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POINTS AT INFINITY OF COMPLETE OPEN RIEMANNIAN MANIFOLDS

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ABSTRACT. For a complete open Riemannian manifold, the ideal boundary consists of points at infinity. The so-called Busemann-functions play the role of distance functions for points at infinity. We study the similarity and difference between Busemann-functions and ordinary distance functions.

1. INTRODUCTION

In Riemannian geometry, noncompact manifolds have been less popular than compact ones. It is certainly because compact manifolds have bounded geometry and topology, and hence they are easier to handle. When we study a complete open Riemannian manifold M, we sometimes need to consider a compactification of M. If a complete open manifold M is compactified in certain ways, its boundary ∂M is called the ideal boundary. There are several ways to define the ideal boundary of an open manifold, but maybe the most natural definition is by equivalence classes of rays. In the Euclidean case, two straight lines are equivalent if they are parallel to each other. A natural generalization of parallel lines in an open Riemannian manifold should be asymptotic rays, but the asymptotic relation is not an equivalence relation in general. In some earlier works on this subject, in order to avoid this problem, a point at infinity is defined by an equivalence relation artificially constructed from the asymptotic relation. We will discuss these problems in detail.

The concept of ideal boundary has been widely used in the study of non-positively curved manifolds. In this case, the universal covering spaces are Hadamard manifolds, on which asymptotic rays behave almost like parallel lines in the Euclidean space, and the ideal boundary is defined in a natural way. In fact, an inner metric called Tits' metric can be defined on the ideal boundary, and the geometry of ideal

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boundary is closely related to the global structure of manifolds Ballmann, Gromov & Schroeder [1], Gromov [7]. When the ideal boundary of an open manifold is properly defined, the so-called Busemann-function corresponding to a ray plays the role of a distance function from a point at infinity. In many aspects, the property of Busemann-function resembles that of an ordinary distance function, but there are also differences. The concept of Busemann-function is also proven useful in the study of mainfolds without conjugate points. Busemann-functions are only continuous in general, but for manifolds without conjugate points they have some differentiability Eberlein [4], Yim [14]. An interesting and also useful tool for this case is the stable Jacobi tensor, which is a Jacobi tensor along a ray satisfying certain boundary conditions. We will show how this concept is related to points at infinity.

Aside from non-positively curved manifolds, the asymptotic geometry of open manifolds has not been very successful in attracting a lot of attention. It is partly because the asymptotic structure of an open manifold is in general very complicated and there are not many analytic tools one can use. The study of asymptotic rays on open manifolds and related topics was pioneered by Busemann. Some early results on this subject can be found in Busemann [2], Nasu [11, 12], and tools developed in these works are used in Lewis [10] to study the cut locus of a point at infinity for surfaces. In Busemann [3], Innami [9], these concepts are further developed and studied, and in particular the structure of cut locus and the differentiability of Busemann-functions are considered. There are other related topics studied by many authors, and in this paper we will summarize some of these facts in a coherent way and introduce some new concepts with new results. In particular, we define well-known concepts such as cut locus, conjugate locus, and convexity for the points at infinity. They are defined in a similar way as in the case of ordinary interior points and have some similar properties, but there are also differences and we will discuss these problems.

We refer to Busemann [2, 3] for details about asymptotic rays, and to Innami [9], Shiohama [13] for technical facts on Busemann-functions.

2. IDEAL BOUNDARY

In this section, we recall some known facts about the ideal boundary of open manifolds, and study the property of space at infinity.

The ideal boundary or the space at infinity is by definition the set of points at infinity. There are several ways to define this space at infinity, but we first follow the lead of Ballmann, Gromov & Schroeder [1], Gromov [8]. For a Riemannian manifold M let $d(\cdot,\cdot)$ denote the distance function induced by the Riemannian metric, and for each $p \in M$ let d_p be the continuous function $q \mapsto d(p,q)$. The map $p \mapsto d_p$ defines an embedding of M into $C(M)$, the set of continuous functions on M with the topology of uniform convergence on compact sets.

We consider $C_*(M) := C(M)/(\text{constant functions})$ with the induced topology and the induced embedding $i : M \to C_*(M)$ defined by $i(p) = [d_p]$, the equivalence class of d_p. The boundary ∂M is defined as $Cl(M) \setminus i(M)$, where $Cl(M)$ is the closure of $i(M)$ in $C_*(M)$. A point in ∂M is an equivalence class of functions called *horofunctions*, which are well-defined up to constants. This definition of ideal boundary has the benefit of generality that it can be defined for any non-compact metric spaces, but it is rather difficult to study the geometry of manifolds in terms of horofunctions alone.

For a non-compact complete Riemannian manifold M, the ideal boundary can be defined in a more geometric way. We define a point at infinity by a direction, which is an equivalence class of rays. We will assume once and for all that all geodesics on M are parametrized by arclength. A ray $\gamma : [0,\infty) \to M$ is defined as an isometric imbedding of $[0,\infty)$ in M. For a given ray γ, another ray σ is said to be asymptotic to γ if there is a sequence of minimizing geodesics $\{\sigma_j : [0,l_j] \to M\}$, such that $\lim_{j\to\infty} \sigma_j(0) = \sigma(0)$ and $\sigma_j(l_j) = \gamma(t_j)$ for some divergent sequence $\{t_j\}$ and $\sigma'(0) = \lim_{j\to\infty} \sigma_j'(0)$, where $\sigma'(0)$ is the tangent vector to σ at 0. We denote by $\sigma \succ \gamma$ when σ is asymptotic to γ. If a ray σ is a sub-ray of a ray γ, it is denoted by $\sigma \subset \gamma$.

In the Euclidean space, parallel lines are pointing same direction, and hence are said to define a point at infinity. It is therefore quite tempting to say that in an open Riemannian manifold asymptotic rays define a point at infinity. The problem with this argument is that in general the asymptotic relation is not an equivalence relation and some additional considerations are needed. In Lewis [10], Nasu [11], this problem with the asymptotic relation was circumvented as follows: A point p_∞ at infinity is defined as a maximal set of rays such that a ray asymptotic to one ray in p_∞ is asymptotic to each ray in p_∞. Denote by M_∞ the set of all points at infinity, which is also called the ideal boundary of M. This definition of ideal boundary looks more geometric, but the way equivalence relation is defined from the asymptotic relation is rather artificial and its geometric meaning is not easy to comprehend at first glance.

The above two definitions of ideal boundary do not seem to have much in common. In order to demonstrate the relationship between these two concepts, we need a special kind of horofunctions, the so-called Busemann-functions, defined by rays. The Busemann-function b_γ corresponding to a ray γ is defined by

$$b_\gamma(x) = \lim_{t\to\infty}(t - d(\gamma(t), x)).$$

Since $t \mapsto (t - d(\gamma(t), x))$ is monotone increasing for $t > 0$ and bounded above by $d(\gamma(0), x)$, it converges uniformly on every compact subset of M. Since distance functions are Lipschitz continuous with Lipschitz constant 1, so are Busemann-functions. For any ray γ, it is clear that the Busemann-function b_γ is a horofunction. In fact, b_γ is a limit of distance functions $[-d_{\gamma(t)}]$ as $t \to \infty$ in $C_*(M)$.

A Busemann-function is not a smooth function in general. The differentiability of Busemann-functions and horofunctions is studied in Eberlein [4], Yim [14]. The level set $H_\gamma(t) = b_\gamma^{-1}(t)$ of the Busemann function of a ray γ is called *a horosphere of* γ. A horosphere is not always a smooth submanifold, but it is in some sense a limit of metric spheres Busemann [2], Shiohama [13] such as

$$H_\gamma(a) = \lim_{t\to\infty} S_{\gamma(t)}(t-a), \quad a \in b_\gamma(M).$$

The basic properties of asymptotic rays and Busemann-functions are studied in many literatures, and we summarize some necessary results as follows. (See Busemann [2, 3] and Shiohama [13] for details.)

Proposition 1. *Let M be a complete open Riemannian manifold and $\gamma : [0,\infty) \to M$ a ray in M.*

(1) *For each $p \in M$, there is a ray σ_p asymptotic to γ with $\sigma(0) = p$.*
(2) *If $\sigma \succ \gamma$ and $\gamma_1 \subset \gamma \subset \gamma_2$, then $\sigma \succ \gamma_1$ and $\sigma \succ \gamma_2$.*
(3) *If $\sigma_n \succ \gamma$ for all $n = 1, 2, 3, \ldots$ and $\sigma_n \to \sigma$, then $\sigma \succ \gamma$.*
(4) *A ray σ is asymptotic to a ray γ if and only if for all $t > 0$*

$$b_\gamma(\sigma(t)) = b_\gamma(\sigma(0)) + t.$$

In particular, $\nabla b_\gamma = \sigma'$ along $\sigma(0,\infty)$, and $\sigma'(t) \perp H_\gamma(\sigma(t))$ if H_γ is regular at $\sigma(t)$.

(5) *If $\sigma \succ \gamma$, then for $q = \sigma(t)$, $t > 0$, there exists a unique ray $\sigma_q \succ \gamma$ with $\sigma_q(0) = q$ and we have $\sigma_q \subset \sigma$.*

Let's for a moment denote by $B(M) \subset \partial M$ the set of all Busemann-functions defined on M. Note that the horofunctions in ∂M are in fact defined up to constants and this rule applies to Busemann-functions too.

Theorem 1. *For a complete noncompact Riemannian manifold M, there is a one-to-one correspondence between M_∞ and the set of all Busemann-functions $B(M)$ on M up to constants. In other words, $b_\gamma - b_\sigma =$ constant for two rays γ and σ if and only if γ and σ belong to the same point in M_∞.*

Proof. Assume that γ and σ are two rays such that $b_\gamma - b_\sigma =$ constant. By Proposition 1(4), for any ray ρ asymptotic to γ, we have $b_\gamma(\rho(t)) - b_\gamma(\rho(0)) = t$ for each $t \geq 0$. However, by the assumption, we have $b_\gamma(\rho(t)) - b_\gamma(\rho(0)) = b_\sigma(\rho(t)) - b_\sigma(\rho(0))$, and again by Proposition 1(4) ρ is asymptotic to σ. Therefore, if a ray is asymptotic to either one of γ and σ, then it is asymptotic to the other as well, which clearly means γ and σ belong to same equivalence class in M_∞.

We now assume that γ and σ represent the same point at infinity. By the definition of equivalence relation for M_∞, we know that a ray asymptotic to either one of these two rays must be asymptotic to the other as well. Therefore by Proposition 1(4), we know that an asymptotic direction of a ray coincides with the gradient direction of the corresponding Busemann-function. Hence we have $\nabla b_\gamma = \nabla b_\sigma$ whenever both functions are differentiable. We now use a standard argument to prove that these two Busemann-functions agree up to a constant. For the sake of completeness, we provide a proof.

Put $u = b_\gamma - b_\sigma$. The Busemann functions b_γ and σ are Lipschitz continuous and hence differentiable almost everywhere on M. We therefore see that u is also differentiable almost everywhere and $\nabla u = 0$ whenever it exists in M. Let $B \subset M$ be a convex open ball. Then B is diffeomorphic to an open ball in $\mathbf{R}^n$. Since the space of smooth functions $C^\infty(B)$ is dense in the Sobolev space $H^{1,2}(B)$ and $u \in H^{1,2}(B)$, there is a sequence of smooth functions $\{u_n\}$ which converges to u in $H^{1,2}(B)$. In particular, we have $\lim_{n\to\infty} ||\nabla u_n||_2 = 0$, where $||\cdot||$ denotes the L^2 norm on B. For any two points x, $y \in B$, let α_y be the minimizing geodesic in B such that $\alpha_y(0) = x$, $\alpha_y(1) = y$. Then we have $u_n(x) - u_n(y) = \int_0^1 \nabla u_n(\alpha_y(t))\alpha_y'(t)dt$, and hence

$$\begin{aligned}\int_B (u_n(x) - u_n(y))dy &= \int_B \int_0^1 \nabla u_n(\alpha_y(t))\alpha_y'(t)dtdy \\ &= \int_0^1 \int_B \nabla u_n(\alpha_y(t))\alpha_y'(t)dydt.\end{aligned}$$

Since $\lim_{n\to\infty} ||\nabla u_n||_2 = 0$, applying Hölder's inequality, it is not difficult to see that for any $t \in [0,1]$

$$\lim_{n\to\infty} \int_B \nabla u_n(\alpha_y(t))\alpha_y'(t)dy = 0.$$

Hence the left hand side of the above equation vanishes as $n \to \infty$, and we have

$$\begin{aligned} 0 &= \lim_{n\to\infty} \int_B (u_n(x) - u_n(y))dy \\ &= \lim_{n\to\infty} \Big(u_n(x)\operatorname{vol}(B) - \int_B u_n(y)dy\Big) \\ &= u(x)\operatorname{vol}(B) - \lim_{n\to\infty} \int_B u_n(y)dy. \end{aligned}$$

Since $x \in B$ is arbitrary, u is constant on B, and hence u is a constant function on M. □

From this theorem, we see that a point $p_\infty \in M_\infty$ is actually an equivalence class of rays which define same Busemann-function up to constants. If b_γ is a Busemann-function such that p_∞ is identified with the class $[b_\gamma]$, we will denote by $p_\infty = [\gamma]$ the equivalence class of rays defining the same Busemann-function up to constants.

Corollary 1. *For a complete open Riemannian manifold M, there is an embedding $M_\infty \hookrightarrow \partial M$.*

There are examples of horofunctions which are not Busemann-functions for any ray, and hence the above inclusion may not be surjective (see Yim [15]). In general these non-Busemann horofunctions do not carry much of geometry in Riemannian manifolds, and we will mostly concentrate on M_∞.

In the Euclidean space, a ray is a straight line and an asymptotic ray is a parallel line. Therefore the asymptotic relation or the parallelism is an equivalence relation, and asymptotic lines define a direction which corresponds to a point at infinity. For a general Riemannian manifold, one may want to try the same argument. In fact, on Hadamard spaces, which are simply connected manifolds with non-positive curvature, the asymptotic relation is indeed an equivalence relation and defines an ideal boundary. However, for more general Riemannian manifolds, the asymptotic relation is not an equivalence relation and it has been a major obstacle in the study of asymptotic geometry of open manifolds. It is therefore meaningful to know the optimal condition for the asymptotic relation to be an equivalence relation.

A compact domain Ω in a Riemannian manifold is called *convex* if a minimal geodesic between any two points in Ω is contained in Ω. We can generalize this concept to a compactified open manifold, and consider a connection between an interior point and a point at infinity. We then have the following definition.

Definition 1. The ideal boundary M_∞ of a noncompact complete Riemannian manifold M is called *convex* if for any $q \in M$ and $p_\infty = [\gamma] \in M_\infty$ there is a ray σ such that $b_\gamma - b_\sigma =$ constant and $\sigma(0) = q$.

If $b_\gamma - b_\sigma =$ constant, then $\sigma \in [\gamma]$ and we say that the point $q = \sigma(0)$ and the point $[\gamma] \in M_\infty$ are connected by the minimal geodesic σ. So the convexity of the ideal boundary of a complete non-compact Riemannian manifold is a sort of geodesic completeness on $M \cup M_\infty$. Note that in Yim [15] a surface satisfying the above condition are called weakly convex, and a convex ideal boundary satisfies a stronger condition that every horofunction in ∂M can be connected to each interior point. But, as mentioned earlier, we will only consider M_∞ and our definition of convex ideal boundary as above suffices.

Theorem 2. *For a complete open Riemannian manifold M, the ideal boundary M_∞ is convex if and only if the asymptotic relation $\succ$ is an equivalence relation.*

Proof. Assume that $\succ$ is an equivalence relation. For any $q \in M$ and $p_\infty \in M_\infty$, let $\sigma : [0, \infty) \to M$ be a ray such that $\sigma(0) = q$ and $\sigma \succ \gamma$ for each ray $\gamma \in p_\infty = [\gamma]$. Then by the symmetry for any $\gamma \in p_\infty$ we have $\gamma \succ \sigma$. If a ray ρ is asymptotic to any ray $\gamma \in p_\infty$, by the transitivity, we have $\rho \succ \sigma$. Then by the definition of M_∞ we have $\sigma \in [\gamma]$, and hence M_∞ is convex.

Suppose M_∞ is convex. For any ray γ, let $\sigma : [0, \infty) \to M$ be a ray such that $\sigma \succ \gamma$ and consider a point $q = \sigma(t_0)$ for some $t_0 > 0$. By the convexity of M_∞ there exist a ray $\sigma_q : [0, \infty) \to M$ such that $\sigma_q(0) = q$ and $b_{\sigma_q} - b_\gamma =$ constant. Then, by Proposition 1, we see that $\sigma_q \succ \gamma$ and $\sigma_q \subset \sigma$. Therefore $b_\sigma - b_{\sigma_q} = t_0$ is a constant and so is $b_\sigma - b_\gamma$. We can now conclude that if $\gamma \succ \sigma$ then $b_\sigma - b_\gamma$= constant. Since $b_\sigma - b_\gamma$= constant implies $\gamma \succ \sigma$ and $\sigma \succ \gamma$, it is clear that $\succ$ is an equivalence relation. □

As a consequence of this theorem, if the ideal boundary M_∞ is convex then it can be identified as the set of all asymptotic classes of rays. In other words, a point at infinity is a direction defined by mutually asymptotic rays.

3. A POINT AT INFINITY

In this section, we study the geometric property of a point at infinity. In particular, we compare the points at infinity with the ordinary interior points and study the similarity and difference.

For an ordinary interior point $p \in M$, a point $q \in M$ is called a *cut point* of p if an extension of a minimal geodesic between p and q is not minimal anymore. A ray plays the role of minimal geodesic between an interior point and a point at infinity, and we can consider an analogous concept as the cut point for a point at infinity. A ray $\sigma : [0, \infty) \to M$ asymptotic to γ is said to be *maximal* if for any $\varepsilon > 0$ its extension as a geodesic to $\sigma_\varepsilon : [-\varepsilon, \infty) \to M$ is not an asymptotic ray to γ. The initial point $\sigma(0)$ of a maximal asymptotic ray is called a *cut point* of γ. We denote by $\text{Cut}(\gamma)$ the cut locus of a ray γ, which is by definition the set of all cut points of γ. By the definition of M_∞, we see that a cut point of a ray depends only on the class, the point at infinity. In other words, if $[\gamma] = [\sigma] = p_\infty$ then $\text{Cut}(\gamma) = \text{Cut}(\sigma)$ and we may consider $\text{Cut}(p_\infty)$.

For a cut point q of p_∞, let $\sigma : [0, \infty) \to M$ is a maximal ray asymptotic to $\gamma \in p_\infty$ such that $\gamma(0) = q$ (Note that such γ may not be unique.). Then there are two possibilities. Either the extension σ_ε is not a ray or it is a ray but not asymptotic to γ. If it is not a ray then it can not be a connection between the point $\sigma_\varepsilon(-\varepsilon)$ and any point at infinity, but if it is a ray not asymptotic to γ then it can be a connection between $\sigma_\varepsilon(-\varepsilon)$ and some point at infinity other than p_∞. In the first case, where the maximal asymptotic ray is not a sub-ray of another ray, the maximal ray is called *terminal* and its initial point is called a terminal point of γ. It had been a question of some interest, first asked by Busemann, whether a maximal asymptotic ray can be a proper sub-ray of another ray and therefore the above distinction of two cases is really meaningful. This interesting phenomenon is something quite different from the ordinary point case, and the following theorem answers when the two cases coincide.

Theorem 3. *If the ideal boundary M_∞ is convex, then each cut point of a ray γ is a terminal point of an asymptotic ray to γ.*

Proof. For $q \in \text{Cut}(\gamma)$ let $\sigma : [0, \infty) \to M$ be a ray asymptotic to γ such that $\sigma(0) = p$. Suppose q is not a terminal point of σ. Then σ can be extended as a ray $\sigma_\varepsilon : [-\varepsilon, \infty) \to M$. Then by Proposition 1(5), the sub-ray $\sigma_{\frac{\varepsilon}{2}}$ is the unique ray asymptotic to σ at $\sigma_{\frac{\varepsilon}{2}}(-\frac{\varepsilon}{2})$. Since M_∞ is convex, by Theorem 2, the asymptotic relation $\succ$ is transitive and hence $\sigma_{\frac{\varepsilon}{2}} \succ \gamma$, which implies that p is not a cut point of γ and it is a contradiction. □

We now study in detail the structure of the cut locus of a point at infinity. For an ordinary point the cut locus is where the distance function fails to be differentiable, and it has many interesting properties. Busemann-functions play the role of distance functions from points at infinity, and in many aspects they behave much like distance functions.

For an ordinary point $p \in M$ the cut locus $\mathrm{Cut}(p)$ is a closed set and it is the image of the exponential map from the cut locus in the tangent space T_pM. For a point $p_\infty \in M_\infty$, however, nothing even similar to the exponential map can be defined and in fact there exists an example of cut locus which is not even closed (see Nasu [11]). This example of cut locus which is not closed demonstrates that the structure of the cut locus of a point at infinity can be quite different from that of an ordinary point, and it is very difficult to use any analytic tools to study its geometry. For $p \in M$ the distance function $d_p : M \to \mathbf{R}$ is a smooth function on $M \setminus \mathrm{Cut}(p)$. Unfortunately, Busemann-functions are horofunctions, which are limits of distance functions in $C_*(M)$, and hence they are only Lipschitz continuous.

Since Busemann-functions are only continuous, without any extra conditions the first order differentiability is the only possibility one can hope for. If there are two distinct rays emanating from a point p and asymptotic to a ray γ, then b_γ is not differentiable at p. The set of all such points is denoted by $C_2(\gamma)$. In other words, for each point $q \in C_2(\gamma)$ there are at least two rays emanating from q and asymptotic to γ. Let $S(\gamma)$ denote the set of points where the Busemann-function b_γ is not differentiable. It is quite clear that all these concepts only depend on the equivalence class in M_∞ represented by γ, and they also enjoy the following properties (See Busemann [3], Innami [9]).

Proposition 2. *Let M be a complete noncompact Riemannian manifold and let γ be a ray. Then*

(1) $C_2(\gamma) \subset S(\gamma) \subset \mathrm{Cut}(\gamma)$.
(2) $C_2(\gamma)$ *is dense in* $\mathrm{Cut}(\gamma)$.
(3) $S(\gamma)$ *is of measure zero.*

In Proposition 1(5), we see that if $\sigma \succ \gamma$ then a sub-ray $\sigma_q : [0,\infty) \to M$ with $\sigma_q(0) = q = \sigma(t)$ for $t > 0$ is a unique ray asymptotic to γ at q, which means $q \notin C_2(\gamma)$. In fact, a stronger statement is still true, and it is not very difficult to prove the following statement using Proposition 1(4) (See Shiohama [13]).

Proposition 3. *Let M be a complete noncompact Riemannian manifold and let γ be a ray. If $\sigma \succ \gamma$, then $\sigma(t) \notin S(\gamma)$ for all $t > 0$.*

As mentioned earlier, the cut locus of a point at infinity may not be closed. However, if it is closed, we have the following.

Proposition 4. *Let M be a noncompact complete Riemannian manifold. If the cut locus $\mathrm{Cut}(\gamma)$ of a ray γ is closed, then the Busemann function b_γ of γ is C^1 on $M \setminus \mathrm{Cut}(\gamma)$.*

Proof. For any $p \notin \mathrm{Cut}(\gamma)$, since $S(\gamma) \subset \mathrm{Cut}(\gamma)$ and $\mathrm{Cut}(\gamma)$ is closed, there exists an open neighborhood U of p on which the Busemann-function b_γ is differentiable. For any $q \in U$, let $q_i \to q$ be a sequence in U such that

$$\nabla b_\gamma(q_i) \to v \in T_qM.$$

Let $\gamma_i : [0,\infty) \to M$ be a ray such that $\gamma_i \succ \gamma$ and $\gamma_i(0) = q_i$. Then $\nabla b_\gamma(q_i) = \gamma_i'(0)$ and a subsequence of $\{\gamma_i\}$ converges to a ray γ_0 such that $\gamma_o'(0) = v$. Since a limit of asymptotic rays is again asymptotic, γ_0 is asymptotic to γ. Since $C_2(\gamma) \subset S(\gamma)$ and $q \notin S(\gamma)$, a ray asymptotic to γ is unique at q. Hence $v = \nabla b_\gamma(q)$ and ∇b_γ is continuous at q. □

In the finite case, the cut locus consists of first conjugate points and points with multiple minimal connections. In the case of a point at infinity, it is not easy to define a conjugate point because the exponential map is not defined and in general a variation through asymptotic rays does not give rise to a Jacobi-field along a ray. We therefore use an indirect method to define the first conjugate point of a point at infinity. Let $\gamma : [0,\infty) \to M$ be a ray in M and $R_\gamma = R_\gamma(t)$ be the tensor field along γ defined by $R_\gamma(t)V = R(V, \gamma'(t))\gamma'(t)$ for any vector field $V(t)$ orthogonal to $\gamma'(t)$, where $R(\cdot,\cdot)$ is the curvature tensor of M. Then a matrix solution to the differential equation

$$D'' + R_\gamma D = 0$$

is called a *Jacobi tensor*, where D' denotes the covariant derivative of D along γ. Each Jacobi tensor, applied to a parallel normal vector field along γ, gives rise to an $(n-1)$-dimensional space of Jacobi-fields along γ when the dimension of M is n.

If $\gamma : [0,\infty) \to M$ is a ray, for any $s \in (0,\infty)$, the geodesic segment $\gamma[0,s]$ has no conjugate point on it and there exists a unique Jacobi-tensor D_s such that

$$D_s(0) = \mathrm{Id}, \quad D_s(s) = 0.$$

It is then easy to see that for each s the tensor $D_s'(0)$ is symmetric and $0 < s < t$ implies $D_s'(0) < D_t'(0)$, where $T_1 < T_2$ means $T_2 - T_1$ is positive definite for tensors T_1 and T_2. Therefore, as $s \to \infty$, $D_s'(0)$ increases monotonically and $\lim_{s\to\infty} D_s'(0)$ exists if and only if $D_s'(0)$ is uniformly bounded for all $s > 0$.

If $\lim_{s\to\infty} D_s'(0)$ exists, then it is easy to see that $D_\infty \overset{\text{def}}{=} \lim_{s\to\infty} D_s$ also exists and $D_\infty'(0) = \lim_{s\to\infty} D_s'(0)$. In this case, the tensor field $D_\infty(t)$ along γ is called the *stable Jacobi tensor* in some literatures, for example in Eschenburg [5], Eschenburg & O'Sullivan [6]. If the Busemann-function b_γ is C^2-differentiable, then $D_\infty'(0)$ is the shape operator of the horosphere $b_\gamma^{-1}(0)$, and one may want to think the stable Jacobi tensor as a tensor field defined by the variation through rays asymptotic to γ. However this is not true in general because Busemann-functions are not C^2, and it is also quite difficult to integrate this field in order to study the behavior of neighboring asymptotic rays. Nevertheless, the following definition still makes sense.

Definition 2. For a ray $\gamma : [0,\infty) \to M$ in a complete open Riemannian manifold M, $q = \gamma(0)$ is called *a conjugate point of* γ *if* $\lim_{s\to\infty} D_s'(0)$ does not converge.

As mentioned earlier, the tensor $D_s'(0)$ is symmetric for each $s > 0$, and if $q = \gamma(0)$ is a conjugate point of γ then there exists a vector $v \in T_qM$ such that

$$\lim_{s\to\infty} \langle v, D_s'(0)(v)\rangle \nearrow \infty.$$

If we consider a finite geodesic, this definition of conjugate points clearly coincides with that of first conjugate points in the usual sense. The conjugate locus of a ray γ, the set of all conjugate points of γ, will be denoted by $\mathrm{Conj}(\gamma)$. As in the case of ordinary conjugate points in M, a conjugate point of a ray is also a cut point.

Theorem 4. *Let M be a noncompact complete Riemannian manifold. Then for any ray γ in M,*

$$\mathrm{Conj}(\gamma) \subset \mathrm{Cut}(\gamma).$$

Proof. Assume $q = \gamma(0)$ is a conjugate point of γ and let $\{D_s\}$ be as above. It then suffices to show the extension $\gamma_\varepsilon : [-\varepsilon, \infty) \to M$ of γ is not a ray for $\varepsilon > 0$. If the geodesic segment $\gamma_\varepsilon[-\varepsilon, 0]$ has a conjugate point on it, then γ_ε is clearly not a ray. We therefore assume that $\gamma_\varepsilon[-\varepsilon, 0]$ has no conjugate point and hence there exist a unique Jacobi-tensor $D_{-\varepsilon}$ along $\gamma_\varepsilon[-\varepsilon, 0]$ such that $D_{-\varepsilon}(-\varepsilon) = 0$ and $D_{-\varepsilon}(0) = \mathrm{Id}$. Since $D'_s(0)$ increases monotonically and unbounded as $s \to \infty$, for large enough $s > 0$, there exists a vector $v \in T_qM$ such that

$$\langle v, (D'_s(0) - D'_{-\varepsilon}(0))(v)\rangle > 0.$$

Let $v(t)$ be a parallel vector field along $\gamma_\varepsilon[-\varepsilon, s]$ such that $v(0) = v$, and let $V(t)$, $-\varepsilon < t < s$, be a piecewise smooth vector field defined by

$$V(t) = D_{-\varepsilon}(t)(v(t)) \text{ for } -\varepsilon \le t \le 0,$$
$$V(t) = D_s(t)(v(t)) \text{ for } 0 \le t \le s.$$

Then the index form $\mathrm{Ind}(V, V)$ of V is

$$\begin{aligned}\mathrm{Ind}(V, V) &= \int_{-\varepsilon}^{s} (\langle V', V\rangle' - \langle V'' + R_\gamma V, V\rangle)dt \\ &= \langle v, (D'_{-\varepsilon}(0) - D'_s(0))(v)\rangle < 0.\end{aligned}$$

Therefore $\gamma_\varepsilon[-\varepsilon, s]$ is not a minimal geodesic, and hence γ_ε is not a ray. □

For an ordinary interior point, a cut point is either a point with multiple minimal connections or a first conjugate point. The proof of this fact involves the exponential map or Jacobi-fields along minimal geodesics. These concepts are not natural for a point at infinity, and it looks quite difficult to characterize points in the cut locus any further without more conditions. It is not even known whether $\mathrm{Cut}(\gamma)$ is of measure zero while $S(\gamma)$ is dense in $\mathrm{Cut}(\gamma)$ and of measure zero (see Proposition 2).

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