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ON n-TUPLES OF TENSOR PRODUCTS OF p-HYPONORMAL OPERATORS

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ABSTRACT. The operator $A \in L(\mathcal{H}_i)$, the Banach algebra of bounded linear operators on the complex infinite dimensional Hilbert space $\mathcal{H}_i$, is said to be p-hyponormal if $(A^*A)^p \geq (AA^*)^p$ for $p \in (0,1]$. Let $\widehat{\mathcal{H}} = \mathcal{H}_1 \widehat{\otimes \cdots \otimes} \mathcal{H}_n$ denote the completion of $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$ with respect to some crossnorm. Let I_i be the identity operator on $\mathcal{H}_i$. Letting $T_i := I_1 \otimes \cdots \otimes I_{i-1} \otimes A_i \otimes \cdots \otimes I_n$ on $\widehat{\mathcal{H}}$, where each A_i is p-hyponormal, it is proved that the commuting n-tuple $\mathbf{T} = (T_1, \ldots, T_n)$ satisfies Bishop's condition (β) and that if $\mathbf{T}$ is Weyl then there exists a non-singular commuting n-tuple $\mathbf{S}$ such that $\mathbf{T} = \mathbf{S} + \mathbf{F}$ for some n-tuple $\mathbf{F}$ of compact operators.

1. INTRODUCTION

Let $L(\mathcal{H})$ denote the Banach algebra of bounded linear operators acting on a complex infinite dimensional Hilbert space $\mathcal{H}$. Let $\mathbf{T} = (T_1, \ldots, T_n)$ denote a commuting n-tuple of operators in $L(\mathcal{H})$. Recall (Curto [2], Taylor [11]) that $\mathbf{T}$ is said to be *non-singular* if the *Koszul complex* for $\mathbf{T}$, denoted by $K(\mathbf{T}, \mathcal{H})$, is exact at every stage. Also, $\mathbf{T}$ is said to be *Fredholm* if the Koszul complex $K(\mathbf{T}, \mathcal{H})$ is Fredholm, *i. e.*, all homology spaces of $K(\mathbf{T}, \mathcal{H})$ are finite dimensional. In this case the *index* of $\mathbf{T}$, denoted ind$(\mathbf{T})$, is defined as the *Euler characteristic* of $K(\mathbf{T}, \mathcal{H})$, *i. e.*, as the alternating sum of dimensions of all homology spaces of $K(\mathbf{T}, \mathcal{H})$. If $\mathbf{T} \in L(\mathcal{H})$ is Fredholm with index zero, then we say that $\mathbf{T}$ is *Weyl.* We shall write $\sigma_T(\mathbf{T})$, $\sigma_{Te}(\mathbf{T})$, and $\sigma_{Tw}(\mathbf{T})$ for the *Taylor spectrum*, the *Taylor essential spectrum*, and *Taylor-Weyl spectrum* of $\mathbf{T}$, respectively: thus,

$$\sigma_T(\mathbf{T}) = \{\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n : \mathbf{T} - \lambda \text{ is singular}\},$$
$$\sigma_{Te}(\mathbf{T}) = \{\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n : \mathbf{T} - \lambda \text{ is not Fredholm}\},$$

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and

$$\sigma_{Tw}(\mathbf{T}) = \{\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n : \mathbf{T} - \lambda \text{ is not Weyl}\}.$$

For any open polydisk $D \subset \mathbb{C}^n$, let $\mathcal{O}(D, \mathcal{H})$ denote the Frechét space of $\mathcal{H}$-valued analytic functions on D. Then we say (Eschmeier & Putinar [6]) that a commuting n-tuple $\mathbf{T}$ has the *single valued extension property*, shortened to SVEP, if the Koszul complex $\mathcal{K}(\mathbf{T} - \lambda, \mathcal{O}(D, \mathcal{H}))$ is exact in positive degrees and $\mathbf{T}$ has *Bishop's condition (β)* if it has the SVEP and its Koszul complex has also separated homology in degree zero. Obviously, the following implication holds:

$$\text{Bishop's condition}(\beta) \implies \text{the SVEP}.$$

For more details, see (Eschmeier & Putinar [6]).

We shall write

$$\sigma_p(\mathbf{T}) = \Big\{\lambda \in \mathbb{C}^n : \text{ there exists a non-zero vector } x \in \mathcal{H} \text{ such that } x \in \bigcap \ker(T_i - \lambda_i)\Big\}$$

for the eigenvalues of $\mathbf{T}$,

$$p_{00}(\mathbf{T}) = \sigma_T(\mathbf{T}) \setminus \{\sigma_{Te}(\mathbf{T}) \cup \operatorname{acc}\sigma(\mathbf{T})\}$$

for the *Riesz points* of $\sigma_T(\mathbf{T})$, and $\operatorname{iso}\sigma_T(\mathbf{T})$ for all isolated points of $\sigma_T(\mathbf{T})$, respectively. Recall (Aluthge [1], Duggal [4, 5], Jeon & Duggal [8], Yingbin & Zikun [14]) that an operator $T \in L(\mathcal{H})$ is said to be *p-hyponormal* if

$$(T^*T)^p - (TT^*)^p \geq 0 \text{ for } p \in (0, 1].$$

If $p = 1$, T is just hyponormal. We shall denote the class of p-hyponormal operators by $\mathfrak{H}(p)$; $\mathfrak{H}\mathfrak{U}(p)$ shall denote the class of those p-hyponormal operators for which the partial isometry U in the polar decomposition $T = U|T|$ is unitary.

Throughout this paper, for complex infinite dimensional Hilbert spaces $\mathcal{H}_i$ ($1 \leq i \leq n$), we let $\widehat{\mathcal{H}} = \mathcal{H}_1 \widehat{\otimes \cdots \otimes} \mathcal{H}_n$ denote the completion of $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$ with respect to some crossnorm. Let I_i be the identity operator on $\mathcal{H}_i$. For $A_i \in L(\mathcal{H}_i)$, let

$$T_i := I_1 \otimes \cdots \otimes I_{i-1} \otimes A_i \otimes \cdots \otimes I_n \text{ on } \widehat{\mathcal{H}}.$$

Then $\mathbf{T} = (T_1, \ldots, T_n)$ is a commuting (in fact, doubly commuting) n-tuple of operators on $\widehat{\mathcal{H}}$.

2. RESULTS

Theorem 1. *Let $A_i \in \mathfrak{H}(p)$ and let $\mathbf{T} = (T_1, \ldots, T_n)$ be the n-tuple of operators*

$$T_i := I_1 \otimes \cdots \otimes I_{i-1} \otimes A_i \otimes \cdots \otimes I_n \text{ on } \widehat{\mathcal{H}}.$$

Then there exist an n-tuple $\mathbf{S} = (S_1, \ldots, S_n)$ with

$$S_i := I_1 \otimes \cdots \otimes I_{i-1} \otimes B_i \otimes \cdots \otimes I_n \text{ on } \widehat{\mathcal{H}} \text{ for some } B_i \in \mathfrak{H}(1),$$

a quasiaffinity X and an injection Y such that both $\mathbf{T}$ and $\mathbf{S}$ have Bishop's condition (β) and

$$X\mathbf{T} = \mathbf{S}X \text{ and } \mathbf{T}Y = Y\mathbf{S}.$$

Proof. Given an $A_i \in \mathfrak{H}(p)$, we decompose A_i into its normal and pure parts by $A_i = A_{i0} \oplus A_{i1}$ with respect to the decomposition $\mathcal{H}_i = \mathcal{H}_{i0} \oplus \mathcal{H}_{i1}$. Then $A_{i1} \in \mathfrak{H}(p)$. Let A_{i1} have the polar decomposition $A_{i1} = U_{i1}|A_{i1}|$ where U_{i1} is an isometry. Define the Aluthge transform $\widehat{A}_{i1}$ of A_{i1} by $\widehat{A}_{i1} = |A_{i1}|^{1/2} U_{i1} |A_{i1}|^{1/2}$. Let $\widehat{A}_{i1}$ have the polar decomposition $\widehat{A}_{i1} = V_{i1}|\widehat{A}_{i1}|$ where V_{i1} is (also) an isometry. Again, define the Aluthge transform $\widetilde{A}_{i1}$ of $\widehat{A}_{i1}$ by $\widetilde{A}_{i1} = |\widehat{A}_{i1}|^{1/2} V_{i1} |\widehat{A}_{i1}|^{1/2}$. Then $\widetilde{A}_{i1} \in \mathfrak{H}(1)$ by Aluthge [1]. Let $X_{i1} = |\widehat{A}_{i1}|^{1/2}|A_{i1}|^{1/2}$ and $Y_{i1} = U_{i1}|A_{i1}|^{1/2} V_{i1} |\widehat{A}_{i1}|^{1/2}$. Then X_{i1} is a quasiaffinity and Y_{i1} is an injection such that

$$X_{i1}A_{i1} = \widetilde{A}_{i1}X_{i1} \text{ and } A_{i1}Y_{i1} = Y_{i1}\widetilde{A}_{i1}.$$

Let $B_i = A_{i0} \oplus \widetilde{A}_{i1}$. Then $B_i \in \mathfrak{H}(1)$. Defining the quasiaffinity X_i by $X_i = I_{\mathcal{H}_{i0}} \oplus X_{i1}$ and the injection Y_i by $Y_i = I_{\mathcal{H}_{i0}} \oplus Y_{i1}$, it follows that

$$X_iA_i = B_iX_i \text{ and } A_iY_i = Y_iB_i.$$

Considering the tensor products of operators X and Y,

$$X := X_i \otimes \cdots \otimes X_n \text{ and } Y := Y_1 \otimes \cdots \otimes Y_n,$$

it is then seen that X is a quasiaffinity, Y is an injection and

$$X\mathbf{T} = \mathbf{S}X \text{ and } \mathbf{T}Y = Y\mathbf{S}.$$

Recall from Duggal [4, Theorem 3] that if $B_1, B_2 \in L(\mathcal{H}_i)$, then

$$B_1 \otimes B_2 \in \mathfrak{H}(p) \text{ if and only if } B_1, B_2 \in \mathfrak{H}(p). \tag{2.1}$$

Hence it follows from a finite induction argument that

$$T_i \in \mathfrak{H}(p) \text{ if and only if } A_i \in \mathfrak{H}(p) \text{ for all } i = 1, \ldots, n. \tag{2.2}$$

Since each T_i has Bishop's condition (β) (see Duggal [5, Theorem 1], Yingbin & Zikun [14, Theorem 7]), it follows from Wolff [13, Corollary 2.2] that the n-tuple $\mathbf{T} = (T_1, \ldots, T_n)$ has Bishop's condition (β). Similarly, the n-tuple $\mathbf{S} = (S_1, \ldots, S_n)$ also has Bishop's condition (β). □

Corollary 2. *We first notice that both X and Y constructed in proof of Theorem 1 are quasiaffinities when each A_i belongs to $\mathfrak{H}\mathfrak{U}(p)$.*

$$\sigma_*(\mathbf{T}) = \sigma_*(\mathbf{S}), \text{ where } \sigma_* \text{ stands for either of } \sigma_T,\ \sigma_{Te},\ \sigma_{Tw}. \tag{2.3}$$

Proof. We first notice that X and Y constructed in proof of Theorem 1 become both quasiaffinities when each A_i belongs to $\mathfrak{H}\mathfrak{U}(p)$. Thus $\mathbf{T}$ and $\mathbf{S}$ are (jointly) quasisimilar n-tuples. Since $\mathbf{T}$ and $\mathbf{S}$ have Bishop's condition (β) by Theorem 1, Putinar [9, Theorem 1] implies (2.3). This completes the proof. □

Fredholm n-tuples enjoy most of the properties single Fredholm operators possess (see Curto [3]). It is well known that a Fredholm operator of index zero (*i. e.*, Weyl operator) can be perturbed by a compact operator to an invertible operator. Thus one may ask if this property holds in several variables (*cf.* Curto [2, Problem 3]). As it turns out, this perturbation property fails in several variables (see Gelca [7] for an example). Despite the failure of this property for the general case, the following result gives a positive answer to the question in case of tensor products considered here.

Theorem 3. *Let $A_i \in \mathfrak{H}(p)$ and let $\mathbf{T} = (T_1, \ldots, T_n)$ be an n-tuple of operators*

$$T_i := I_1 \otimes \cdots \otimes I_{i-1} \otimes A_i \otimes \cdots \otimes I_n \text{ on } \widehat{\mathcal{H}}.$$

If $\mathbf{T}$ is Weyl and singular, then there exists a non-singular commuting n-tuple $\mathbf{S} = (S_1, \ldots, S_n)$ such that $\mathbf{T} = \mathbf{S} + \mathbf{F}$ for some n-tuple of compact operators F_i $(i = 1, \ldots, n)$.

Proof. Since $\mathbf{T}$ has Bishop's condition (β), if $\mathbf{T}$ is Weyl and singular, then Putinar [10, Theorem 1] implies $0 \in p_{00}(\mathbf{T})$. Let f be the characteristic function of $0 \in \mathrm{iso}\,\sigma_T(\mathbf{T})$. Since f is analytic in a neighborhood of $\sigma_T(\mathbf{T})$, Taylor [11, Theorem 4.8] implies the existence of an idempotent $P_0 = f(\mathbf{T}) \in L(\widehat{\mathcal{H}})$ such that $P_0 T_i = T_i P_0$, T_i is quasinilpotent on $\mathrm{ran} P_0$, and

$$0 \notin \sigma_T(\mathbf{T}|_{\ker P_0}). \tag{2.4}$$

Since the restriction of a p-hyponormal operator to an invariant subspace is again p-hyponormal Uchiyama ([12, Lemma 4]) and p-hyponormal operators are normaloid, we see that $T_i|_{\text{ran}P_0} = 0$. Then the fact that $0 \in p_{00}(\mathbf{T})$ implies that the subspace $\text{ran}P_0$ is finite dimensional, and so P_0 is a compact operator on $\widehat{\mathcal{H}}$. Considering $\mathbf{F} = (P_0, \ldots, P_0)$ and $\mathbf{S} = \mathbf{T} - \mathbf{F} = (T_1 - P_0, \ldots, T_n - P_0)$, it now follows that $\mathbf{S}$ is a commuting n-tuple. This by [Curto [3] p. 39] implies that

$$\sigma_T(\mathbf{S}) = \sigma_T((\mathbf{T} - \mathbf{F})|_{\text{ran}P_0}) \cup \sigma_T((\mathbf{T} - \mathbf{F})|_{\text{ker}P_0}).$$

Obviously, $0 \notin \sigma_T((\mathbf{T} - \mathbf{F})|_{\text{ran}P_0})$ and by (2.4)

$$0 \notin \sigma_T((\mathbf{T} - \mathbf{F})|_{\text{ker}P_0}) = \sigma_T(\mathbf{T}|_{\text{ker}P_0}).$$

Thus $0 \notin \sigma_T(\mathbf{S})$, *i. e.*, $\mathbf{S} = \mathbf{T} - \mathbf{F}$ is non-singular, and hence $\mathbf{T} = \mathbf{S} + \mathbf{F}$. □

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