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Sharp Expectation Bounds on Extreme Order Statistics from Possibly Dependent Random Variables[1]

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Abstract

In this paper, we derive sharp upper and lower expectation bounds on the extreme order statistics from possibly dependent random variables whose marginal distributions are only known. The marginal distributions of the considered random variables may not be the same and the expectation bounds are completely determined by the marginal distributions only.

Keywords : Expectation bounds, extreme order statistics, dependent random variables.

1. Introduction

Suppose that we are given a set of possibly dependent random variables and that only their marginal distributions are known. Then it is generally not possible to compute the expectations of their products or their order statistics. Knowing bounds on these expectations therefore plays an important role in various fields of probability theory and statistics. The Cauchy-Schwarz inequality is one example of these.

Upper bounds on the expectations of order statistics from an independent sample were dealt with by Moriguti (1953), Hartley and David (1954), Gumbel (1954), Balakrishnan (1990) and Huang (1998). Some of these results were extended to the case of a dependent sample by Lai and Robbins (1978) and Arnold (1985). The most general form for expectation bounds on order statistics from a dependent sample was given by Papadatos (2001). Assuming that $X_1, \ldots, X_n$ are a possibly dependent sample from a distribution function F with finite mean, he established that $E(X_{k:n})$, the expectation of the kth smallest order statistic from the sample $X_1, \ldots, X_n$, satisfies

$$\frac{n}{k} \int_0^{k/n} F^{-1}(u)\, du \le E(X_{k:n}) \le \frac{n}{n+1-k} \int_{(k-1)/n}^{1} F^{-1}(u)\, du, \quad k=1, \ldots, n, \tag{1.1}$$

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where F^{-1} denotes the generalized inverse function of F and is defined by

$$F^{-1}(u) := \inf\{x \in \mathbb{R} : F(x) \geq u\}, \quad u \in (0,1).$$

In fact, Papadatos (2001) used the concept of maximal and minimal stability to derive expectation bounds on order statistics from arbitrary random variables. Let $X_1, \ldots, X_n$ be arbitrary, possibly dependent, random variables with possibly different marginal distributions. We say that $X_1, \ldots, X_n$ are maximally (or minimally) stable of order j for some $j = 1, \ldots, n$ if the distribution of $\max\{X_{k_1}, \ldots, X_{k_j}\}$ (or $\min\{X_{k_1}, \ldots, X_{k_j}\}$) is the same for all $1 \leq k_1 < \cdots < k_j \leq n$. In particular, Papadatos (2001) showed that if $X_1, \ldots, X_n$ are maximally (or minimally) stable of order j for some $j = 1, \ldots, n$ and $E|X_i| < \infty$ for all $i = 1, \ldots, n$, then

$$E(X_{n:n}) \geq \int_0^1 F_{(j)}^{-1}(u)\, du \quad \left(\text{or } E(X_{1:n}) \leq \int_0^1 G_{(j)}^{-1}(u)\, du\right), \tag{1.2}$$

where $F_{(j)}^{-1}$ and $G_{(j)}^{-1}$ denote the generalized inverse functions of $F_{(j)}$ and $G_{(j)}$, respectively, which are defined by

$$F_{(j)}(x) = P\{X_1 \leq x, \ldots, X_j \leq x\}, \quad x \in \mathbb{R},$$
$$G_{(j)}(x) = 1 - P\{X_1 > x, \ldots, X_j > x\}, \quad x \in \mathbb{R}.$$

To compute the expectation bounds in (1.2), one has to know the explicit forms of the dependent functions $F_{(j)}$ and $G_{(j)}$, which are in fact the distribution functions of $\max\{X_1, \ldots, X_j\}$ and $\min\{X_1, \ldots, X_j\}$, respectively. For a larger j, one may easily expect that the bounds in (1.2) become sharper, since more information about the dependent structure of the random variables is then utilized. Note that for $j = n$, the inequalities in (1.2) should be replaced by equalities, i.e.

$$E(X_{n:n}) = \int_0^1 F_{(n)}^{-1}(u)\, du, \quad E(X_{1:n}) = \int_0^1 G_{(n)}^{-1}(u)\, du,$$

since $F_{(n)}$ and $G_{(n)}$ are actually the distribution functions of $X_{n:n}$ and $X_{1:n}$, respectively.

In this paper, we deal with arbitrary, possibly dependent, random variables $X_1, \ldots, X_n$ with possibly different marginal distributions. Unlike the results in (1.2), we here assume that the dependent structure of the random variables is totally unknown. Without using any information about the dependent structure, we derive sharp upper and lower expectation bounds on their extreme order statistics $X_{n:n}$ and $X_{1:n}$ based on their marginal distributions only. When the marginal distributions are all equal, our bounds coincide with the bounds in (1.1). The result, for instance, might answer whether it is possible to construct a random vector $(X_1, \ldots, X_n)$ with given marginal distributions such that the expectation of their maximum $X_{n:n}$ or minimum $X_{1:n}$ coincides with a specified value.

2. Main Results

We begin with the case of two possibly dependent random variables X_1 and X_2 with possibly different marginal distributions. By $x_1 \vee x_2$ and $x_1 \wedge x_2$ we denote $\max\{x_1, x_2\}$ and $\min\{x_1, x_2\}$, respectively.

Theorem 2.1. *Let X_1 and X_2 be two possibly dependent random variables with possibly different distribution functions F_1 and F_2, respectively. Assume that $E|X_1| < \infty$ and $E|X_2| < \infty$. Then, we have*

(i) $E(F_1^{-1}(U) \vee F_2^{-1}(U)) \le E(X_1 \vee X_2) \le E(F_1^{-1}(U) \vee F_2^{-1}(1-U)),$ (2.1)

(ii) $E(F_1^{-1}(U) \wedge F_2^{-1}(1-U)) \le E(X_1 \wedge X_2) \le E(F_1^{-1}(U) \wedge F_2^{-1}(U)),$ (2.2)

where U is a uniform(0,1) random variable.

Proof. (i) From the Bonferroni inequality, we have

$$(F_1(x) + F_2(x) - 1) \vee 0 \le G(x) := P\{X_1 \vee X_2 \le x\} \le F_1(x) \wedge F_2(x), \quad x \in \mathbb{R}. \tag{2.3}$$

Let G_0 be the distribution function of the random variable $F_1^{-1}(U) \vee F_2^{-1}(1-U)$. Then

$$\begin{aligned} G_0(x) &= P\{F_1^{-1}(U) \vee F_2^{-1}(1-U) \le x\} = P\{F_1^{-1}(U) \le x, F_2^{-1}(1-U) \le x\} \\ &= P\{U \le F_1(x), 1-U \le F_2(x)\} = (F_1(x) + F_2(x) - 1) \vee 0, \quad x \in \mathbb{R}. \end{aligned}$$

Substituting $G_0(x)$ for the lower bound of $G(x)$ in (2.3) and taking generalized inverses there, we thus have

$$F_1^{-1}(u) \vee F_2^{-1}(u) \le G^{-1}(u) \le G_0^{-1}(u), \quad u \in (0,1).$$

Thus,

$$E(F_1^{-1}(U) \vee F_2^{-1}(U)) \le E(G^{-1}(U)) \le E(G_0^{-1}(U)),$$

which completes the proof of (2.1) since $E(G^{-1}(U)) = E(X_1 \vee X_2)$ and $E(G_0^{-1}(U)) = E(F_1^{-1}(U) \vee F_2^{-1}(1-U))$.

(ii) It is easy to show that

$$F_1(x) \vee F_2(x) \le H(x) := P\{X_1 \wedge X_2 \le x\} \le (F_1(x) + F_2(x)) \wedge 1, \quad x \in \mathbb{R}. \tag{2.4}$$

Let H_0 be the distribution function of the random variable $F_1^{-1}(U) \wedge F_2^{-1}(1-U)$. Then

$$\begin{aligned} H_0(x) &= P\{F_1^{-1}(U) \wedge F_2^{-1}(1-U) \le x\} = P\{F_1^{-1}(U) \le x \text{ or } F_2^{-1}(1-U) \le x\} \\ &= P\{F_1^{-1}(U) \le x\} + P\{F_2^{-1}(1-U) \le x\} - P\{F_1^{-1}(U) \le x, F_2^{-1}(1-U) \le x\} \\ &= F_1(x) + F_2(x) - (F_1(x) + F_2(x) - 1) \vee 0 = (F_1(x) + F_2(x)) \wedge 1, \quad x \in \mathbb{R}. \end{aligned}$$

Substituting $H_0(x)$ for the upper bound of $H(x)$ in (2.4) and taking generalized inverses there, we thus have

$$H_0^{-1}(u) \le H^{-1}(u) \le F_1^{-1}(u) \wedge F_2^{-1}(u), \quad u \in (0,1).$$

Thus,

$$E(H_0^{-1}(U)) \le E(H^{-1}(U)) \le E(F_1^{-1}(U) \wedge F_2^{-1}(U)),$$

which completes the proof of (2.2) since $E(H^{-1}(U)) = E(X_1 \wedge X_2)$ and $E(H_0^{-1}(U)) = E(F_1^{-1}(U) \wedge F_2^{-1}(1-U))$. □

The bounds in Theorem 2.1 are sharp in the sense that for instance, the upper and lower bounds in (2.1) are attainable when $X_1 = F_1^{-1}(U)$, $X_2 = F_2^{-1}(1-U)$ and $X_1 = F_1^{-1}(U)$, $X_2 = F_2^{-1}(U)$, respectively. If $F_1 = F_2$, then the bounds in Theorem 2.1 coincide with the bounds in (1.1) with $n=2$ and $F = F_1 = F_2$.

Corollary 2.1. *For any two events* A *and* B, *we have*

(i) $P(A) \vee P(B) \le P(A \cup B) \le (P(A) + P(B)) \wedge 1$, (2.5)

(ii) $(P(A) + P(B) - 1) \vee 0 \le P(A \cap B) \le P(A) \wedge P(B)$. (2.6)

Proof. Let U be a uniform(0,1) random variable and write $p = P(A^c)$ and $q = P(B^c)$. Let I_A denote the indicator function of the set A.

(i) Since $P(A \cup B) = E(I_A \vee I_B)$, we have from (2.1)

$$\begin{aligned} P(A \cup B) &\le E(I_{(p,1)}(U) \vee I_{(q,1)}(1-U)) = E(I_{(p,1)}(U) \vee I_{(0,1-q)}(U)) \\ &= 1 + (1-p-q) \wedge 0 = (P(A) + P(B)) \wedge 1 \end{aligned}$$

and

$$P(A \cup B) \ge E(I_{(p,1)}(U) \vee I_{(q,1)}(U)) = 1 - p \wedge q = P(A) \vee P(B),$$

which completes the proof of (2.5).

(ii) Since $P(A \cap B) = E(I_A \wedge I_B)$, we have from (2.2)

$$P(A \cap B) \le E(I_{(p,1)}(U) \wedge I_{(q,1)}(U)) = E(I_{(p \vee q,1)}(U)) = 1 - p \vee q = P(A) \wedge P(B)$$

and

$$\begin{aligned} P(A \cap B) &\ge E(I_{(p,1)}(U) \wedge I_{(q,1)}(1-U)) = E(I_{(p,1)}(U) \wedge I_{(0,1-q)}(U)) \\ &= (1-q-p) \vee 0 = (P(A) + P(B) - 1) \vee 0, \end{aligned}$$

which completes the proof of (2.6). □

We briefly introduce two examples.

Example 2.1. Let X_1 and X_2 be two possibly dependent uniform(0,1) random variables. Let

F be the common distribution function of X_1 and X_2. Then $F^{-1}(U)=U$, where $U\sim$ uniform(0,1), so that

$$E(F^{-1}(U)\vee F^{-1}(U))=E(U)=1/2,$$
$$E(F^{-1}(U)\vee F^{-1}(1-U))=E(U\vee(1-U))=3/4,$$
$$E(F^{-1}(U)\wedge F^{-1}(U))=1/2,$$
$$E(F^{-1}(U)\wedge F^{-1}(1-U))=1/4.$$

Thus, by Theorem 2.1, we have

$$1/2\leq E(X_1\vee X_2)\leq 3/4,\quad 1/4\leq E(X_1\wedge X_2)\leq 1/2.$$

In particular, if X_1 and X_2 are independent, one can easily find that $E(X_1\vee X_2)=2/3$ and $E(X_1\wedge X_2)=1/3$.

Example 2.2. Let $X_1\sim$uniform(0,1) and $X_2\sim$exponential(1) be possibly dependent. Let F_1 and F_2 be the distribution functions of X_1 and X_2, respectively. Then $F_1^{-1}(U)=U$ and $F_2^{-1}(U)=-\log(1-U)$, where $U\sim$uniform(0,1). Using the program of Mathematica, we get

$$E(F_1^{-1}(U)\vee F_2^{-1}(U))=E(U\vee(-\log(1-U)))=1,$$
$$E(F_1^{-1}(U)\vee F_2^{-1}(1-U))=E(U\vee(-\log U))=1.22797,$$
$$E(F_1^{-1}(U)\wedge F_2^{-1}(U))=0.5,$$
$$E(F_1^{-1}(U)\wedge F_2^{-1}(1-U))=0.272031.$$

Thus, by Theorem 2.1, we have

$$1\leq E(X_1\vee X_2)\leq 1.22797,\quad 0.272031\leq E(X_1\wedge X_2)\leq 0.5.$$

In particular, if X_1 and X_2 are independent, one can easily find that $E(X_1\vee X_2)=3/2-e^{-1}=1.13212$ and $E(X_1\wedge X_2)=e^{-1}=0.367879$.

The result of Theorem 2.1 can be extended to the general case of n $(n\geq 3)$ possibly dependent random variables with possibly different marginal distributions if the marginal distributions are all continuous. By $\vee_{i=1}^{n}x_i$ and $\wedge_{i=1}^{n}x_i$ we denote $\max\{x_1,\ldots,x_n\}$ and $\min\{x_1,\ldots,x_n\}$, respectively.

Theorem 2.2. *Let $X_1,\ldots,X_n$ $(n\geq 3)$ be possibly dependent random variables with possibly different distribution functions $F_1,\ldots,F_n$, respectively. Assume that $E|X_i|<\infty$ for all $i=1,\ldots,n$. If $F_1,\ldots,F_n$ are continuous, then we have*

(i) $E(\vee_{i=1}^{n}F_i^{-1}(U))\leq E(\vee_{i=1}^{n}X_i)\leq E(\vee_{i=1}^{n}Y_i),$ (2.7)

(ii) $E(\wedge_{i=1}^{n} Z_i) \le E(\wedge_{i=1}^{n} X_i) \le E(\wedge_{i=1}^{n} F_i^{-1}(U)),$ (2.8)

where U *is a uniform(0,1) random variable, and* $\{Y_1, \ldots, Y_n\}$ *and* $\{Z_1, \ldots, Z_n\}$ *are defined recursively by*

$Y_1 = F_1^{-1}(U), \quad M_1 = Y_1, \quad G_1 = F_1,$

$Y_k = F_k^{-1}(1 - G_{k-1}(M_{k-1})), \quad M_k = \vee_{i=1}^{k} Y_i, \quad G_k$*: distribution function of* M_k, $k = 2, \ldots, n$

and

$Z_1 = F_1^{-1}(U), \quad N_1 = Z_1, \quad H_1 = F_1,$

$Z_k = F_k^{-1}(1 - H_{k-1}(N_{k-1})), \quad N_k = \wedge_{i=1}^{k} Z_i, \quad H_k$*: distribution function of* N_k, $k = 2, \ldots, n.$

Proof. (i) Due to the Fréchet bounds, we have

$$(\sum_{i=1}^{n} F_i(x) - n + 1) \vee 0 \le G(x) := P\{\vee_{i=1}^{n} X_i \le x\} \le \wedge_{i=1}^{n} F_i(x), \quad x \in \mathbb{R}. \tag{2.9}$$

By induction, we will show that for $k = 1, \ldots, n$,

$$G_k(x) = (\sum_{i=1}^{k} F_i(x) - k + 1) \vee 0, \quad x \in \mathbb{R}. \tag{2.10}$$

Clearly, (2.10) holds for $k = 1$. Suppose (2.10) holds for $k = j \in \{1, \ldots, n-1\}$. Then

$$\begin{aligned} G_{j+1}(x) &= P\{M_{j+1} \le x\} = P\{M_j \le x, Y_{j+1} \le x\} \\ &= P\{M_j \le x, F_{j+1}^{-1}(1 - G_j(M_j)) \le x\} \\ &= P\{G_j^{-1}(U) \le x, F_{j+1}^{-1}(1 - G_j(G_j^{-1}(U))) \le x\} \\ &= P\{U \le G_j(x), 1 - U \le F_{j+1}(x)\} \\ &= (G_j(x) + F_{j+1}(x) - 1) \vee 0 = (\sum_{i=1}^{j+1} F_i(x) - j) \vee 0, \quad x \in \mathbb{R}, \end{aligned}$$

since $G_j(G_j^{-1}(U)) = U$ from the continuity of G_j. This implies that (2.10) also holds for $k = j+1$. Thus (2.10) holds for $k = 1, \ldots, n$. Substituting $G_n(x)$ for the lower bound of $G(x)$ in (2.9) and taking generalized inverses there, we have

$$\vee_{i=1}^{n} F_i^{-1}(u) \le G^{-1}(u) \le G_n^{-1}(u), \quad u \in (0,1).$$

Thus,

$$E(\vee_{i=1}^{n} F_i^{-1}(U)) \le E(G^{-1}(U)) \le E(G_n^{-1}(U)),$$

which completes the proof of (2.7) since $E(G^{-1}(U)) = E(\vee_{i=1}^{n} X_i)$ and $E(G_n^{-1}(U)) = E(M_n) = E(\vee_{i=1}^{n} Y_i)$.

(ii) We start with an obvious relation

$$\vee_{i=1}^{n} F_i(x) \le H(x) := P\{\wedge_{i=1}^{n} X_i \le x\} \le (\sum_{i=1}^{n} F_i(x)) \wedge 1, \quad x \in \mathbb{R}. \tag{2.11}$$

By induction, we will show that for $k = 1, \ldots, n$,

$$H_k(x) = (\sum_{i=1}^{k} F_i(x)) \wedge 1, \quad x \in \mathbb{R}. \tag{2.12}$$

Clearly, (2.12) holds for $k=1$. Suppose (2.12) holds for $k=j \in \{1, \dots, n-1\}$. Then

$$\begin{aligned} H_{j+1}(x) &= P\{N_{j+1} \le x\} = P\{N_j \le x \text{ or } Z_{j+1} \le x\} \\ &= P\{N_j \le x\} + P\{Z_{j+1} \le x\} - P\{N_j \le x, Z_{j+1} \le x\} \\ &= H_j(x) + F_{j+1}(x) - (H_j(x) + F_{j+1}(x) - 1) \vee 0 \\ &= (H_j(x) + F_{j+1}(x)) \wedge 1 = (\sum_{i=1}^{j+1} F_i(x)) \wedge 1, \quad x \in \mathbb{R}, \end{aligned}$$

since

$$\begin{aligned} P\{N_j \le x, Z_{j+1} \le x\} &= P\{N_j \le x, F_{j+1}^{-1}(1 - H_j(N_j)) \le x\} \\ &= P\{H_j^{-1}(U) \le x, F_{j+1}^{-1}(1 - H_j(H_j^{-1}(U))) \le x\} \\ &= P\{U \le H_j(x), 1 - U \le F_{j+1}(x)\} \\ &= (H_j(x) + F_{j+1}(x) - 1) \vee 0 \end{aligned}$$

from the continuity of H_j. This implies that (2.12) also holds for $k=j+1$. Thus (2.12) holds for $k=1, \dots, n$. Substituting $H_n(x)$ for the upper bound of $H(x)$ in (2.11) and taking generalized inverses there, we have

$$H_n^{-1}(u) \le H^{-1}(u) \le \wedge_{i=1}^{n} F_i^{-1}(u), \quad u \in (0,1).$$

Thus,

$$E(H_n^{-1}(U)) \le E(H^{-1}(U)) \le E(\wedge_{i=1}^{n} F_i^{-1}(U)),$$

which completes the proof of (2.8) since $E(H^{-1}(U)) = E(\wedge_{i=1}^{n} X_i)$ and $E(H_n^{-1}(U)) = E(N_n)$ $= E(\wedge_{i=1}^{n} Z_i)$. □

The bounds in Theorem 2.2 are sharp. For instance, the upper and lower bounds in (2.7) are attainable when $X_i = Y_i$, $i=1, \dots, n$ and $X_i = F_i^{-1}(U)$, $i=1, \dots, n$, respectively. Notice that for each $i=1, \dots, n$, Y_i and Z_i have the same distribution function F_i. If $F_1 = \cdots = F_n$, then the bounds in Theorem 2.2 coincide with the bounds in (1.1) as seen in the next corollary.

Corollary 2.2. *Let $X_1, \dots, X_n$ $(n \ge 3)$ be a possibly dependent sample from a distribution function F with finite mean. If F is continuous, then we have*

$$\text{(i)} \quad \int_0^1 F^{-1}(u)\,du \le E(\vee_{i=1}^{n} X_i) \le n \int_{(n-1)/n}^{1} F^{-1}(u)\,du, \tag{2.13}$$

$$\text{(ii)} \quad n \int_0^{1/n} F^{-1}(u)\,du \le E(\wedge_{i=1}^{n} X_i) \le \int_0^1 F^{-1}(u)\,du. \tag{2.14}$$

Proof. Let U be a uniform(0,1) random variable.

(i) Since $G_n(x)=(nF(x)-n+1)\vee 0$, $x\in\mathbb{R}$ (see (2.10)), we have

$$G_n^{-1}(u)=F^{-1}((u+n-1)/n),\quad u\in(0,1).$$

From (2.7), we thus have

$$E(F^{-1}(U))\le E(\vee_{i=1}^n X_i)\le E(\vee_{i=1}^n Y_i)=E(G_n^{-1}(U))=E(F^{-1}((U+n-1)/n)),$$

which completes the proof of (2.13).

(ii) Since $H_n(x)=(nF(x))\wedge 1$, $x\in\mathbb{R}$ (see (2.12)), we have $H_n^{-1}(u)=F^{-1}(u/n)$, $u\in(0,1)$. From (2.8), we thus have

$$E(F^{-1}(U))\ge E(\wedge_{i=1}^n X_i)\ge E(\wedge_{i=1}^n Z_i)=E(H_n^{-1}(U))=E(F^{-1}(U/n)),$$

which completes the proof of (2.14). □

We end this section with one more example.

Example 2.3. A random variable is said to have a BurrIV(α,γ) distribution for some $\alpha>0$ and $\gamma>0$ if its distribution function is given by $F(x)=[(\alpha/x-1)^{1/\alpha}+1]^{-\gamma}$, $0<x<\alpha$. Let $X_1\sim$uniform(0,1), $X_2\sim$BurrIV(1,2) and $X_3/2\sim$uniform(0,1) be possibly dependent. Then the distribution functions of X_1, X_2 and X_3 are respectively $F_1(x)=x$ and $F_2(x)=x^2$ for $x\in[0,1]$ and $F_3(x)=x/2$, $0\le x\le 2$, and so $F_1^{-1}(u)=u$, $F_2^{-1}(u)=\sqrt{u}$ and $F_3^{-1}(u)=2u$ for $u\in(0,1)$. Thus,

$$\vee_{i=1}^3 F_i^{-1}(u)=\begin{cases}\sqrt{u} & \text{if } 0<u\le 1/4,\\ 2u & \text{if } 1/4<u<1,\end{cases}$$

and $\wedge_{i=1}^3 F_i^{-1}(u)=u$, $0<u<1$. Also,

$$G_3(x)=\Big(\sum_{i=1}^3 F_i(x)-2\Big)\vee 0=\begin{cases}x^2+3x/2-2 & \text{if } (-3+\sqrt{41})/4\le x\le 1,\\ x/2 & \text{if } 1<x\le 2,\end{cases}$$

and $H_3(x)=\Big(\sum_{i=1}^3 F_i(x)\Big)\wedge 1=x^2+3x/2$, $0\le x\le 1/2$, so that

$$G_3^{-1}(u)=\begin{cases}(-3+\sqrt{41+16u})/4 & \text{if } 0<u\le 1/2,\\ 2u & \text{if } 1/2<u<1,\end{cases}$$

and $H_3^{-1}(u)=(-3+\sqrt{9+16u})/4$, $0<u<1$. An easy computation yields that

$$E(\vee_{i=1}^3 F_i^{-1}(U))=49/48=1.02083,$$

$$E(G_3^{-1}(U))=(379-41\sqrt{41})/96=1.21325,$$

$$E(\wedge_{i=1}^3 F_i^{-1}(U))=0.5,$$

$$E(H_3^{-1}(U))=13/48=0.270833,$$

where $U \sim$ uniform(0,1). Thus, by Theorem 2.2, we have

$$1.02083 \le E(\vee_{i=1}^{3} X_i) \le 1.21325, \quad 0.270833 \le E(\wedge_{i=1}^{3} X_i) \le 0.5.$$

3. Conclusion

In this paper, we considered possibly dependent random variables $X_1, \ldots, X_n$ with possibly different distribution functions $F_1, \ldots, F_n$, respectively, and derived upper and lower expectation bounds on the extreme order statistics $\vee_{i=1}^{n} X_i$ and $\wedge_{i=1}^{n} X_i$ based only on $F_1, \ldots, F_n$. The expectation bounds are sharp in the sense that they are attainable. Our formulas for the expectation bounds are particularly helpful to notice when they are attainable and how the corresponding dependent structures of $X_1, \ldots, X_n$ are composed of. We did not cover the expectation bound on an arbitrary order statistic from $X_1, \ldots, X_n$, which is left as a further research topic.

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