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CONVERGENCE OF WEIGHTED U-EMPIRICAL PROCESSES†

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ABSTRACT

In this paper, we define the weighted U-empirical process for simple linear model and show the weak convergence to a Gaussian process under some conditions. Then we illustrate the usage of our result with examples. In the appendix, we derive the variance of the weighted U-empirical distribution function.

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1. INTRODUCTION

Consider the simple linear model

$$Y_i = \beta_0 + \beta_1 x_i + \varepsilon_i, \quad 1 \le i \le n,$$

where $\varepsilon_i{}'s$ are independent and identically distributed random variables with a common unknown distribution function F, $x_i{}'s$ are known covariates, β_1 is the parameter of our interest and β_0, the nuisance parameter. Without loss of generality, we may assume that $x_1 \le x_2 \le \cdots \le x_n$ with at least one strict inequality. Also we assume that the distribution function F is uniformly continuous. Several nonparametric procedures for the inferences about β_1 have been carried out based on the following differences

$$\varepsilon_j - \varepsilon_i = Y_j - Y_i - \beta_1(x_j - x_i) \tag{1.1}$$

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such as Wilcoxon rank sum procedure for the two sample problem and Sen (1968)'s procedure for the regression setting. From now on, we assume that $\beta_0 = 0$ since β_0 disappears in the expression (1.1). Sievers (1978) also considered the inferences about β_1 based on (1.1) but used the following weighted rank statistics defined by

$$S_n(\beta_1) = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} w_{ij} I[Y_j - Y_i \leq \beta_1(x_j - x_i)],$$

where the weights $w_{ij} \geq 0$ with $w_{ij} = 0$ whenever $x_i = x_j$. We note that $S_n(0)$ is the Wilcoxon rank sum statistic or Sen's statistic for testing $H_0 : \beta_1 = 0$ if $w_{ij} = 1$ for $x_i < x_j$. Sievers considered obtaining the point and interval estimates and proposed test procedures for β_1 based on $S_n(\beta_1)$ by varying the weights. Now we note that $(Y_j - Y_i)/(x_j - x_i)$ can be considered as a kernel for β_1 since $(Y_j - Y_i)/(x_j - x_i)$ is an unbiased estimate of β_1 for $x_i \neq x_j$. For this reason, Serfling (1980) named $S_n(\beta_1)$ the weighted U-statistics. Silverman (1983) considered a class of empirical processes having the structure of U-statistics for one sample setting and showed the weak convergence of the processes to a continuous Gaussian process. Also O'Neil and Redner (1993) and Major (1994) considered obtaining limiting distributions for the weighted U-statistics based on the *iid* setting. In this paper, we consider the weak convergence of the processes having the weighted U-statistics structure under the linear model. In the following, we will assume that $\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} w_{ij} = 1$. Then we may define

$$G_n(y) = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} w_{ij} I[Y_j - Y_i \leq y + \beta_1(x_j - x_i)]$$

as the weighted U-empirical distribution function. Before we proceed further, we introduce several notation for the later use. For each k, $1 \leq k \leq n$, let $w_{k\cdot} = \sum_{i=k+1}^{n} w_{ki}$ and $w_{\cdot k} = \sum_{i=1}^{k-1} w_{ik}$ with the notations that $w_{\cdot 1} = 0$ and $w_{n\cdot} = 0$. Also let $w_{1n}^2 = \sum_{k=1}^{n} w_{k\cdot}^2$, $w_{2n}^2 = \sum_{k=1}^{n} w_{\cdot k}^2$ and $w_{12n} = \sum_{k=1}^{n} w_{k\cdot} w_{\cdot k}$. Finally, let $w_n^2 = \sum_{k=1}^{n} (w_{k\cdot} - w_{\cdot k})^2$. Then we define the weighted U-empirical process as follows: For each $y \in (-\infty, \infty)$,

$$W_n(y) = \frac{1}{w_n}\{G_n(y) - G(y)\},$$

where $w_n = (w_n^2)^{1/2}$ and $G(y) = E\{I[Y_j - Y_i \leq y + \beta_1(x_j - x_i)]\}$ is the distribution function of $\varepsilon_j - \varepsilon_i = Y_j - Y_i - \beta_1(x_j - x_i)$.

In the next section, we show the weak convergence of W_n to W, which is a Gaussian process on $D(-\infty, \infty)$.

2. WEAK CONVERGENCE

Before we prove the weak convergence of W_n to a Gaussian process W, first of all, we show the convergence of finite dimensional distribution of W_n and obtain the covariance function. For this purpose, we employ the method of projection (*cf.* Hájek, 1968). For each k, let

$$\begin{aligned} h(y;u) &= E\{G_n(y) - G(y)|Y_k = u\} \\ &= \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} w_{ij}E\{I[Y_j - Y_i \le y + \beta_1(x_j - x_i)] - G(y)|Y_k = u\} \\ &= \sum_{i=k+1}^{n} w_{ki}\{F(y+u-\beta_1 x_k) - G(y)\} \\ &\qquad - \sum_{i=1}^{k-1} w_{ik}\{F(-y+u-\beta_1 x_k)^- - (1-G(y))\} \\ &= w_{k\cdot}\{F(y+u-\beta_1 x_k) - G(y)\} - w_{\cdot k}\{F(-y+u-\beta_1 x_k)^- - (1-G(y))\}, \end{aligned}$$

where $F(-y+u-\beta_1 x_k)^- = P\{Y_j - \beta_1 x_j < -y+u-\beta_1 x_k\}$. Then we note that by the change-of-variable technique,

$$\begin{aligned} E\{F(y+Y_k-\beta_1 x_k)\} &= G(y), \\ \mathrm{Var}\{F(y+Y_k-\beta_1 x_k)\} &= \int_{-\infty}^{\infty}\{F(y+u) - G(y)\}^2 dF(u), \qquad (2.1) \\ E\{F(-y+Y_k-\beta_1 x_k)^-\} &= 1 - G(y) \end{aligned}$$

and

$$\mathrm{Var}\{F(-y+Y_k-\beta_1 x_k)^-\} = \int_{-\infty}^{\infty}\{F(-y+u)^- - (1-G(y))\}^2 dF(u). \qquad (2.2)$$

Thus we have that for each k,

$$E\{h(y;Y_k)\} = 0$$

and

$$\begin{aligned} &\mathrm{Var}\{h(y;Y_k)\} \\ &= w_{k\cdot}^2 \int_{-\infty}^{\infty}\{F(y+u) - G(y)\}^2 dF(u) \qquad (2.3) \\ &\qquad -2w_{k\cdot}w_{\cdot k}\int_{-\infty}^{\infty}\{F(y+u)-G(y)\}\{F(-y+u)^- - (1-G(y))\}dF(u) \\ &\qquad +w_{\cdot k}^2\int_{-\infty}^{\infty}\{F(-y+u)^- - (1-G(y))\}^2 dF(u). \end{aligned}$$

Now let for each $y \in (-\infty, \infty)$,

$$W_{1n}(y) = \frac{1}{w_{1n}} \sum_{k=1}^{n} w_{k\cdot}\{F(y + Y_k - \beta_1 x_k) - G(y)\}$$

and

$$W_{2n}(y) = \frac{1}{w_{2n}} \sum_{k=1}^{n} w_{\cdot k}\{F(-y + Y_k - \beta_1 x_k)^- - (1 - G(y))\},$$

where $w_{in} = (w_{in}^2)^{1/2}$ for each $i = 1, 2$. Also let $W_n^*(y) = \sum_{k=1}^n h(y; Y_k)/w_n$. Then we note that

$$\begin{aligned} W_n^*(y) &= \frac{w_{1n}}{w_n} \frac{1}{w_{1n}} \sum_{k=1}^{n} w_{k\cdot}\{F(y + Y_k - \beta_1 x_k) - G(y)\} \\ &\quad - \frac{w_{2n}}{w_n} \frac{1}{w_{2n}} \sum_{k=1}^{n} w_{\cdot k}\{F(-y + Y_k - \beta_1 x_k)^- - (1 - G(y))\} \\ &= \frac{w_{1n}}{w_n} W_{1n} - \frac{w_{2n}}{w_n} W_{2n}. \end{aligned}$$

We note that $W_n^*(y)$ is the projection of $W_n(y)$ onto the space of sum of independent random variables. In order to show the weak convergence of $W_n^*(y)$ to $W(y)$, we need the following two assumptions.

ASSUMPTION 1. For all i and j and for all n,

$$w_{ij} = O(n^{-2}).$$

ASSUMPTION 2. As $n \to \infty$,

$$\frac{w_{1n}}{w_n} \to \lambda_1 \quad \text{and} \quad \frac{w_{2n}}{w_n} \to \lambda_2$$

for some real numbers $\lambda_1 > 0$ and $\lambda_2 > 0$.

From Assumption 1, Noether's condition follows immediately with the assumption that $\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} w_{ij} = 1$. Therefore we have that as $n \to \infty$,

$$\max_{1 \le i \le n} \frac{w_{i\cdot}^2}{w_{1n}^2} \to 0 \quad \text{and} \quad \max_{1 \le j \le n} \frac{w_{\cdot j}^2}{w_{2n}^2} \to 0. \tag{2.4}$$

Also Assumption 2 with the definitions for w_{1n}, w_{2n}, w_{12n} and w_n implies that

$$\frac{w_{12n}}{w_n^2} \to \lambda_{12}$$

for some real number $\lambda_{12} \ge 0$. Then we have the following result.

LEMMA 2.1. *With Assumptions 1 and 2, $W_n^*(y)$ converges weakly to a Gaussian process $W(y)$ on $D(-\infty,\infty)$ with covariance function $C(y_1,y_2)$, where*

$$\begin{aligned}
&C(y_1,y_2)\\
&= \lambda_1^2\int_{-\infty}^{\infty}\{F(y_1+u)-G(y_1)\}\{F(y_2+u)-G(y_2)\}dF(u)\\
&\quad -\lambda_{12}\int_{-\infty}^{\infty}\Big[\{F(y_1+u)-G(y_1)\}\{F(-y_2+u)^- -(1-G(y_2))\}\\
&\quad +\{F(-y_1+u)^- -(1-G(y_1))\}\{F(y_2+u)-G(y_2)\}\Big]dF(u)\\
&\quad +\lambda_2^2\int_{-\infty}^{\infty}\{F(-y_1+u)^- -(1-G(y_1))\}\{F(-y_2+u)^- -(1-G(y_2))\}dF(u).
\end{aligned}$$

PROOF. The covariance structure follows immediately from (2.1) and (2.2). Also the finite dimensional convergence of $(W_n^*(y_1),\dots,W_n^*(y_p))$ to $(W(y_1),\dots,W(y_p))$ is obvious since $W_n^*(y)$ consists of independent and bounded random variables. Therefore it is enough to show the tightness. For this, we note that for any given $\varepsilon>0$ and $\delta>0$,

$$\begin{aligned}
P\Big\{\sup_{|t-s|<\delta}|W_n^*(t)-W_n^*(s)|>\varepsilon\Big\} &\le P\Big\{\sup_{|t-s|<\delta}\frac{w_{1n}}{w_n}|W_{1n}(t)-W_{1n}(s)|>\frac{\varepsilon}{2}\Big\}\\
&\quad +P\Big\{\sup_{|t-s|<\delta}\frac{w_{2n}}{w_n}|W_{2n}(t)-W_{2n}(s)|>\frac{\varepsilon}{2}\Big\}\\
&\le P\Big\{\sup_{|t-s|<\delta}|W_{1n}(t)-W_{1n}(s)|>\frac{\varepsilon}{2(\lambda+1)}\Big\}\\
&\quad +P\Big\{\sup_{|t-s|<\delta}|W_{2n}(t)-W_{2n}(s)|>\frac{\varepsilon}{2(\lambda+1)}\Big\}
\end{aligned}$$

for all sufficiently large n from the definitions for w_{1n}, w_{2n} and w_n, where $\lambda=\max\{\lambda_1,\lambda_2\}$. Also we note that $\sum_{k=1}^n(w_{k\cdot}/w_{1n})^2=1$ and $\sum_{k=1}^n(w_{\cdot k}/w_{2n})^2=1$ for all n. Therefore the conditions (N1) and (N2) in Theorem 2.2a.1 of Koul (1992, p. 11) are satisfied with (2.4). Thus the tightness follows from Theorem 2.2a.1 of Koul by noting that $W_{in}(y)$ consists of independent and bounded random variables for each $i=1,2$. □

We note that in Lemma 2.1, the two empirical processes $W_{1n}(y)$ and $W_{2n}(y)$ are orthogonal if $w_{12n}=0$. Now we show the asymptotic equivalence between $W_n(y)$ and $W_n^*(y)$ in the following sense.

LEMMA 2.2. *With Assumption 1,*

$$\lim_{n\to\infty} \sup_{-\infty<y<\infty} E[\{W_n(y) - W_n^*(y)\}^2] = 0.$$

PROOF. First of all, we note that with the double expectation theorem (*cf.* Bickel and Doksum, 1977), for each $y \in (-\infty, \infty)$,

$$\begin{aligned} E[\{W_n(y) - W_n^*(y)\}^2] &= E[\{W_n(y)\}^2] + E[\{W_n^*(y)\}^2] - 2E[W_n(y)W_n^*(y)] \\ &= E[\{W_n(y)\}^2] - E[\{W_n^*(y)\}^2] \end{aligned}$$

since

$$\begin{aligned} E\{W_n(y)W_n^*(y)\} &= \frac{1}{w_n} E\Big\{W_n(y)\sum_{k=1}^n h(y;Y_k)\Big\} \\ &= \frac{1}{w_n}\sum_{k=1}^n E\{W_n(y)h(y;Y_k)\} \\ &= \frac{1}{w_n}\sum_{k=1}^n E[h(y;Y_k)E\{W_n(y)|Y_k\}] \\ &= \frac{1}{w_n^2}\sum_{k=1}^n E\{h^2(y;Y_k)\} \\ &= E[\{W_n^*(y)\}^2] \end{aligned}$$

with the fact that $E\{\sum_{k=1}^n h(y;Y_k)\}^2 = \sum_{k=1}^n E\{h^2(y;Y_k)\}$.

Since

$$E[\{W_n(y)\}^2] = \mathrm{Var}\{W_n(y)\} = \mathrm{Var}\Big\{\frac{1}{w_n}G_n(y)\Big\}$$

and

$$E[\{W_n^*(y)\}^2] = \mathrm{Var}\{W_n^*(y)\} = \mathrm{Var}\Big\{\frac{1}{w_n}\sum_{k=1}^n h(y;Y_k)\Big\},$$

we have from Appendix,

$$\begin{aligned} &E[\{W_n(y)\}^2] - E[\{W_n^*(y)\}^2] \\ &= \frac{1}{w_n^2}\sum_{i=1}^{n-1}\sum_{j=i+1}^n w_{ij}^2\Big[G(y)(1-G(y)) - \int_{-\infty}^{\infty}\{F(y+u) - G(y)\}^2 dF(u) \\ &\qquad - \int_{-\infty}^{\infty}\{F(-y+u)^- - (1-G(y))\}^2 dF(u)\Big]. \end{aligned}$$

Therefore we may conclude with Assumption 1 that

$$\begin{aligned}
&\lim_{n\to\infty}\sup_{-\infty<y<\infty} E[\{W_n(y)-W_n^*(y)\}^2]\\
&\quad=\lim_{n\to\infty}\sup_{-\infty<y<\infty}\left\{E[\{W_n(y)\}^2]-E[\{W_n^*(y)\}^2]\right\}\\
&\quad\le\lim_{n\to\infty}\frac{1}{w_n^2}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} w_{ij}^2\\
&\quad=0
\end{aligned}$$

from the definition of w_n and Assumption 1. □

Therefore we now arrive at the following conclusion.

THEOREM 2.1. *Under Assumptions 1 and 2, $W_n(y)$ converges weakly to a Gaussian process $W(y)$ on $D(-\infty,\infty)$.*

PROOF. The finite dimensional convergence from Slutsky's Theorem is obvious with Lemma 2.2. Therefore it is enough to show the tightness. For this matter, first of all, for any $\delta>0$ and for any two real numbers s and t such that $|t-s|<\delta$, we define the modulus of continuity of W_n as follows:

$$\Omega_n(\delta)=\sup_{|t-s|\le\delta}|W_n(t)-W_n(s)|.$$

Then it is enough to show that for any $\varepsilon>0$,

$$\lim_{\delta\downarrow 0}\overline{\lim_{n\to\infty}}P\{\Omega_n(\delta)\ge\varepsilon\}=0.$$

For this, let $\alpha=(\alpha(1),\dots,\alpha(n))$ be an arbitrary permutation of $(1,\dots,n)$. Then we make pairs such as $(\alpha(2j-1),\alpha(2j))$ with consecutive two permutational numbers, where $j=1,\dots,[n/2]$, where $[a]$ is the largest integer part of a which does not exceed the real number a. If n is even, then we can obtain the complete $n/2$ number of pairs whereas if n odd, then we discard the last one $\alpha(n)$. Then we define independent random variables as follows: For each j, $j=1,\dots,[n/2]$ if $x_{\alpha(2j)}>x_{\alpha(2j-1)}$, then

$$V_j^\alpha(y)=I[Y_{\alpha(2j)}-Y_{\alpha(2j-1)}\le y+\beta_1(x_{\alpha(2j)}-x_{\alpha(2j-1)})].$$

If $x_{\alpha(2j)}<x_{\alpha(2j-1)}$, then

$$V_j^\alpha(y)=I[Y_{\alpha(2j-1)}-Y_{\alpha(2j)}\le y+\beta_1(x_{\alpha(2j-1)}-x_{\alpha(2j)})].$$

Finally, if $x_{\alpha(2j)} = x_{\alpha(2j-1)}$, then $V_j^\alpha(y) = 0$.
Also let

$$U_n^\alpha(y) = \sum_{j=1}^{[n/2]} w_j^\alpha [V_j^\alpha(y) - G(y)],$$

where w_j is the corresponding weight such as $w_j^\alpha = w_{ij}$ if $V_j^\alpha(y) = I[Y_j - Y_i \leq y + \beta_1(x_j - x_i)]$. Then we note that

$$\sum_{\text{all } \alpha} U_n^\alpha(y) = 2!(n-2)! \left[\frac{n}{2}\right] \{G_n(y) - G(y)\}$$

or

$$G_n(y) - G(y) = \frac{1}{2!(n-2)![n/2]} \sum_{\text{all } \alpha} U_n^\alpha(y).$$

Therefore

$$\begin{aligned} W_n(y) &= \frac{1}{2!(n-2)![n/2]} \frac{1}{w_n} \sum_{\text{all } \alpha} U_n^\alpha(y) \\ &= \frac{1}{n!} \frac{\binom{n}{2}}{[n/2]} \frac{1}{w_n} \sum_{\text{all } \alpha} U_n^\alpha(y) \\ &= \frac{1}{n!} \sum_{\text{all } \alpha} W_n^\alpha(y), \end{aligned}$$

where

$$W_n^\alpha(y) = \frac{\binom{n}{2}}{[n/2]} \frac{1}{w_n} U_n^\alpha(y).$$

Also let

$$\Omega_n^\alpha(\delta) = \sup_{|t-s| \leq \delta} |W_n^\alpha(t) - W_n^\alpha(s)|$$

for the modulus of continuity of W_n^α for each permutation α. We note that W_n^α consists of independent random variables for any particular permutation α, whose number of elements is at most $[n/2]$. Also we note that

$$\frac{\binom{n}{2}}{[n/2]} \frac{1}{w_n} = O(n^{3/2}).$$

Then we have that with triangle inequality,

$$\begin{aligned} E\{\Omega_n(\delta)\} &\leq \frac{1}{n!} \sum_{\text{all } \alpha} E\{\Omega_n^\alpha(\delta)\} \\ &\leq \max_\alpha E\{\Omega_n^\alpha(\delta)\}. \end{aligned}$$

Therefore by Chebyshev's inequality, we have for any $\varepsilon > 0$

$$\begin{aligned}\lim_{\delta\downarrow 0}\overline{\lim_{n\to\infty}} P\{\Omega_n(\delta)\geq\varepsilon\} &\leq \lim_{\delta\downarrow 0}\overline{\lim_{n\to\infty}}\frac{1}{\varepsilon}E\{\Omega_n(\delta)\}\\ &\leq \lim_{\delta\downarrow 0}\overline{\lim_{n\to\infty}}\frac{1}{\varepsilon}\max_{\alpha}E\{\Omega_n^{\alpha}(\delta)\}\\ &= 0,\end{aligned}$$

since for any permutation α, W_n^{α} is a weighted empirical process, which weakly converges to a normal process. □

In order to illustrate the usage of our result, we consider the following examples. First of all, suppose that the first n_1 number of observations has been taken from control group and the last $n_2 = n - n_1$ number of observations taken from the treatment group. Then this is just the two-sample location translation problem. Since the sum of weights should be one, $w_{ij} = 1/(n_1 n_2)$ if we use the uniform weights for all i and j. We note that $w_{ij} = O(n^{-2})$, which satisfies Assumption 1. Then we easily obtain that

$$w_{k\cdot} = \begin{cases} n_1^{-1}, & \text{for } 1\leq k\leq n_1,\\ 0, & \text{for } n_1+1\leq k\leq n\end{cases}$$

and

$$w_{\cdot k} = \begin{cases} 0, & \text{for } 1\leq k\leq n_1,\\ n_2^{-1}, & \text{for } n_1+1\leq k\leq n.\end{cases}$$

Thus $w_{1n}^2 = 1/n_1$, $w_{2n}^2 = 1/n_2$ and $w_{12n} = 0$. Since $w_n^2 = 1/n_1 + 1/n_2 = n/(n_1 n_2)$, we have that

$$\begin{aligned}\frac{1}{w_n^2}\mathrm{Var}\left\{\sum_{k=1}^{n} h(y;Y_k)\right\} &= \frac{n_2}{n}\int_{-\infty}^{\infty}\{F(y+u)-G(y)\}^2 dF(u)\\ &\quad+\frac{n_1}{n}\int_{-\infty}^{\infty}\{F(-y+u)^{-}-(1-G(y))\}^2 dF(u),\end{aligned}$$

which will converge to $\mathrm{Var}(W(y))$ for all y with Assumption 2 that $(n_1/n)^{1/2}\to\lambda_1$ and $(n_2/n)^{1/2}\to\lambda_2$ as $n\to\infty$. Especially when $y=0$, we obtain that

$$\begin{aligned}&\frac{1}{w_n^2}\mathrm{Var}\left\{\sum_{k=1}^{n} h(0;Y_k)\right\}\\ &= \frac{n_2}{n}\int_{-\infty}^{\infty}\{F(u)-G(0)\}^2 dF(u) + \frac{n_1}{n}\int_{-\infty}^{\infty}\{F(u)^{-}-(1-G(0))\}^2 dF(u)\end{aligned}$$

$$= \frac{1}{12}\left(\frac{n_1}{n} + \frac{n_2}{n}\right),$$

which is the variance of Wilcoxon rank sum statistic $W_n(0)$ under $H_0 : \beta_1 = 0$ and converges to $(\lambda_1^2 + \lambda_2^2)/12$ of $\mathrm{Var}(W(0))$.

As another example, we consider the regression setting. For testing $H_0 : \beta_1 = 0$ based on $S_n(\beta_1)$, the limiting variance of $W_n(0)$ would be again

$$\frac{1}{12}(\lambda_1^2 - 2\lambda_{12} + \lambda_2^2)$$

from Lemma 2.1 with Assumptions 1 and 2. Therefore the inferences about β_1 can be performed with the fact that $W(0)$ is normally distributed with mean 0 and variance $(\lambda_1^2 - 2\lambda_{12} + \lambda_2^2)/12$.

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APPENDIX

In this appendix, first of all, we derive the variance of $G_n(y)$ in the following manner. First of all, we note that

$$\begin{aligned}\mathrm{Var}\{w_{ij}I[Y_j - Y_i \le y + \beta_1(x_j - x_i)]\} &= w_{ij}^2\mathrm{Var}\{I[Y_j - Y_i \le y + \beta_1(x_j - x_i)]\} \\ &= w_{ij}^2 G(y)(1 - G(y)).\end{aligned}$$

For the covariance, we will consider the following four cases separately for the pairs (i, j) and (k, l):

(1) For $i = k$,

$$\begin{aligned}&\mathrm{Cov}\{w_{ij}I[Y_j - Y_i \le y + \beta_1(x_j - x_i)],\ w_{il}I[Y_l - Y_i \le y + \beta_1(x_l - x_i)]\} \\ &\quad = w_{ij}w_{il}\int\{F(y+u) - G(y)\}^2 dF(u).\end{aligned}$$

(2) For $j = k$,

$$\begin{aligned}&\mathrm{Cov}\{w_{ij}I[Y_j - Y_i \le y + \beta_1(x_j - x_i)],\ w_{jl}I[Y_l - Y_j \le y + \beta_1(x_l - x_j)]\} \\ &\quad = -w_{ij}w_{jl}\int\{[F(-y+u)^- - (1 - G(y))\}\{F(y+u) - G(y)\}dF(u).\end{aligned}$$

(3) For $i = l$,

$$\begin{aligned}
&\mathrm{Cov}\{w_{ij}I[Y_j - Y_i \le y + \beta_1(x_j - x_i)],\ w_{ki}I[Y_i - Y_k \le y + \beta_1(x_i - x_k)]\} \\
&\quad = -w_{ij}w_{ki}\int \{F(-y+u)^- - (1 - G(y))\}\{F(y+u) - G(y)\}dF(u).
\end{aligned}$$

(4) For $j = l$,

$$\begin{aligned}
&\mathrm{Cov}\{w_{ij}I[Y_j - Y_i \le y + \beta_1(x_j - x_i)],\ w_{kj}I[Y_j - Y_k \le y + \beta_1(x_j - x_k)]\} \\
&\quad = w_{ij}w_{kj}\int \{F(-y+u)^- - (1 - G(y))\}^2 dF(u).
\end{aligned}$$

Then we may obtain $\mathrm{Var}[G_n(y)]$ as follows:

$$\mathrm{Var}[G_n(y)] = \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} w_{ij}^2 G(y)(1 - G(y)) + C_1 - C_2 - C_3 + C_4.$$

The C's are expressed as follows:

$$\begin{aligned}
C_1 &= \sum_{i=1}^{n-2}\sum\sum_{i+1\le j\ne l\le n} w_{ij}w_{il}\int \{F(y+u) - G(y)\}^2 dF(u) \\
&= \left\{\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\sum_{l=i+1}^{n} w_{ij}w_{il} - \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} w_{ij}^2\right\}\int \{F(y+u) - G(y)\}^2 dF(u) \\
&= \left\{\sum_{i=1}^{n-1} w_{i\cdot}\sum_{j=i+1}^{n} w_{ij} - \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} w_{ij}^2\right\}\int \{F(y+u) - G(y)\}^2 dF(u) \\
&= \left\{\sum_{i=1}^{n-1} w_{i\cdot}^2 - \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} w_{ij}^2\right\}\int \{F(y+u) - G(y)\}^2 dF(u) \\
&= \left\{\sum_{i=1}^{n} w_{i\cdot}^2 - \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} w_{ij}^2\right\}\int \{F(y+u) - G(y)\}^2 dF(u) \\
&= \left\{w_{1n}^2 - \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} w_{ij}^2\right\}\int \{F(y+u) - G(y)\}^2 dF(u),
\end{aligned}$$

$$\begin{aligned}
C_2 &= \sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\sum_{l=j+1}^{n} w_{ij}w_{jl}\int \{F(-y+u)^- - (1 - G(y))\} \\
&\qquad \times\{F(y+u) - G(y)\}dF(u)
\end{aligned}$$

$$= \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} w_{ij} w_{j\cdot} \int \{[F(-y+u)^- - (1-G(y))\}\{F(y+u) - G(y)\} dF(u)$$
$$= \sum_{j=2}^{n-1} w_{\cdot j} w_{j\cdot} \int \{F(-y+u)^- - (1-G(y))\}\{F(y+u) - G(y)\} dF(u)$$
$$= \sum_{j=1}^{n} w_{\cdot j} w_{j\cdot} \int \{F(-y+u)^- - (1-G(y))\}\{F(y+u) - G(y)\} dF(u),$$

with the facts that $w_{\cdot 1} = 0$ and $w_{n\cdot} = 0$. Also C_3 and C_4 may be obtained with similar fashion such as

$$C_3 = \sum_{i=2}^{n-1} \sum_{j=i+1}^{n} \sum_{k=1}^{i-1} w_{ij} w_{ki} \int \{F(-y+u)^- - (1-G(y))\}\{F(y+u) - G(y)\} dF(u)$$
$$= \sum_{j=1}^{n} w_{j\cdot} w_{\cdot j} \int \{F(-y+u)^- - (1-G(y))\}\{F(y+u) - G(y)\} dF(u)$$

and

$$C_4 = \sum \sum_{1 \le i \ne k \le j} \sum_{j=3}^{n} w_{ij} w_{kj} \int \{F(-y+u)^- - (1-G(y))\}^2 dF(u)$$
$$= \left\{ w_{2n}^2 - \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} w_{ij}^2 \right\} \int \{F(-y+u)^- - (1-G(y))\}^2 dF(u).$$

Then we have the following relation between the variance of $G_n(y)$ and $\sum_{k=1}^{n} h(y; Y_k)$ with (2.3).

$$\mathrm{Var}\{G_n(y)\} - \mathrm{Var}\left\{ \sum_{k=1}^{n} h(y; Y_k) \right\}$$
$$= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} w_{ij}^2 G(y)(1-G(y)) - \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} w_{ij}^2 \int \{F(y+u) - G(y)\}^2 dF(u)$$
$$- \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} w_{ij}^2 \int \{F(-y+u)^- - (1-G(y))\}^2 dF(u)$$
$$= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} w_{ij}^2 \Big[G(y)(1-G(y)) - \int \{F(y+u) - G(y)\}^2 dF(u)$$
$$- \int \{F(-y+u)^- - (1-G(y))\}^2 dF(u) \Big].$$

REFERENCES

BICKEL, P. J. AND DOKSUM, K. A. (1977). *Mathematical Statistics : Basic Ideas and Selected Topics*, Holden-Day, San Francisco.

HÁJEK, J. (1968). "Asymptotic normality of simple linear rank statistics under alternatives", *The Annals of Mathematical Statistics*, **39**, 325–346.

KOUL, H. L. (1992). "Weighted empiricals and linear models", *IMS Lecture Notes Monograph Series*, Vol. 21, Institute of Mathematical Statistics, Hayward, California.

MAJOR, P. (1994). "Asymptotic distributions for weighted U-statistics", *The Annals of Probability*, **22**, 1514–1535.

O'NEIL, K. A. AND REDNER, R. A. (1993). "Asymptotic distributions of weighted U-statistics of degree 2", *The Annals of Probability*, **21**, 1159–1169.

SEN, P. K. (1968). "Estimates of the regression coefficient based on Kendall's tau", *Journal of the American Statistical Association*, **63**, 1379–1389.

SERFLING, R. J. (1980). *Approximation Theorems of Mathematical Statistics*, John Wiley & Sons, New York.

SIEVERS, G. L. (1978). "Weighted rank statistics for simple linear regression", *Journal of the American Statistical Association*, **73**, 628–631.

SILVERMAN, B. W. (1983). "Convergence of a class of empirical distribution functions of dependent random variables", *The Annals of Probability*, **11**, 745–751.