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Superior Mandelbrot Set

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Mandelbrot sets and its generalizations have been extensively studied by using the Picard iterations. The purpose of this paper is to study superior Mandelbrot sets, a new class of Mandelbrot sets by introducing the Mann iterative procedure for polynomials $Q_c(z) := z^n + c$. We generate some superior Mandelbrot sets for different values of n (≥ 2) and these new figures are exciting and fascinating.

Keywords: Julia set, Mandelbrot set, Mann sequence, superior criterions.
MSC2000 Classification: 90C32

1. INTRODUCTION

Perhaps the Mandelbrot set is the most popular object in fractal theory. It is believed that it is not only the most beautiful object, which has been made visible but the most complex also. This object was given by Mandelbrot (1979) in 1979 and has been the subject of intense research right from its advent.

It is known to us that all the complex quadratic functions are topologically conjugate to the complex quadratic function $Qc(z) = z^2 + c$. Every Julia set for $Qc(z) = z^2 + c$ is either connected or totally disconnected. The Mandelbrot set works as a locator for the two types of Julia sets. Each point in the Mandelbrot set represents a c-value for which the Julia set is connected and each point in its complement represents a c-value for which the Julia set is totally disconnected (*cf.* Crownover 1995; Devaney 1992; Holmgren 1994; Peitgen, Jürgens & Saupe 1992c, 1992a).

In the early 1980's, Douady & Hubbard (1984) started the study of structure of the Mandelbrot set. They proved that the Mandelbrot set is connected. They also gave an outstanding conjecture that this set is locally connected (see also Peitgen & Richter 1986), which was later proved by Shishikura (1998) in 1991 (*cf.* Peitgen, Jürgens & Saupe 1992c, p. 425).

The Mandelbrot set for quadratics has been studied rigorously (see Beardon 1991; Crownover 1995; Devaney 1992; Holmgren 1994; Peitgen, Jurgens & Saupe 1992c, 1992a; Steinmnetz 1993). Branner & Hubbard (1988), in the study of the dynamics of cubic polynomials, produced the first extensive study of the analog of the Mandelbrot set for cubics. As there are two critical orbits for cubics, the study of the Mandelbrot set for cubics is much complicated than that of quadratic functions (*cf.* Devaney 1992).

In 2000, Rochon (2000) gave generalizations of the Mandelbrot set in three and four dimensions. He also proved that the generalized Mandelbrot set in four dimensions is connected.

Instead of using Picard iteration procedure in functions to generate Mandelbrot sets, we introduce the Mann iterative procedure to define superior Mandelbrot sets. Superior escape criterions in the context of superior Julia sets are studied in Rani & Kumar (2004). These criterions are applicable in generation of superior Mandelbrot sets as well.

The purpose of this paper is to present some Mandelbrot sets for $Q_c(z) = z^n + c$, for $n = 2, 3, 4, 5$.

2. PRELIMINARIES

Basically there are two types of feedback machines (*cf.* Peitgen, Jürgens & Saupe 1992b, 1992c, 1992a).

(i) One-step machine,

(ii) Two-step machine.

One step machines are characterized by Peano-Picard iterations (generally called Picard or function iterations) formula

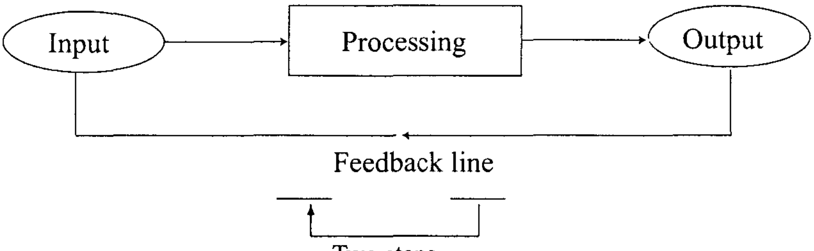
$$\underline{x_{n+1}} = f(\underline{x_n}),$$

One step

where $f(x)$ can be any function of x. It requires one number x_n as input and returns a new number x_{n+1} as better understood by the following diagram.

In two-step feedback machines, output may be computed by a formula of the type

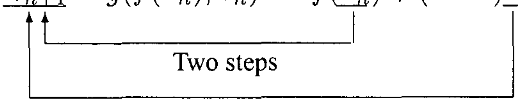
$$x_{n+1} = g(x_n, x_{n-1}).$$



It requires two numbers x_n and x_{n-1} as input and returns a new number x_{n+1}.

Mann iterations (*cf.* Definition 3.1) may be considered excellent examples of two-step feedback processes. In the Mann iterative procedure, we use a parameter $s \in [0,1] \subset \mathbb{R}$. Here, in function g, in the place of x_n and x_{n-1} we use $f(x_n)$ and x_n respectively. Thus

$$\underline{x_{n+1}} = g(f(x_n), x_n) = sf(\underline{x_n}) + (1-s)\underline{x_n}.$$



At $s = 0$, there is no change in the input and at $s = 1$, two-step machine works as a one-step machine.

We adhere to the standard notations and definitions from Beardon (1991), Devaney (1992), Holmgren (1994), Peitgen, Jürgens & Saupe (1992c, 1992a) and Rani & Kumar (2004). In all that follows $\mathbb{C}$ stands for the complex plane.

Let X be a non-empty set and $f : X \to X$ be any function. For a point x_0 in X, the *Picard orbit* (generally called *orbit* or *trajectory*) of f is the set of all iterates of a point x_0, that is,

$$\{x_n : x_n = f(x_{n-1}), n = 1, 2, \ldots\}.$$

Definition 2.1. The *Mandelbrot set M for the quadratic $Q_c(z) = z^2 + c$* is defined as the collection of all $c \in \mathbb{C}$ for which the orbit of the point 0 is bounded, that is,

$$M = \{c \in \mathbb{C} : \{Q_c^k(0) : k = 0, 1, 2, \ldots\} \text{ is bounded}\},$$

where $Q_c^0(z) = z$, $Q_c^{k+1}(z) = Q_c(Q_c^k(z))$ for $k = 0, 1, 2, \ldots$.

We choose the initial point 0, as 0 is the only critical point of Q_c (*i. e.*, the only point for which $Q_c'(z) = 0$) and the orbit of 0 is called the *critical orbit*. To check whether the orbit is bounded or not we have the following well-known result.

Theorem 2.1. *If $|c| > 2$ and $|z| \geq c$, then the orbit of z escapes to ∞. In particular, the point c is not in M.*

Remark. Recall that if $-2 \leq c \leq 0.25$, the iteration of the critical point 0 is bounded and the Julia set is connected. Thus, the interval $[-2, 0.25]$ on the real axis belongs to the Mandelbrot set M. In fact, $c = -2$ is the only point in M which has an absolute value equal to 2 (see Crownover 1995; Peitgen, Jürgens & Saupe 1992c, 1992a).

3. SUPERIOR MANDELBROT SET

Let A be a subset of complex numbers such that $f : A \to A$. For $x_0 \in A$, construct a sequence $\{x_n\}$ in the following manner.

$$\begin{aligned} x_1 &= s_1 f(x_0) + (1 - s_1)x_0, \\ x_2 &= s_2 f(x_1) + (1 - s_2)x_1, \\ &\vdots \\ x_n &= s_n f(x_{n-1}) + (1 - s_n)x_{n-1}, \\ &\vdots \end{aligned}$$

where $0 < s_n \leq 1$ and $\{s_n\}$ is convergent to a non-zero number.

Definition 3.1 (Mann (1953)). The sequence $\{x_n\}$ constructed above is called the *Mann sequence of iterates* or the *superior sequence of iterates*. We may denote it by $\mathrm{SO}(f, x_0, \{s_n\})$.

In all that follows, we take $s_n = s$, $n = 1, 2, \ldots$ and hence

$$x_n = sf(x_{n-1}) + (1 - s)x_{n-1}, \text{ for } n = 1, 2, \ldots.$$

Now we define the Mandelbrot set for

$$Q_c(z) = z^n + c,$$

where $n = 2, 3, 4, \ldots$ with respect to superior iterates.

Definition 3.2. *Superior Mandelbrot set* SM for the n-th degree polynomial $Q_c(z) = z^n + c$ is defined as the collection of all $c \in \mathbb{C}$ for which the orbit of the critical point 0 is bounded, *i. e.*,

$$\mathrm{SM} = \{c \in \mathbb{C} : \{Q_c^k(0) : k = 0, 1, 2, \ldots\} \text{ is bounded}\},$$

where $Q_c^0(z) = z$, $Q_c^{k+1}(z) = sQ_c(Q_c^k(z)) + (1 - s)Q_c^k(z)$ for $k = 0, 1, 2, \ldots$.

4. THE SUPERIOR CRITERIONS

The escape criterions play a crucial role in the analysis (and generation) of Mandelbrot and superior Mandelbrot sets. The escape criterions for superior filled Julia sets (*cf.* Rani & Kumar 2004) are applicable in the study and generation of superior Mandelbrot sets as well.

Followings are the escape criterions for some of the superior Julia sets.

(i) The escape criterion for $Q_c(z) = z^2 + c$ is $\max\{|c|, (2/s)\}$.
(ii) The escape criterion for $Q_c(z) = z^3 + c$ is $\max\{|c|, (2/s)^{1/2}\}$.
(iii) The escape criterion for $Q_c(z) = z^4 + c$ is $\max\{|c|, (2/s)^{1/3}\}$.
(iv) The escape criterion for $Q_c(z) = z^5 + c$ is $\max\{|c|, (2/s)^{1/4}\}$.

We remark that although the escape criterion (i) for $z^2 + c$ works well for cubics, biquadratics and polynomials of higher degrees as well, yet further refined escape criterions (ii), (iii) and (iv) are preferable as it decreases execution time when a computer program executes to generate superior Mandelbrot sets. However, as n becomes large, hopefully the difference of execution times of $z^n + c$ and $z^{n+1} + c$ becomes negligible.

5. GENERATION OF SUPERIOR MANDELBROT SETS

To generate a Mandelbrot set for quadratic functions, an algorithm may be found in Devaney (1992), Holmgren (1994) and Peitgen, Jürgens & Saupe (1992c, 1992a). See a Mandelbrot set in Figure 1.

For each function of any degree, by varying the parameter s of the superior iterative procedure, different Mandelbrot sets are generated. We have written programs in C++ to generate superior Mandelbrot sets. Some of the exciting figures are presented as well.

5.1. Superior mandelbrot sets for $Q_c(z) = z^2 + c$

For $s = 1$, we get usual Mandelbrot set. See superior Mandelbrot sets for $s = 0.8, 0.5, 0.3$ and 0.01 in Figures 2, 3, 4 and 5, respectively.

5.2. Superior mandelbrot sets for $Q_c(z) = z^3 + c$

When $s = 1$, we obtain the usual Mandelbrot set (see Figure 6). Superior Mandelbrot sets are constructed in Figure 7 and Figure 8 when $s = 0.5$ and $s = 0.1$ respectively. Notice that a dotted horizontal line drawn through any of these figures (as shown in Figure 6) divides these figures into two equal parts.

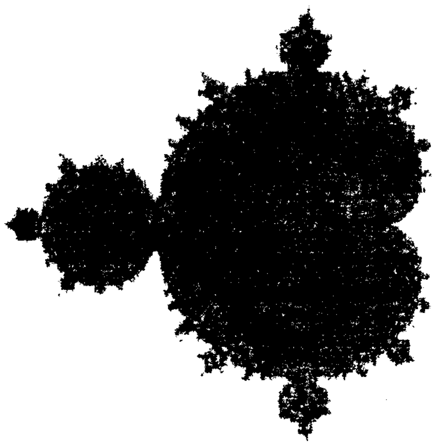


Figure 1. Mandelbrot set for quadratic polynomial

Figure 2. Superior Mandelbrot set for quadratic polynomial with $s = 0.8$

5.3. Superior mandelbrot sets for $Q_c(z) = z^4 + c$

Superior Mandelbrot sets for the biquadratic function in Figure 9 and Figure 10 correspond to $s = 1$ and 0.1 respectively. Note that Figure 9 and Figure 10 may be divided into three equal parts. Dotted lines in Figure 9 show the subdivisions.

Figure 3. Superior Mandelbrot set for quadratic polynomial with $s = 0.5$

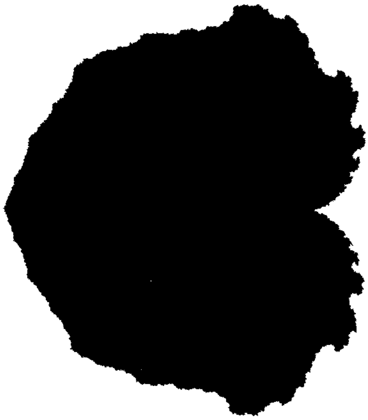


Figure 4. Superior Mandelbrot set for quadratic polynomial with $s = 0.3$

5.4. Superior mandelbrot sets for $Q_c(z) = z^5 + c$

We have generated superior Mandelbrot sets for $z^5 + c$ taking $s = 1$ and $s = 0.1$ (see Figure 11 and Figure 12 respectively). Notice that both the figures may be divided equally into four parts. This has been shown in Figure 12 by drawing dotted lines.

A glance at Figures 1–12 suggests a striking feature that each figure is formed of $(n-1)$ equal parts, where n is the degree of the polynomial $Q_c(z) = z^n + c$. So, if $n \to \infty$ in the superior Mandelbrot set, then the total number of equal components (of this set) will tend to ∞.

Figure 5. Superior Mandelbrot set for quadratic polynomial with $s = 0.01$

Figure 6. Superior Mandelbrot set for cubic polynomial with $s = 1$

Should we conjecture that the superior Mandelbrot sets for $Q_c(z) = z^n + c$ will look like a circle when n is large?

Figure 7. Superior Mandelbrot set for cubic polynomial with $s = 0.5$

Figure 8. Superior Mandelbrot set for cubic polynomial with $s = 0.1$

6. CONCLUDING REMARKS

We have used the superior escape criterions (*cf.* Section 4) to generate corresponding superior Mandelbrot sets for $z^n + c$, $n = 2, 3, 4, 5$. Unlike the usual Mandelbrot set for a

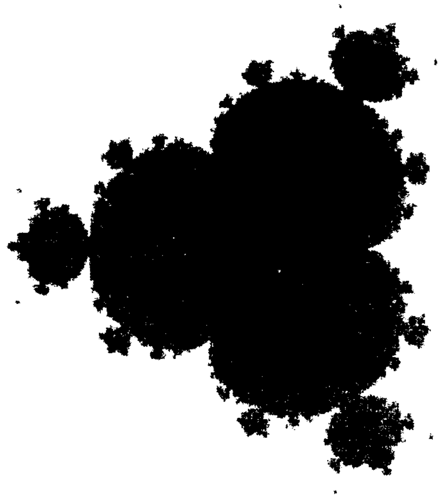


Figure 9. Superior Mandelbrot set for biquadratic polynomial with $s = 1$

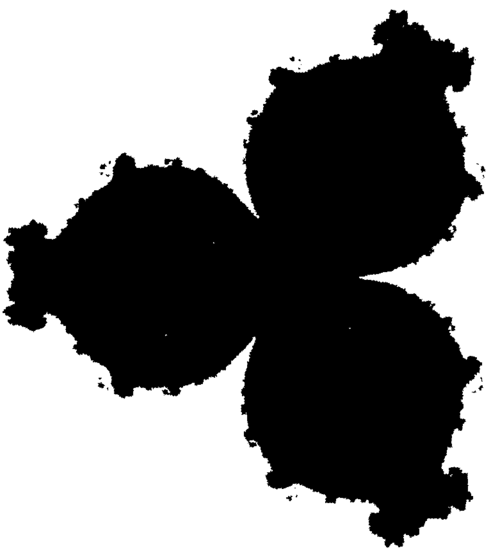


Figure 10. Superior Mandelbrot set for biquadratic polynomial with $s = 0.1$

function, we obtain more than one superior Mandelbrot set for the function, depending on the value of s. Notice that for each value of s *in a particular function*, there are different domains of c. Consequently, an entirely different superior Mandelbrot set is obtained for

Figure 11. Superior Mandelbrot set for fifth degree polynomial with $s = 1$

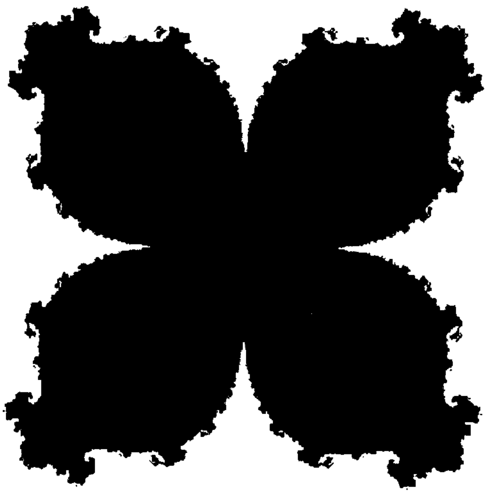


Figure 12. Superior Mandelbrot set for fifth degree polynomial with $s = 0.1$

each s. This new concept of many superior Mandelbrot sets for a function includes all the prejudices as a special case only.

In the study of the function $z^n + c$ $(n \geq 2)$, we notice that each superior Mandelbrot set is a composition of $n - 1$ equal parts. So, when $n \to \infty$, total number of equal parts in the superior Mandelbrot set also tends to infinite. This raises a question. What will be the shape of the limiting superior Mandelbrot set for $z^n + c$ as $n \to \infty$. Should we conjecture a circle?

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