) \subset R(X)$ and $ST(X) \subset R(X)$ then from Proposition 2.1,

$$\{x_n\} = O(R^{-1}AB, R^{-1}ST, x_0),$$
$$\{y_n\} = O(STR^{-1}, ABR^{-1}, ABx_0).$$

LEMMA 2.2. (Dhage [5]) *Let $\{x_n\} \subset X$ be bounded with D-bound M satisfying*

$$D(x_n, x_{n+1}, x_m) \leq f^n(M)$$

for all $n \in \mathbb{N}$ and all $m > n+1$, where $f : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying $\sum_{n=1}^{\infty} f^n(t) < \infty$ for each $t \in \mathbb{R}^+$. Then $\{x_n\}$ is a D-Cauchy sequence.

Let Φ denote the set of functions $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying (1) ϕ is non-decreasing, (2) $\phi(t) < t$ for $t > 0$, and (3) $\sum_{n=1}^{\infty} \phi^n(t) < \infty$.

LEMMA 2.3. *Let P, Q and R be three self-maps of a D-metric space (X, D) satisfying* (i) *$P(X) \subset R(X)$ and $Q(X) \subset R(X)$,* (ii) *for some $x_0 \in X$ an orbit $O(QR^{-1}, PR^{-1}, Px_0)$ is bounded, and* (iii) *for some $\phi \in \Phi$, some $\alpha \in [0, \frac{1}{3}]$ and $x, y, z \in O(R^{-1}P, R^{-1}Q, x_0)$*

$$\begin{aligned} D(Px, Qy, Rz) \leq \phi \max\{&D(Rx, Ry, Rz), D(Px, Rx, Rz), \\ &D(Qy, Ry, Rz), \alpha D(Px, Ry, Rz), \alpha D(Rx, Qy, Rz)\}. \end{aligned}$$

Then $\{y_n\}$ is a D-Cauchy sequence.

Proof. Let $x_0 \in X$. As seen above we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that $Px_{2n} = Rx_{2n+1} = y_{2n+1}$ and $Qx_{2n-1} = Rx_{2n} = y_{2n}$ for all n. Thus the sequence $\{y_n\} \subset O(QR^{-1}, PR^{-1}, Px_0)$ which is bounded. Let M be a bound of it then $D(x, y, z) \leq M$ for all $x, y, z \in O(QR^{-1}, PR^{-1}, Px_0)$. We show, by induction on n, that for $m \geq n+1$

$$D(y_n, y_{n+1}, y_m) \leq \phi^n(M) \tag{2.1}$$

for all n. Putting $x = x_0, y = x_1$ and $z = x_m$ in the condition (iii), we get

$$D(y_1, y_2, y_m) \leq \phi \max\{D(y_0, y_1, y_m), D(y_0, y_1, y_m), D(y_1, y_2, y_m), \alpha D(y_1, y_1, y_m), \alpha D(y_0, y_2, y_m)\} \leq \phi(M)$$

since each factor is less than M. Hence (2.1) holds for $n = 1$. Again assuming that (2.1) holds for $n \leq p(= 2k$, say), we will show that (2.1) holds for $n = p+1(= 2k+1)$. Putting $x = x_{2k}, y = x_{2k+1}$ and $z = x_m$ in the condition (iii), we get

$$\begin{aligned} D(y_{2k+1}, y_{2k+2}, y_m) \leq{}& \phi \max\{D(y_{2k}, y_{2k+1}, y_m), D(y_{2k}, y_{2k+1}, y_m), \\ & D(y_{2k+1}, y_{2k+2}, y_m, \alpha D(y_{2k+1}, y_{2k+1}, y_m, \alpha D(y_{2k}, y_{2k+2}, y_m)\} \\ ={}& \phi \max\{D(y_{2k}, y_{2k+1}, y_m), \alpha D(y_{2k+1}, y_{2k+1}, y_m), \\ & \alpha D(y_{2k}, y_{2k+2}, y_m)\}, \end{aligned}$$

since $D(y_{2k+1}, y_{2k+2}, y_m)$ can not be maximum as for $t > 0$ it will give $t < t$ which is a contradiction. If $t = 0$, then $\{y_n\}$ is a constant sequence form $n = 2k$ onwards.

Case 1:

$$\begin{aligned} D(y_{2k+1}, y_{2k+2}, y_m) &\leq \phi\{D(y_{2k}, y_{2k+1}, y_m)\} \\ &\leq \phi\phi^{2k}(M) = \phi^{2k+1}(M). \end{aligned}$$

Case 2:

$$\begin{aligned} D(y_{2k+1}, y_{2k+2}, y_m) &\leq \phi\{\alpha D(y_{2k+1}, y_{2k+1}, y_m)\} \\ &\leq \phi(\alpha\{D(y_{2k+1}, y_{2k+1}, y_{2k}) + D(y_{2k+1}, y_{2k}, y_m) \\ &+ D(y_{2k}, y_{2k+1}, y_m)\}) \\ &\leq \phi(3\alpha\phi^{2k}(M)) \leq \phi^{2k+1}(M). \end{aligned}$$

Case 3:

$$\begin{aligned}D(y_{2k+1}, y_{2k+2}, y_m) &\leq \phi(\alpha D(y_{2k}, y_{2k+2}, y_m))\\&\leq \phi(\alpha\{D(y_{2k}, y_{2k+2}, y_{2k+1}) + D(y_{2k}, y_{2k+1}, y_m)\\&+ D(y_{2k+1}, y_{2k+2}, y_m)\})\\&= \phi(\alpha\{D(y_{2k}, y_{2k+1}, y_{2k+2}) + D(y_{2k}, y_{2k+1}, y_m)\\&+ D(y_{2k+1}, y_{2k+2}, y_m)\})\\&\leq \phi(2\alpha\phi^{2k}(M) + \alpha D(y_{2k+1}, y_{2k+2}, y_m))\\&< 2\alpha\phi^{2k+1}(M) + \alpha D(y_{2k+1}, y_{2k+2}, y_m)\end{aligned}$$

implies $(1-\alpha)D(y_{2k+1}, y_{2k+2}, y_m) \leq 2\alpha\phi^{2k+1}(M)$. Since $\alpha \leq \frac{1}{3}$, $\frac{2\alpha}{1-\alpha} \leq 1$. So we get $D(y_{2k+1}, y_{2k+2}, y_m) \leq \phi^{2k+1}(M)$, which is (2.1) for $n = 2k+1$. Hence the equation (2.1) holds for $n = 2k+1$, if it is true for $n \leq 2k$. Similarly, we can show that the equation (2.1) holds for $n = 2k+2$, if it is true for $n \leq 2k+1$. Therefore, by induction on n, the equation (2.1) holds for all n. By Lemma 2.2, $\{y_n\}$ is a D-Cauchy sequence in $O(QR^{-1}, PR^{-1}, Px_0)$. □

Remark 2.1. If we substitute $P = AB$ and $Q = ST$, then (iii) becomes

$$\begin{aligned}D(ABx, STy, Rz) \leq \phi \max\{&D(Rx, Ry, Rz), D(ABx, Rx, Rz),\\&D(STy, Ry, Rz), \alpha D(ABx, Ry, Rz), \alpha D(Rx, STy, Rz)\}\end{aligned}$$

for some $\phi \in \Phi$, some $\alpha \in [0, \frac{1}{3}], x, y, z \in O(R^{-1}AB, R^{-1}ST, x_0)$, and $\{y_n\} = O(STR^{-1}, ABR^{-1}, ABx_0)$ is a D-Cauchy sequence.

Remark 2.2. If for some $x, y, z \in X, D(x, y, z) \leq \phi(D(x, y, z))$, then $x = y = z$. If $D(x, y, z) > 0$, then

$$D(x, y, z) \leq \phi(D(x, y, z)) < D(x, y, z),$$

i.e., $D(x, y, z) < D(x, y, z)$, which is a contradiction.

THEOREM 2.1. ([4, Theorem 2.1] Let $A, B : X \to X$ be self-maps and let X be orbitally bounded and orbitally complete D metric space and suppose that

$$D(Ax, By, z) \leq \phi \max\{D(x, y, z), D(x, Ax, z), D(y, By, z), \\ \alpha D(x, By, z), \alpha D(Ax, y, z)\}$$

for all $x, y \in X, z \in \overline{O(A, B, x) \cup O(B, A, y)}$, some $\phi \in \Phi$ and $\alpha \in [0, \frac{1}{3}]$. Then A and B have a unique common fixed point in X.

3. Main results

The following theorem generalizes this for five self-maps. It further improves it by restricting the domain and of boundedness and completeness to some orbits only.

THEOREM 3.1. *Let A, B, S, T and R be five self-maps of a D-metric space (X, D) satisfying:*

(3.i) $AB(X) \subset R(X)$ *and* $ST(X) \subset R(X)$.

(3.ii) *For some* $x_0 \in X$, *the orbit* $O(STR^{-1}, ABR^{-1}, ABx_0)$ *is bounded and complete.*

(3.iii) *For some* $\phi \in \Phi$, $z \in O(R^{-1}AB, R^{-1}ST, x_0)$, *some* $\alpha[0, \frac{1}{3}]$, *and all* $x, y \in X$

$$D(ABx, STy, Rz) \leq \phi \max\{D(Rx, Ry, Rz), D(ABx, Rx, Rz), \\ D(STy, Ry, Rz), \alpha D(ABx, Ry, z), \alpha D(Rx, STy, Rz)\}.$$

(3.iv) The pairs (AB, R) and $T(ST, R)$ are weak compatible.

(3.v) $AB = BA, RB = BR, ST = TS, RT = TR$.

Then A, B, S, T and R have a unique common fixed point.

Proof. For some $x_0 \in X$, construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that $ABx_{2n} = Rx_{2n+1} = y_{2n+1}, STx_{2n+1} = Rx_{2n+2} = y_{2n+2}$ for all $n = 0, 1, 2, \cdot$. Substituting $P = AB$ and $Q = ST$ in Lemma 2.3, then by Remark 2.2, $\{y_n\}$ is D-Cauchy in $O(STR^{-1}, ABR^{-1}, ABx_0)$, which is complete. Therefore, $\{y_n\}$ converges to some $u \in X$. Also its subsequences $\{ABx_{2n}\}, \{STx_{2n+1}\}, \{Rx_{2n}\}$ and $\{Rx_{2n+1}\}$ also converge to u. Hence

$$\{ABx_{2n}\} \to u \quad \& \quad \{STx_{2n+1}\} \to u \tag{3.1}$$

$$\{Rx_{2n}\} \to u \quad \& \quad \{Rx_{2n+1}\} \to u. \tag{3.2}$$

Step 1: Put $x = x_{2n}$ and $y = x_{2n+1}$ in the condition (3.iii), we get

$$\begin{aligned} D(ABx_{2n}, STx_{2n+1}, Rz) \leq \phi \max\{&D(Rx_{2n}, Rx_{2n+1}, Rz), \\ &D(ABx_{2n}, Rx_{2n}, Rz), D(STx_{2n+1}, Rx_{2n+1}, Rz), \\ &\alpha D(ABx_{2n}, Rx_{2n+1}, Rz), \alpha D(Rx_{2n}, STx_{2n+1}, Rz)\}. \end{aligned}$$

Letting $n \to \infty$ and using the equations (3.1) and (3.2), we get

$$\begin{aligned} D(u, u, Rz) \leq \phi \max\{&D(u, u, Rz), D(u, u, Rz), D(u, u, Rz), \\ &\alpha D(u, u, Rz), \alpha D(u, u, Rz)\}. \end{aligned}$$

Therefore, by Remark 2.2, $D(u, u, Rz) = 0$. Hence $u = Rz$ for all $z \in O(R^{-1}AB, R^{-1}ST, x_0)$. So

$$u = Rx_0. \tag{3.3}$$

Step 2: Put $x = x_{2n}, y = x_0$ and $z = x_0$ in the condition (3.iii), we get

$$\begin{aligned} D(ABx_{2n}, STx_0, Rx_0) \leq \phi \max\{&D(Rx_{2n}, Rx_0, Rx_0), \\ &D(ABx_{2n}, Rx_{2n}, Rx_0), D(STx_0, Rx_0, Rx_0), \\ &\alpha D(ABx_{2n}, Rx_0, Rx_0), \alpha D(Rx_{2n}, STx_0, Rx_0)\}. \end{aligned}$$

Letting $n \to \infty$ and using the equations (3.1), (3.2) and (3.3), we get

$$D(u, STx_0, u) \le \phi(D(u, STx_0, u)).$$

Therefore, by Remark 2.2, $D(u, STx_0, u) = 0$, which gives $STx_0 = u$. Hence $Rx_0 = STx_0 = u$. Since (ST, R) is weak-compatible, we get

$$STu = Ru \tag{3.4}$$

Step 3: Put $x = x_0, y = x_{2n+1}$ and $z = x_0$ in the condition (3.iii), we get

$$\begin{aligned} D(ABx_0, STx_{2n+1}, Rx_0) \le \phi \max\{&D(Rx_0, Rx_{2n+1}, Rx_0), \\ &D(ABx_0, Rx_0, Rx_0), D(STx_{2n+1}, Rx_{2n+1}, Rx_0), \\ &\alpha D(ABx_0, Rx_{2n+1}, Rx_0), \alpha D(Rx_0, STx_{2n+1}, Rx_0)\}. \end{aligned}$$

Letting $n \to \infty$ and using the equations (3.1), (3.2) and (3.3), we get $D(ABx_0, u, u) \le \phi(D(ABx_0, u, u))$. Therefore, by Remark 2.2, $D(ABx_0, u, u) = 0$, which gives $ABx_0 = u$. Hence $Rx_0 = ABx_0 = u$. Since (AB, R) is weak-compatible, we get $Au = Ru$. Therefore, by the equation (3.4)) we have

$$ABu = Ru = STu. \tag{3.5}$$

Step 4: Put $x = u, y = u$ and $z = x_0$ in the condition (3.iii), we get

$$\begin{aligned} D(ABu, STu, Rx_0) \le \phi \max\{&D(Ru, Ru, Rx_0), D(ABu, Ru, Rx_0), \\ &D(STu, Ru, Rx_0), \\ &\alpha D(ABu, Ru, Rx_0), \alpha D(Ru, STu, Rx_0)\}. \end{aligned}$$

Using the equation (3.5), we get

$$D(ABu, ABu, u) \le \phi(D(ABu, ABu, u)).$$

Therefore, by Remark 2.2, $D(ABu, ABu, u) = 0$, which gives $ABu = u$. Therefore, $Ru = STu = ABu = u$, i.e., u is a common fixed point of R, ST and AB.

Step 5: Put $x = x_{2n}, y = Tu$ and $z = x_0$ in the condition (3.iii), we get

$$\begin{aligned} D(ABx_{2n}, ST(Tu), Rx_0) \leq \phi \max\{&D(Rx_{2n}, R(Tu), Rx_0), \\ &D(ABx_{2n}, Rx_{2n}, Rx_0), D(STx0, R(Tu), Rx_0), \\ &\alpha D(ABx_{2n}, R(Tu), Rx_0), \alpha D(Rx_{2n}, ST(Tu), Rx_0)\}. \end{aligned}$$

Since $ST = TS$ and $RT = TR$, we have

$$\begin{aligned} D(ABx_{2n}, T(STu), Rx_0) \leq \phi \max\{&D(Rx_{2n}, T(Ru), Rx_0), \\ &D(ABx_{2n}, Rx_{2n}, Rx_0), D(STx_0, T(Ru), Rx_0), \\ &\alpha D(ABx_{2n}, R(Tu), Rx_0), \alpha D(Rx_{2n}, T(STu), Rx_0)\}. \end{aligned}$$

Letting $n \to \infty$ and using the equations (3.1), (3.2) and (3.3), we get $D(u, Tu, u) \leq \phi(D(u, Tu, u))$, which gives $Tu = u$. Now $STu = u$ gives $Su = u$. Hence

$$Su = Tu = Ru = u = ABu = STu. \tag{3.6}$$

Step 6: Put $x = Bu, y = x_{2n+1}$ and $z = x_0$ in the condition (3.iii), we get

$$\begin{aligned} D(AB(Bu), STx_{2n+1}, Rx_0) \leq \phi \max\{&D(R(Bu), Rx_{2n+1}, Rx_0), \\ &D(AB(Bu), Rx_{2n+1}, Rx_0), D(STx_{2n+1}, R(Bu), Rx_0), \\ &\alpha D(AB(Bu), Rx_{2n+1}, Rx_0), \alpha D(R(Bu), STx_{2n+1}, Rx_0)\}. \end{aligned}$$

Since $AB = BA$ and $RB = BR$, we have

$$\begin{aligned} D(B(ABu), STx_{2n+1}, Rx_0) \leq \phi \max\{&D(B(Ru), Rx_{2n+1}, Rx_0), \\ &D(B(ABu), Rx_{2n+1}, Rx_0), D(STx_{2n+1}, B(Ru), Rx_0), \\ &\alpha D(B(ABu), Rx_{2n+1}, Rx_0), \alpha D(B(Ru), STx_{2n+1}, Rx_0)\}. \end{aligned}$$

Letting $n \to \infty$ and using the equations (3.1), (3.2) and (3.6), we get $D(Bu, u, u) \leq \phi(D(Bu, u, u))$, which gives $Bu = u$. Now $ABu = u$ gives $Au = u$. Hence $Au = Bu = Su = Tu = Ru = u$.

Step 7: (Uniqueness) Let u_1 be another common fixed point of A, B, S, T and R. Then $Ru_1 = Au_1 = Bu_1 = Su_1 = Tu_1 = u_1$. Put $x = u_1, y = u_1$ and $z = x_0$ in the condition (3.iii), we get $D(u_1, u_1, u) \leq \phi D(u_1, u_1, u)$, which gives $u_1 = u$.

Therefore, u is a unique common fixed point of A, B, S and T. $\square$

COROLLARY 3.2. *Let A, B, S and T be four self-maps of a D-metric space (X, D) satisfying:*

(1) *For some $x_0 \in X$, the orbit $\{y_n\} = O(ST, AB, x_0)$ is bounded and complete.*

(2) *For $\phi \in \Phi$, some $\alpha \in [0, \frac{1}{3}]$, all $x, y \in X$ and $z \in O(AB, ST, x_0)$*

$$D(ABx, STy, z) \leq \phi \max\{D(x, y, z), D(ABx, x, z), D(STy, y, z), \alpha D(ABx, y, z), \alpha D(x, STy, z)\}.$$

(3) *$AB = BA$ and $ST = TS$.*

Then A, B, S and T have a unique common fixed point.

Proof. The result follows from Theorem 3.1 by taking $R = I$. $\square$

COROLLARY 3.3. *Let A and B be two self-maps of a D-metric space (X, D) satisfying:*

(1) *For some $x_0 \in X$ and some positive integers a and s, the orbit $\{y_n\} = O(S^s, A^a, x_0)$ is bounded and complete.*

(2) *For $\phi \in \Phi$, some $\alpha \in [0, \frac{1}{3}]$, all $x, y \in X$ and $z \in O(A^a, S^s, x_0)$*

$$D(A^a x, S^s y, z) \leq \phi \max\{D(x, y, z), D(A^a x, x, z), D(S^s y, y, z), \alpha D(A^a x, y, z), \alpha D(x, S^s y, z)\}.$$

Then A and S have a unique common fixed point.

Proof. In Corollary 3.2, if we take $B = A^{a-1}$ and $T = S^{s-1}$, then A^{a-1}, A, S^{s-1} and S have a unique common fixed point, i.e., A and S have a unique common fixed point. □

COROLLARY 3.4. *Let A, S and R be three self-maps of a D-metric space (X, D) satisfying:*

(1) *$A(X) \subset R(X)$ and $S(X) \subset R(X)$.*

(2) *The pairs (A, R) and (S, R) are weak-compatible.*

(3) *For some $x_0 \in X$, the orbit $\{y_n\} = O(SR^{-1}, AR^{-1}, Ax_0)$ is bounded and complete.*

(4) *For some $\phi \in \Phi$, some $\alpha \in [0, \frac{1}{3}]$, $z \in O(R^{-1}A, R^{-1}S, x_0)$, and all $x, y \in X$*

$$D(Ax, Sy, Rz) \leq \phi \max\{D(Rx, Ry, Rz), D(Ax, Rx, Rz), D(Sy, Ry, Rz), \alpha D(Ax, Ry, z), \alpha D(Rx, Sy, Rz)\}.$$

Then A, S and R have a unique common fixed point.

Proof. The result follows from Corollary 3.2 by taking $B = T = I$. □

COROLLARY 3.5. *Let A and R be two self-maps of a D-metric space (X, D) satisfying:*

(1) *$A(X) \in R(X)$.*

(2) *The pair (A, R) is weak-compatible.*

(3) *For some $x_0 \in X$, the orbit $\{y_n\} = O(AR^{-1}, Ax_0)$ is bounded and complete.*

(4) *For $\phi \in \Phi$, some $\alpha \in [0, \frac{1}{3}]$, all $x, y \in X$ and $z \in O(R^{-1}A, x_0)$*

$$D(Ax, Ay, Rz) \leq \phi \max\{D(Rx, Ry, Rz), D(Ax, Rx, Rz), D(Ay, Ry, Rz), \alpha D(Ax, Ry, z), \alpha D(Rx, Ay, Rz)\}.$$

Then A and R have a unique common fixed point.

Proof. The result follows from Corollary 3.4 by taking $S = A = I$. □

Corollary 3.6. *Let A and S be two self-maps of a D-metric space (X, D) such that for some $x_0 \in X$ the orbit $O(A, S, x_0)$ is bounded and complete, and for some $\phi \in \Phi$, some $\alpha \in [0, \frac{1}{3}]$, all $x, y \in X$ and $z \in O(A, S, x_0)$,*

$$D(Ax, Sy, z) \leq \phi \max\{D(x, y, z), D(Ax, x, z), D(Sy, y, z), \alpha D(Ax, y, z), \alpha D(x, Sy, z)\}.$$

Then A and S have a unique common fixed point.

Proof. The result follows from Corollary 3.4 by taking $R = I$, and the orbit $O(SR^{-1}, AR^{-1}, Ax_0)$ becomes $O(S, A, Ax_0)$ which is contained in $O(A, S, x_0)$. □

Remark 3.1. The above corollary improves Theorem 2.1 of Dhage [5] in respect of restricting the domain and of completeness and boundedness. Thus Theorem 3.1 is a generalization of the result of Dhage [5] from two self-maps to four self-maps.

Corollary 3.7. *Let A be a self-map of a D-metric space (X, D) such that for some $x_0 \in X$, the orbit $O(A, x_0)$ is bounded and complete, and for some $\phi \in \Phi$, all $x, y \in X$ and all $z \in O(A, x_0)$,*

$$D(Ax, Ay, z) \leq \phi \max\{D(x, y, z), D(Ax, x, z), D(Ay, y, z), \alpha D(x, Ay, z), \alpha D(Ax, y, z)\}.$$

Then A has a unique fixed point.

Proof. The result follows from Corollary 3.6 by taking $S = A$ and the orbit $O(S, A, x_0) = O(A, x_0)$. □

References

1. Y.J. Cho, *Fixed point in fuzzy metric space*, J. Fuzzy Math. **5** (1997), 949–962.
2. B.C. Dhage, *Generalized metric spaces and mapping with fixed points*, Bull. Cal. Math. Soc. **84** (1992), 329–336.
3. ______, *A common fixed point principle in D-metric spaces*, Bull. Cal. Math. Soc. **91** (1999), 475–480.
4. ______, *Some results on common fixed point I*, Indian J. Pure Appl. Math. **30** (1999), 827–837.
5. B.C. Dhage, A.M. Pathan and B.E. Rhoades, *A general existence principal for fixed point theorem in D-metric space*, Internat. J. Math. Math. Sci. **23** (2000), 441–448.
6. G. Jungck, *Compatible mappings and common fixed point*, Internat. J. Math. Math. Sci. **9** (1986), 771–779.
7. B.E. Rhoades, *A fixed point theorem for generalized metric space*, Internat. J. Math. Math. Sci. **19** (1996), 457–460.
8. S. Sessa, *On a weak commutative condition in fixed point consideration*, Publ. Inst. Math (Beogard) **32** (1982), 146–153.
9. T. Veerapandi and C. Rao, *Fixed point theorem of some multivalued mapping in D-metric space*, Bull. Cal. Math. Soc. **87** (1995), 549–556.

*
School of Studies in Mathematics
Vikram University
Ujjain-456010(M. P.), India

**
M. B. Khalsa College
Rajmohalla, Indore (M. P.)
India

E-mail: shobajain1@yahoo.com