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ON THE HYERS-ULAM-RASSIAS STABILITY OF A MODIFIED ADDITIVE AND QUADRATIC FUNCTIONAL EQUATION

KIL-WOUNG JUN, HARK-MANN KIM, AND DON O LEE

ABSTRACT. In this paper, we solve the general solution of a modified additive and quadratic functional equation $f(x+3y)+3f(x-y)=f(x-3y)+3f(x+y)$ in the class of functions between real vector spaces and obtain the Hyers-Ulam-Rassias stability problem for the equation in the sense of Găvruţa.

1. INTRODUCTION

In 1940, Ulam [17] raised a question concerning the stability of group homomorphism:

> Let G_1 be a group and let G_2 be a metric group with the metric $d(\cdot,\cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a function $f : G_1 \to G_2$ satisfies the inequality $d(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(f(x), H(x)) < \varepsilon$ for all $x \in G_1$?

In other words, we are looking for situations when the homomorphisms are stable, *i. e.*, if a mapping is almost a homomorphism, then there exists a true homomorphism rear it.

It is easy to see that the quadratic function $f(x) = cx^2$ on real field is a solution of the following equation:

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{1.1}$$

So, it is national that the equation (1.1) is called a *quadratic functional equation.* In particular, every solution of the quadratic equation (1.1) is said to be a *quadratic function.* It is well known that a function f between real vector spaces is quadratic if

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and only if there exists a unique symmetric biadditive function B such that $f(x) = B(x,x)$ for all x (see Aczél & Dhombres [1], Kannappan [14]). The biadditive function B is given by

$$B(x,y) = \frac{1}{4}(f(x+y) - f(x-y)). \tag{1.2}$$

The case of approximately additive functions was solved by Hyers [7] and generalized by Rassias [15]. During the last decades, the stability problems of several functional equations have been extensively investigated by a number of authors Baker [2], Hyers, Isac & Rassias [8, 9], Hyers & Rassias [10], Jun & Kim [12], Rassias [16]. A stability problem for the quadratic functional equation (1.1) was solved by a lot of authors Czerwik [4], Grabiec [6], Jung [13]. Further, Jun & Lee [11] proved the generalized Hyers-Ulam stability of the pexiderized quadratic equation (1.1).

Now, we investigate the following new additive and quadratic functional equation,

$$f(x+3y) + 3f(x-y) = f(x-3y) + 3f(x+y). \tag{1.3}$$

In this paper, we obtain the general solution of equation (1.3) in the class of functions between real or complex vector spaces and we establish the Hyers-Ulam-Rassias stability problem for the equation (1.3) in the sense of Găvruţa.

2. General Solution of (1.3)

We here present the general solution of the functional equation (1.3).

Theorem 2.1. *Let X and Y be real vector spaces. A function $f : X \to Y$ satisfies the functional equation (1.3) if and only if there exist functions $Q : X \to Y$, $A : X \to Y$ and a constant c in Y such that $f(x) = Q(x) + A(x) + c$ for all $x \in X$, where Q is quadratic, and A is additive.*

Proof. We first assume that f is a solution of the functional equation(1.3).

Let $A(x) := \frac{1}{2}[f(x) - f(-x)]$ and $Q(x) := \frac{1}{2}[f(x) + f(-x)] - f(0)$. Then it follows that $A(0) = 0$, $A(-x) = -A(x)$ and $Q(0) = 0$, $Q(-x) = Q(x)$. Since f satisfies the functional equation (1.3), we get that

$$A(x+3y) + 3A(x-y) = A(x-3y) + 3A(x+y) \tag{2.1}$$

$$Q(x+3y) + 3Q(x-y) = Q(x-3y) + 3Q(x+y) \tag{2.2}$$

for all $x, y \in X$.

Now we prove that A is additive.

Putting $y = x$ and $x = 0$ in (2.1), separately, we have $A(2x) = 2A(x)$ and $A(3y) = 3A(y)$.

Interchanging x and y in (2.1) and rewriting the resulting equation, we get

$$A(3x+y) + A(3x-y) = A(3x+3y) + A(3x-3y) \tag{2.3}$$

for all $x, y \in X$.

From (2.3), we deduce

$$A(u) + A(v) = A(2u-v) + A(2v-u) \tag{2.4}$$

for all $u, v \in X$.

Replacing y by $-2x+y$ in (2.3) and x by $x-2y$ in (2.1), we get

$$A(x+y) + A(5x-y) = 3A(-x+y) + 3A(3x-y) \tag{2.5}$$

$$A(x+y) + 3A(x-3y) = A(x-5y) + 3A(x-y) \tag{2.6}$$

for all $x, y \in X$. Using (2.5) and (2.6), one obtains

$$A(5x-y) + A(x-5y) = -6A(x-y) + 3A(3x-y) + 3A(x-3y) \tag{2.7}$$

for all $x, y \in X$. Applying the relation (2.4) to the left hand side of (2.7), we have

$$A(3x+y) - A(x+3y) = -2A(x-y) + A(3x-y) + A(x-3y). \tag{2.8}$$

Replacing y by $-y$ in (2.8), we have

$$A(3x-y) - A(x-3y) = -2A(x+y) + A(3x+y) + A(x+3y). \tag{2.9}$$

From (2.8) and (2.9), we arrive at

$$A(3x+y) - A(3x-y) = A(x+y) - A(x-y). \tag{2.10}$$

Adding (2.10) to (2.3), we get

$$A(3x+y) + A(-x+y) = A(2x+2y). \tag{2.11}$$

Letting $\alpha = 3x+y$ and $\beta = -x+y$ in (2.11), then we see

$$A(\alpha) + A(\beta) = A(\alpha+\beta).$$

Therefore A is an additive.

Next, we show that Q is quadratic. By putting $y = x$ and $y = \frac{x}{3}$ in (2.2), respectively, we see that $Q(2x) = 4Q(x)$ and $Q(3x) = 9Q(x)$.

Interchanging x and y in (2.2), we get

$$Q(3x+y) + 3Q(x-y) = Q(3x-y) + 3Q(x+y) \tag{2.12}$$

for all $x, y \in X$.

Replacing x by $2x + y$ in (2.2) and y by $x + 2y$ in (2.12), respectively, we get

$$Q(x+2y) + 3Q(x) = Q(x-y) + 3Q(x+y) \tag{2.13}$$

$$Q(2x+y) + 3Q(y) = Q(x-y) + 3Q(x+y) \tag{2.14}$$

for all $x, y \in X$.

Therefore, we have the following crucial equation from (2.13) and (2.14)

$$Q(2x+y) - Q(x+2y) = 3Q(x) - 3Q(y) \tag{2.15}$$

for all $x, y \in X$.

Now utilizing (2.15) one obtains the following two relations

$$Q(x+y) - Q(x-\frac{y}{2}) = \frac{1}{3}Q\left(3(x+\frac{y}{2})\right) - \frac{1}{3}Q(3x),$$
$$Q(x-y) - Q(x+\frac{y}{2}) = \frac{1}{3}Q\left(3(x-\frac{y}{2})\right) - \frac{1}{3}Q(3x).$$

Since $Q(2x) = 4Q(x), Q(3x) = 9Q(x)$ for all $x \in X$, adding the above two relations we get

$$Q(x+y) + Q(x-y) + 6Q(x) = Q(2x+y) + Q(2x-y), \tag{2.16}$$

which is equivalent to the original quadratic functional equation $Q(x+y)+Q(x-y) = 2Qx) + 2Q(y)$ Chang & Kim [3]. Therefore, Q is quadratic.

That is, if $f : X \to Y$ satisfies the functional equation (1.3), then $f(x) = Q(x) + A(x) + f(0)$ for all $x \in X$, where Q is quadratic and A is additive.

Conversely, if there exist functions $Q : X \to Y$, $A : X \to Y$ and a constant c in Y such that $f(x) = Q(x) + A(x) + c$ for all $x \in X$, where Q is quadratic and A is additive, then it is obvious that f satisfies the equation (1.3). □

3. Stability of (1.3)

Throughout this section X and Y will be a real normed linear space and a real Banach space, respectively. Given $f : X \to Y$, we set

$$Df(x,y) := f(x+3y) + 3f(x-y) - f(x-3y) - 3f(x+y)$$

for all $x, y \in X$.

Let $\varphi : X \times X \to [0,\infty)$ be a mapping satisfying one of the conditions $(\mathcal{A})$, $(\mathcal{B})$ and one of the conditions $(\mathcal{C})$, $(\mathcal{D})$:

$$(\mathcal{A}) \qquad \Phi_1(x,y) := \sum_{k=0}^{\infty} \frac{1}{4^{k-1}} \varphi(2^{k-1}x, 2^{k-1}y) < \infty$$

$$(\mathcal{B}) \qquad \Phi_2(x,y) := \sum_{k=1}^{\infty} 4^{k+1} \varphi(\frac{x}{2^{k+1}}, \frac{y}{2^{k+1}}) < \infty$$

$$(\mathcal{C}) \qquad \Psi_1(x,y) := \sum_{k=0}^{\infty} \frac{1}{2^{k-1}} \varphi(2^{k-1}x, 2^{k-1}y) < \infty$$

$$(\mathcal{D}) \qquad \Psi_2(x,y) := \sum_{k=1}^{\infty} 2^{k+1} \varphi(\frac{x}{2^{k+1}}, \frac{y}{2^{k+1}}) < \infty$$

for all $x, y \in X$.

One of the conditions $(\mathcal{A})$, $(\mathcal{B})$ will be needed to derive a quadratic function and one of the conditions $(\mathcal{C})$, $(\mathcal{D})$ will be needed to derive an additive function in the following theorem.

Theorem 3.1. *If a function $f : X \to Y$ satisfies*

$$||Df(x,y))|| \leq \varphi(x,y) \tag{3.1}$$

for all $x, y \in X$. Then there exist a unique quadratic function $Q : X \to Y$ and a unique additive function $A : X \to Y$ satisfying the equation (1.3) *such that*

$$||f(x) - Q(x) - A(x) - f(0)|| \leq \frac{1}{32}\Big[\Phi_i(x,x) + \Phi_i(-x,-x)\Big] + \frac{1}{8}\Big[\Psi_j(x,x) + \Psi_j(-x,-x)\Big],$$

$$\left\|\frac{f(x)+f(-x)}{2} - Q(x) - f(0)\right\| \leq \frac{1}{32}\Big[\Phi_i(x,x) + \Phi_i(-x,-x)\Big],$$

and

$$\left\|\frac{f(x)-f(-x)}{2} - A(x)\right\| \leq \frac{1}{8}\Big[\Psi_j(x,x) + \Psi_j(-x,-x)\Big]$$

for all $x \in X$ and for $i =$ 1 or 2, $j =$ 1 or 2.

The functions Q and A are given by

$$\begin{cases} Q(x) = \lim_{n\to\infty} \frac{f(2^n x)+f(-2^n x)}{2\cdot 4^n} & \text{if } \mathcal{A} \text{ holds,} \\ Q(x) = \lim_{n\to\infty} 4^n \frac{f(\frac{x}{2^n})+f(-\frac{x}{2^n})-2f(0)}{2} & \text{if } \mathcal{B} \text{ holds,} \\ A(x) = \lim_{n\to\infty} \frac{f(2^n x)-f(-2^n x)}{2^{n+1}} & \text{if } \mathcal{C} \text{ holds,} \\ A(x) = \lim_{n\to\infty} 2^n \frac{f(\frac{x}{2^n})-f(-\frac{x}{2^n})}{2} & \text{if } \mathcal{D} \text{ holds} \end{cases}$$

for all $x \in X$.

Proof. Let $f_1 : X \to Y$ be a function defined by $f_1(x) := (1/2)[f(x)+f(-x)] - f(0)$ for all $x \in X$. Then $f_1(0) = 0$, $f_1(x) = f_1(-x)$, and

$$\begin{aligned} ||Df_1(x,y)|| &= ||f_1(x+3y) + 3f_1(x-y) - f_1(x-3y) - 3f_1(x+y)|| \\ &\le (1/2)[\varphi(x,y) + \varphi(-x,-y)] \end{aligned} \tag{3.2}$$

for all $x, y \in X$. Putting $y = x$ in (3.2) yields

$$||f_1(4x) - 4f_1(2x)|| \le (1/2)[\varphi(x,x) + \varphi(-x,-x)] \tag{3.3}$$

for all $x \in X$.

Case 1. Assume that φ satisfies the condition $(\mathcal{A})$. Dividing both sides of (3.3) by 4 and letting $\frac{x}{2}$ for x, we have

$$\left\|\frac{f_1(2x)}{4} - f_1(x)\right\| \le \frac{1}{8}\left[\varphi\left(\frac{x}{2},\frac{x}{2}\right) + \varphi\left(-\frac{x}{2},-\frac{x}{2}\right)\right] \tag{3.4}$$

for all $x \in X$. Replacing x by $2^{n-1}x$ and dividing by 4^{n-1} in (3.4) we obtain

$$\left\|\frac{f_1(2^n x)}{4^n} - \frac{f_1(2^{n-1}x)}{4^{n-1}}\right\| \le \frac{1}{2\cdot 4^n}\left[\varphi(2^{n-2}x, 2^{n-2}x) + \varphi(-2^{n-2}x, -2^{n-2}x)\right] \tag{3.5}$$

for all $x \in X$ and for all $n \in \mathbb{N}$.

An induction argument implies easily that

$$\left\|\frac{f_1(2^n x)}{4^n} - f_1(x)\right\| \le \frac{1}{32}\sum_{i=0}^{n-1} \frac{1}{4^{i-1}}\left[\varphi(2^{i-1}x, 2^{i-1}x) + \varphi(-2^{i-1}x, -2^{i-1}x)\right] \tag{3.6}$$

for all $x \in X$ and for all $n \in \mathbb{N}$. Hence by (3.6) we obtain that

$$\begin{aligned} \left\|\frac{f_1(2^n x)}{4^n} - \frac{f_1(2^m x)}{4^m}\right\| &\le \frac{1}{4^m}\left\|\frac{f_1(2^n x)}{4^{n-m}} - f_1(2^m x)\right\| \\ &\le \frac{1}{32}\sum_{i=m}^{n-1} \frac{1}{4^{i-1}}\left[\varphi(2^{i-1}x, 2^{i-1}x) + \varphi(-2^{i-1}x, -2^{i-1}x)\right] \end{aligned} \tag{3.7}$$

for all $x \in X$ and for all $n, m \in \mathbb{N}$ with $n > m$. Since the right hand side of (3.7) tends to zero as $m \to \infty$, $\{\frac{f_1(2^n x)}{4^n}\}$ is a Cauchy sequence for all $x \in X$ and thus converges by the completeness of Y. Therefore we can define a function $Q : X \to Y$ by

$$Q(x) = \lim_{n\to\infty} \frac{f_1(2^n x)}{4^n}, \quad x \in X.$$

Note that $Q(0) = 0$, $Q(-x) = Q(x)$ for all $x \in X$.

Replacing x, y in (3.2) by $2^n x, 2^n y$ and dividing both sides by 4^n, and after then taking the limit in the resulting inequality, we have

$$Q(x+3y) + 3Q(x-y) - Q(x-3y) - 3Q(x+y) = 0. \tag{3.8}$$

Since Q is even and $Q(2^n x) = 4^n Q(x)$ for all $n \in \mathbb{N}$, the function Q is quadratic as in the proof of Theorem 2.1.

Taking the limit in (3.6) as $n \to \infty$, we obtain

$$\|f_1(x) - Q(x)\| \le \frac{1}{32}\Big[\Phi_1(x,x) + \Phi_1(-x,-x)\Big] \tag{3.9}$$

for all $x \in X$.

To prove the uniqueness, let Q' be another quadratic function satisfying (3.9). Then $Q'(0) = 0$, $Q'(2^n x) = 4^n Q'(x)$, and $Q'(-x) = Q'(x)$ for all $x \in X$. Thus we have

$$\begin{aligned}\|Q(x) - Q'(x)\| &\le \left\|\frac{Q(2^n x)}{4^n} - \frac{f_1(2^n x)}{4^n}\right\| + \left\|\frac{f_1(2^n x)}{4^n} - \frac{Q'(2^n x)}{4^n}\right\| \\ &\le \frac{1}{4^n}\Big\{\|Q(2^n x) - f_1(2^n x)\| + \|f_1(2^n x) - Q'(2^n x)\|\Big\} \\ &\le \frac{1}{16}\frac{\Phi_1(2^n x, 2^n x) + \Phi_1(-2^n x, -2^n x)}{4^n}.\end{aligned}$$

Taking the limit as $n \to \infty$, we conclude that $Q(x) = Q'(x)$ for all $x \in X$.

Case 2. Assume that φ satisfies the condition $(\mathcal{B})$ (and hence $(\mathcal{D})$).

Replacing x by $\frac{x}{4}$ in (3.3) we get

$$\left\|f_1(x) - 4f_1\Big(\frac{x}{2}\Big)\right\| \le \Big(\frac{1}{2}\Big)\Big[\varphi\Big(\frac{x}{4},\frac{x}{4}\Big) + \varphi\Big(-\frac{x}{4},-\frac{x}{4}\Big)\Big] \tag{3.10}$$

for all $x \in X$.

Replacing x by $\frac{x}{2^{n-1}}$ and multiplying by 4^{n-1} in (3.10) we obtain that

$$\left\|4^{n-1}f_1\Big(\frac{x}{2^{n-1}}\Big) - 4^n f_1\Big(\frac{x}{2^n}\Big)\right\| \le \frac{4^{n+1}}{32}\Big[\varphi\Big(\frac{x}{2^{n+1}},\frac{x}{2^{n+1}}\Big) + \varphi\Big(-\frac{x}{2^{n+1}},-\frac{x}{2^{n+1}}\Big)\Big] \tag{3.11}$$

for all $x \in X$ and for all $n \in \mathbb{N}$. An induction argument implies that

$$\left\|4^n f_1\Big(\frac{x}{2^n}\Big) - f_1(x)\right\| \le \frac{1}{32}\sum_{i=1}^{n} 4^{i+1}\Big[\varphi\Big(\frac{x}{2^{i+1}},\frac{x}{2^{i+1}}\Big) + \varphi\Big(-\frac{x}{2^{i+1}},-\frac{x}{2^{i+1}}\Big)\Big] \tag{3.12}$$

for all $x \in X$ and for all $n \in \mathbb{N}$.

Hence

$$\left\|4^n f_1\left(\frac{x}{2^n}\right) - 4^m f_1\left(\frac{x}{2^m}\right)\right\| \leq \frac{1}{32} \sum_{i=m+1}^{n} 4^{i+1}\left[\varphi\left(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}\right) + \varphi\left(-\frac{x}{2^{i+1}}, -\frac{x}{2^{i+1}}\right)\right] \tag{3.13}$$

for all $x \in X$ and for all $n, m \in \mathbb{N}$ with $n > m$.

This shows that $\{4^n f_1(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$ and thus converges. Therefore we can define a function $Q : X \to Y$ by

$$Q(x) = \lim_{n\to\infty} 4^n f_1(\frac{x}{2^n}).$$

Note that $Q(0) = 0$, $Q(-x) = Q(x)$ for all $x \in X$. By (3.2) we have

$$Q(x+3y) + 3Q(x-y) - Q(x-3y) - 3Q(x+y) = 0 \tag{3.14}$$

for all $x, y \in X$ and thus Q is quadratic.

Taking the limit in (3.12) as $n \to \infty$, we obtain

$$\|f_1(x) - Q(x)\| \leq \frac{1}{32}\left[\Phi_2(x,x) + \Phi_2(-x,-x)\right] \tag{3.15}$$

for all $x \in X$.

Using the similar argument to that of Case 1, we easily have the uniqueness of Q satisfying (3.15).

Now let $f_2 : X \to Y$ be a function defined by $f_2(x) := (1/2)[f(x) - f(-x)]$ for all $x \in X$. Then $f_2(0) = 0$, $f_2(-x) = -f_2(x)$, and the relation (3.1) can be written by

$$\begin{aligned} \|Df_2(x,y)\| &= \|f_2(x+3y) + f_2(x-y) - f_2(x-3y) - 3f_2(x+y)\| \\ &\leq \left(\frac{1}{2}\right)\left[\varphi(x,y) + \varphi(-x,-y)\right] \end{aligned} \tag{3.16}$$

for all $x, y \in X$. Putting $y = x$ in (3.16) yields

$$\|f_2(2x) - 2f_2(x)\| \leq \left(\frac{1}{2}\right)\left[\varphi\left(\frac{x}{2}, \frac{x}{2}\right) + \varphi\left(-\frac{x}{2}, -\frac{x}{2}\right)\right] \tag{3.17}$$

for all $x \in X$.

Case 3. Assume that φ satisfies the condition $(\mathcal{C})$ (and hence $(\mathcal{A})$).

Dividing the inequality (3.17) by 2 we have

$$\left\|\frac{f_2(2x)}{2} - f_2(x)\right\| \leq \left(\frac{1}{4}\right)\left[\varphi\left(\frac{x}{2}, \frac{x}{2}\right) + \varphi\left(-\frac{x}{2}, -\frac{x}{2}\right)\right] \tag{3.18}$$

for all $x \in X$. Replacing x by $2^{n-1}x$ in (3.18) and dividing by 2^{n-1} we obtain

$$\left\|\frac{f_2(2^n x)}{2^n} - \frac{f_2(2^{n-1}x)}{2^{n-1}}\right\| \leq \frac{1}{2^{n+1}}\left[\varphi(2^{n-2}x, 2^{n-2}x) + \varphi(-2^{n-2}x, -2^{n-2}x)\right] \tag{3.19}$$

for all $x \in X$ and for all $n \in \mathbb{N}$.

It follows by an induction argument that

$$\left\|\frac{f_2(2^n x)}{2^n} - f_2(x)\right\| \leq \frac{1}{8}\sum_{i=0}^{n-1}\left[\frac{1}{2^{i-1}}\varphi(2^{i-1}x, 2^{i-1}x) + \frac{1}{2^{i-1}}\varphi(-2^{i-1}x, -2^{i-1})\right] \quad (3.20)$$

for all $x \in X$ and for all $n \in \mathbb{N}$. Hence

$$\left\|\frac{f_2(2^n x)}{2^n} - \frac{f_2(2^m x)}{2^m}\right\| \leq \frac{1}{8}\sum_{i=m}^{n-1}\left[\frac{1}{2^{i-1}}\varphi(2^{i-1}x, 2^{i-1}x) + \frac{1}{2^{i-1}}\varphi(-2^{i-1}x, 2^{i-1} - x)\right] \quad (3.21)$$

for all $x \in X$ and for all $n, m \in \mathbb{N}$ with $n > m$.

This shows that $\{\frac{f_2(2^n x)}{2^n}\}$ is a Cauchy sequence for all $x \in X$ and thus converges in Y. Therefore we can define a function $A : X \to Y$ by

$$A(x) = \lim_{n\to\infty} \frac{f_2(2^n x)}{2^n}, \quad x \in X.$$

Note that $A(0) = 0$, $A(-x) = -A(x)$ for all $x \in X$.

By (3.16) we get

$$A(x+3y) + 3A(x-y) - A(x-3y) - 3A(x+y) = 0$$

for all $x, y \in X$ and thus A is additive as in the proof of Theorem 2.1.

Taking the limit in (3.20) as $n \to \infty$, we obtain

$$\|f_2(x) - A(x)\| \leq \frac{1}{8}\Big[\Psi_1(x, x) + \Psi_1(-x, -x)\Big] \quad (3.22)$$

for all $x \in X$.

If A' is another additive function satisfying the inequality (3.22), then $A'(0) = 0$, $A'(-x) = A'(x)$ and $A'(2^n x) = 2^n A'(x)$ for all $x \in X$. Thus one obtains that by (3.22)

$$\begin{aligned}\|A(x) - A'(x)\| &\leq \frac{1}{2^n}\Big(\|A(2^n x) - f_2(2^n x)\| + \|f_2(2^n x) - A'(2^n x)\|\Big)\\ &\leq \frac{\Psi_1(2^n x, -2^n x) + \Psi_1(-2^n x, 2^n x)}{2^{n+2}}.\end{aligned}$$

Taking the limit as $n \to \infty$, we can conclude that we obtain $A(x) = A'(x)$ for all $x \in X$.

Case 4. Assume that φ satisfies the condition $(\mathcal{D})$.

$$\Big\|f_2(x) - 2f_2\Big(\frac{x}{2}\Big)\Big\| \le \Big(\frac{1}{2}\Big)\Big[\varphi\big(\frac{x}{4},\frac{x}{4}\big) + \varphi\Big(-\frac{x}{4},-\frac{x}{4}\Big)\Big] \tag{3.23}$$

for all $x \in X$.

Replacing x by $\frac{x}{2^{n-1}}$ in (3.23) and multiplying by 2^{n-1} we obtain

$$\Big\|2^{n-1}f_2\Big(\frac{x}{2^{n-1}}\Big) - 2^n f_2\Big(\frac{x}{2^n}\Big)\Big\| \le 2^{n-2}\Big[\varphi\Big(\frac{x}{2^{n+1}},\frac{x}{2^{n+1}}\Big) + \varphi\Big(-\frac{x}{2^{n+1}},-\frac{x}{2^{n+1}}\Big)\Big] \tag{3.24}$$

for all $x \in X$ and for all $n \in \mathbb{N}$. An induction argument implies that

$$\Big\|f_2(x) - 2^n f_2\Big(\frac{x}{2^n}\Big)\Big\| \le \frac{1}{8}\sum_{i=1}^{n} 2^{i+1}\Big[\varphi\Big(\frac{x}{2^{i+1}},\frac{x}{2^{i+1}}\Big) + \varphi\Big(-\frac{x}{2^i},-\frac{x}{2^{i+1}}\Big)\Big] \tag{3.25}$$

holds for all $x \in X$ and for all $n \in \mathbb{N}$.

Hence

$$\Big\|2^n f_2\Big(\frac{x}{2^n}\Big) - 2^m f_2\Big(\frac{x}{2^m}\Big)\Big\| \le \frac{1}{8}\sum_{i=m+1}^{n} 2^{i+1}\Big[\varphi\Big(\frac{x}{2^{i+1}},\frac{x}{2^{i+1}}\Big) + \varphi\Big(-\frac{x}{2^{i+1}},-\frac{x}{2^{i+1}}\Big)\Big] \tag{3.26}$$

for all $x \in X$ and for all $n, m \in \mathbb{N}$ with $n > m$.

This implies that $\{2^n f_2(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$ and thus converges. Therefore we can define a function $A : X \to Y$ by

$$A(x) = \lim_{n\to\infty} 2^n f_2\Big(\frac{x}{2^n}\Big), \quad x \in X.$$

Note that $A(0) = 0$, $A(-x) = -A(x)$ for all $x \in X$ and thus A is additive.

Taking the limit in (3.25) as $n \to \infty$, we obtain

$$\|f_2(x) - A(x)\| \le \frac{1}{8}\Big[\Psi_2(x,x) + \Psi_2(-x,-x)\Big] \tag{3.27}$$

for all $x \in X$.

Similarly we have easily that A is a unique additive mapping subject to (3.27). We complete the proof. □

From the main theorem 3.1, we obtain the following corollary concerning the stability of the equation (1.3).

Corollary 3.2. *Let $p \neq 1$, $p \neq 2$ and $\varepsilon \ge 0$ be real numbers. Assume that a function $f : X \to Y$ satisfies the inequality*

$$\|Df(x,y))\| \le \varepsilon(\|x\|^p + \|y\|^p) \tag{3.28}$$

for all $x, y \in X$ ($x, y \in X \backslash \{0\}$ if $p < 0$). Then there exist a unique quadratic function $Q : X \to Y$ and a unique additive function $A : X \to Y$ satisfying (1.3) such that

$$\|f(x) - Q(x) - A(x) - f(0)\| \le \frac{\varepsilon\|x\|^p}{2^p}\Big(\frac{1}{|4-2^p|} + \frac{1}{|2-2^p|}\Big),$$
$$\Big\|\frac{f(x)+f(-x)}{2} - Q(x) - f(0)\Big\| \le \frac{2\varepsilon\|x\|^p}{2^p|4-2^p|},$$

and

$$\Big\|\frac{f(x)-f(-x)}{2} - A(x)\Big\| \le \frac{2\varepsilon\|x\|^p}{2^p|2-2^p|}$$

for all $x \in X$ ($x \in X\backslash\{0\}$ if $p < 0$).

Proof. Let $\varphi(x,y) := \varepsilon(\|x\|^p + \|y\|^p)$ for all $x, y \in X$. Then $\varphi(x,x) = 2\varepsilon\|x\|^p$ for all $x \in X$ ($x \in X\backslash\{0\}$ if $p < 0$).

If $p < 2$, we have

$$\sum_{n=0}^{\infty} \frac{\varphi(2^{n-1}x, 2^{n-1}y)}{4^{n-1}} = \sum_{n=0}^{\infty} \frac{2^{p(n-1)}\ \varepsilon(\|x\|^p + \|y\|^p)}{4^{n-1}} = \frac{16\varepsilon(\|x\|^p + \|y\|^p)}{2^p(4-2^p)}$$

for all $x, y \in X$ ($x, y \in X\backslash\{0\}$ if $p < 0$). If $p > 2$, we have

$$\sum_{n=1}^{\infty} 4^{n+1}\varphi(2^{-n-1}x, 2^{-n-1}y) = \sum_{n=1}^{\infty} \frac{4^{n+1}\ \varepsilon(\|x\|^p + \|y\|^p)}{2^{p(n+1)}} = \frac{16\varepsilon(\|x\|^p + \|y\|^p)}{2^p(2^p-4)}$$

for all $x, y \in X$. If $p < 1$, we have

$$\sum_{n=0}^{\infty} \frac{\varphi(2^{n-1}x, 2^{n-1}y)}{2^{n-1}} = \sum_{n=0}^{\infty} \frac{2^{(n-1)p}\varepsilon(\|x\|^p + \|y\|^p)}{2^{n-1}} = \frac{4\varepsilon(\|x\|^p + \|y\|^p)}{2^p(2-2^p)}$$

for all $x, y \in X$ ($x, y \in X\backslash\{0\}$ if $p < 0$). If $p > 1$, we have

$$\sum_{n=1}^{\infty} 2^{n+1}\varphi(2^{-n-1}x, 2^{-n-1}y) = \sum_{n=1}^{\infty} \frac{2^{n+1}\ \varepsilon(\|x\|^p + \|y\|^p)}{2^{(n+1)p}} = \frac{4\varepsilon(\|x\|^p + \|y\|^p)}{2^p(2^p-2)}$$

for all $x, y \in X$. Thus applying Theorem 3.1 for the three cases $p < 1$, $1 < p < 2$ and $2 < p$, we obtain easily the results. □

Corollary 3.3. *Assume that for some $\theta > 0$, a function $f : X \to Y$ satisfies the inequality*

$$\|Df(x,y))\| \le \theta \tag{3.29}$$

for all $x, y \in X$. Then there exist a unique quadratic function $Q : X \to Y$ and a unique additive function $A : X \to Y$ satisfying (1.3) such that

$$\|f(x) - Q(x) - A(x) - f(0)\| \le \frac{4}{3}\theta,$$
$$\left\|\frac{f(x) + f(-x)}{2} - Q(x) - f(0)\right\| \le \frac{1}{3}\theta,$$

and

$$\|\frac{f(x) - f(-x)}{2} - A(x)\| \le \theta$$

for all $x \in X$.

Proof. Putting $\varphi(x, y) := \theta$, we get immediately the result. □

REFERENCES

1. J. Aczél & J. Dhombres: *Functional equations in several variables,* With applications to mathematics, information theory and to the natural and social sciences. Cambridge University Press, Cambridge, 1989. MR **90h**:39001
2. J. Baker: The stability of the cosine equation. *Proc. Amer. Math. Soc.* **80** (1980), no. 3, 411–416. MR **81m**:39015
3. I. S. Chng & H. M. Kim: On the Hyers-Ulam stability of quadratic functional equations. *J. Inequal. Pure Appl. Math.* **3** (2002), no. 3, Article 33, 12 pp. (electronic). MR **2003f**:39092
4. S. Czerwik: On the stability of the quadratic mapping in normed spaces. *Abh. Math. Sem. Univ. Hamburg* **62** (1992), 59–64. MR **94e**:39026
5. P. Găvruta: A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings. *J. Math. Anal. Appl.* **184** (1994), no. 3, 431–436. MR **95e**:47089
6. A. Grabiec: The generalized Hyers-Ulam stability of a class of functional equations. *Publ. Math. Debrecen* **48** (1996), no. 3–4, 217–235. MR **98a**:39027
7. D. H. Hyers: On the stability of the linear functional equation. *Proc. Nat. Acad. Sci. U. S. A.* **27** (1941). 222–224. MR **2**,315a
8. D. H. Hyers, G. Isac & Th. M. Rassias: *Stability of functional equations in several variables.* Birkhauser Boston, Inc., Boston, MA, 1998.
9. ———: On the asymptoticity aspect of Hyers-Ulam stability of mappings. *Proc. Amer. Math. Soc.* **126** (1998), no. 2, 425–430. MR **98d**:39004
10. D. H. Hyers & Th. M. Rassias: Approximate homomorphisms. *Aequationes Math.* **44** (1992), no. 2–3, 125–153. MR **93i**:39007
11. K. W. Jun & Y. H. Lee: On the Hyers-Ulam-Rassias stability of a Pexiderized quadratic inequality. *Math. Inequal. Appl.* **4** (2001), no. 1, 93–118. MR **2001j**:39040

12. K. W. Jun & H. M. Kim: The generalized Hyers-Ulam-Rassias stability of a cubic functional equation. *J. Math. Anal. Appl.* To appear.
13. S. -M. Jung: On the Hyers-Ulam stability of the functional equations that have the quadratic property. *J. Math. Anal. Appl.* **222** (1998), no. 1, 126–137. MR **99e**:39095
14. Pl. Kannappan: Quadratic functional equation and inner product spaces. *Results Math.* **27** (1995), no. 3–4, 368–372. MR **96h**:39011
15. Th. M. Rassias: On the stability of the linear mapping in Banach spaces. *Proc. Amer. Math. Soc.* **72** (1978), no. 2, 297–300. MR **80d**:47094
16. ______: On the stability of functional equations in Banach spaces. *J. Math. Anal. Appl.* **251** (2000), no. 1, 264–284. MR **2003b**:39036
17. S. M. Ulam: *Problems in modern mathematics.* Science Editions John Wiley & Sons, Inc., New York 1964. MR **43**#6031

(K. -W. Jun) Department of Mathematics, Chungnam National University, 220 Gung-Dong, Yuseong-Gu, Daejeon 305-335, Korea
Email address: kwjun@math.cnu.ac.kr

(H. -M. Kim) Department of Mathematics, Chungnam National University, 220 Gung-Dong, Yuseong-Gu, Daejeon 305-335, Korea
Email address: hmkim@math.cnu.ac.kr

(D. O Lee) Information Center for Mathematical Sciences, Korea Advanced Institute of Science and Technology, 373-1 Guseong-Dong, Yuseong-Gu, Daejeon 305-701, Korea
Email address: donolee@icms.kaist.ac.kr