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두 계층 공급사슬 모형에서 발주정책에 대한 수요 변동성 영향*

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Demand Variability Impact on the Replenishment Policy in a Two-Echelon Supply Chain Model*

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■ Abstract ■

We consider a supply chain model with a make-to-order production facility and a single supplier. The model we treat here is a special case of a two-echelon inventory model. Unlike classical two-echelon systems, the demand process at the supplier is affected by production process at the production facility as well as customer order arrival process. In this paper, we address that how the demand variability impacts on the optimal replenishment policy. To this end, we incorporate Erlang and phase-type demand distributions into the model. Formulating the model as a Markov decision problem, we investigate the structure of the optimal replenishment policy. We also implement a sensitivity analysis on the optimal policy and establish its monotonicity with respect to system cost parameters.

Keyword : Supply Chain Management, Multi-Echelon Inventory Model, Make-to-Order, Erlang, Markov Decision Processes.

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1. 서 론

We consider a make-to-order inventory-production system that produces products based on customer orders. Raw material used in production is supplied by an outside supplier. If raw material is depleted, production at the facility is not provided until it is available. Our model is a special case of two echelon inventory system consisting of a single supplier and a single manufacturer. $(S-1, S)$ and (Q, r) are typical inventory control policies used in continuous review two-echelon models. The analysis of such models has been well studied by Axsäter [2] and Buzacott and Shanthikumar [10]. The main distinction of our model compared to classical two-echelon models is that the demand at the supplier depends on production process as well as demand process at the production facility. This paper studies inventory management at the production facility when system parameters such as demand inter-arrival and production times are probabilistic.

Several important strategic issues for inventory management are raised from the model of interest. The first issue is how to schedule the replenishment of raw material at the production facility. Unlike classical inventory models (see [19]), a replenishment decision of raw material is affected by the queue of outstanding customer orders as well as the inventory of raw material. Second, the model raises a question of how the randomness in both demand and production processes affects the replenishment decision. The third issue is what is the impact on the ongoing replenishment policy if system parameters including cost parameters are changed.

This paper deals with these issues in the context of the Markovian system. Even though the Markovian system may not be an appropriate model for the real world problems, it often provides us with insights into the effective control policies that can be applicable to the real problems. As a starting point for the analysis, we restrict our attention to a facility which produces a single class of product. The facility is assumed to possess Erlang demand and exponential production processes subject to the costs of inventory holding, service delay, and replenishment setup costs. We formulate the problem as a Markov decision problem and characterize an optimal replenishment policy using dynamic programming.

Several recent papers have considered a model similar to one studied in this paper (See Berman and Kim [3-5], Berman and Sapna [6-8], He and Jewkes [14], He et al. [15, 16]). They worked on the characterization of the optimal replenishment policy ([3-5]), the development of algorithms for finding optimal policies ([14-16]), and the optimal performance evaluation of system variables ([6-8]). All the papers assumed that demand occurs according to a Poisson process, that is, demand inter-arrival times are exponentially distributed.

Our paper is closest in approach to the analysis of the optimal policy by [3]. Berman and Kim [3] considered a model with Poisson arrival; exponential production time, and zero lead time and characterized an optimal replenishment policy. They showed that the optimal policy is never to replenish either when the system is empty or when the inventory level is positive, and a threshold type of policy is optimal when inventory is empty. The optimal replenishment policy defined in [3] can be viewed as a dynamic version of (Q, r), that is, the reorder point is a function of the queue of outstanding customer orders.

The major contribution of our work to the above literature is to present an explicit analysis of the optimal replenishment policy under an Erlang demand arrival process. The Erlang demand process allows us to investigate the impact of the variability in demand process on the replenishment policy as well as the system cost and order quantity. It is common practice for some companies such as GM and Toyota to assist their suppliers in variability reduction through a long-term relationship or contractual agreement. Recent research papers (see [1, 11] for detailed references) report that if a company succeeds in reducing variability in demand arrival process through either information sharing or partnership with customers, it can remarkably lower inventory holding costs through just-in-time replenishment from the supplier. However, few papers deal with the issue of how the effort of reducing variability in demand, production, or lead time process could affect the ongoing inventory control or production scheduling ([12]).

We also discuss the model with more generalized phase-type order arrival processes. In many service and manufacturing environments, the quality of arriving orders is uncertain. This leads the company to inspect the quality of those demand orders before finally accepting them as confirmed production (service) orders. Usually, this job is done in sequential stages each of which may require different amounts of time for inspection and have a different variability in inspection time spent. In this case, phase-type arrival processes can become an appropriate modeling tool.

The other important contribution of our work is a marginal analysis of the optimal replenishment policy with respect to system parameters, which was not treated in the above references ([3-8], [14-16]). In this paper, we study how the change in cost parameters affects the control of inventory replenishment. In particular, we establish a monotonicity of the optimal policy with respect to these parameters.

The paper is organized as follows. In the next section, we provide a formulation of our model. Analysis of the optimal replenishment policy is given in Section 3. In Section 4, we establish a monotonicity of optimal performance with respect to system parameters. Section 5 presents the results of numerical analysis. Formulation of phase-type demand arrival process is defined in Section 6. Finally we state our conclusions in the last section.

2. Model Description

We consider the optimal inventory control of a facility which produces a single class of product. Demand (customer order) for product arrives according to a K-Erlang distribution with phase rate $K\lambda$ (We will later extend the K-Erlang arrival process to the case with more generalized phase type distributions). In other words, the inter-arrival time of customer orders follows the distribution of the sum of K independent and identically distributed exponential random variables with rate $K\lambda$ and its mean is given by λ^{-1}. Each order incurs a cost at the rate of c_1 per unit time in queue for being processed. c_1 may be serve as the cost of delaying production of demands.

To process each order, one unit of raw material is taken from inventory. A holding cost is incurred with rate c_2 for each unit of raw material in inventory. Each replenishment setup with order quantity Q incurs a cost of S. We assume that production of a unit of product takes an exponential amount of time with mean μ^{-1}.

A policy specifies, at each decision epoch, whether or not the facility places a replenishment order. The set of decision epochs corresponds to be the set of each demand arrival phase and production completion epochs. Of course, in practice, it would not be the case that the decision maker can observe each phase completion of the arrival process. However, the decision maker can still solve the formulation we give below and then use only the solution at the final phase of each demand arrival as an approximation.

The goal of this paper is to find a control policy that minimizes the expected discounted cost over an infinite horizon when the continuous interest rate is β. We can formulate the optimal replenishment control problem as a Markov decision process. After following the same uniformization process in Lippman [17], the original continuous time Markov decision problem (MDP) can be formulated with an equivalent discrete time MDP with a transition rate $\gamma \triangleq K\lambda + \mu$ and a discount factor $\frac{\gamma}{\beta+\gamma}$. Because it is always possible to redefine the time scale, we assume that $\beta+\gamma=1$ without any loss of generality.

We define the state (x_1, x_2, n) as the number of demands in production queue, inventory level, and indicator variable describing the phase status of arrival process, respectively. If $n=0$, no demands are in arrival process; if $n=k$, an arriving demand is in phase k and will join the production queue after $K-k$ phase transition(s).

We let $J(x_1, x_2, n)$ be the optimal expected discounted cost function when the initial state is given by (x_1, x_2, n). Then, we can write

$$J(x_1, x_2, n) = \min\{T_u J(x_1, x_2, n), T_p J(x_1, x_2, n)\} \tag{1}$$

where

$$T_u J(x_1, x_2, n) = \sum_{i=1}^{2} c_i x_i + K\lambda J(A(x_1, x_2, n)) + \mu J(D(x_1, x_2, n)),$$

$$T_p J(x_1, x_2, n) = S + T_u J(x_1, x_2 + Q, n),$$

$$A(x_1, x_2, n) = \begin{cases} (x_1, x_2, n+1) & \text{if } n < K-1 \\ (x_1+1, x_2, n) & \text{otherwise,} \end{cases}$$

$$D(x_1, x_2, n) = \begin{cases} (x_1-1, x_2-1, n) & \\ & \text{if } x_1 > 0 \text{ and } x_2 > 0 \\ (x_1, x_2, n) & \text{otherwise.} \end{cases}$$

In Equation (1), T_u and T_p are value iteration operators corresponding to *Do not replenish* and *Replenish* action, respectively. The operator $A(\cdot)$ and $D(\cdot)$ represent a state transition triggered by arrival phase and production completion, respectively. We note that production can be possible only when there is a demand in production queue and raw material is available.

3. The structure of the optimal replenishment policy

In order to establish the structural properties of the optimal replenishment policy, it is sufficient to show that certain properties of the functions defined on the state space Γ are preserved under the operator T ([18]). We first introduce the following first difference functions for any function f:

$$\Delta_1 f(x_1, x_2, n) = f(x_1+1, x_2, n) - f(x_1, x_2, n),$$
$$\Delta_3 f(x_1, x_2, n) = f(x_1, x_2, n+1) - f(x_1, x_2, n).$$

Operator Δ_1 implies the marginal cost incurred by holding one more demand in production queue while Δ_3 implies the marginal cost incurred by having one more phase transition in the current arrival process. Let F be the set of all functions defined on the state space Γ such that if $f \in F$, then

$$f(x_1, x_2, n) - f(x_1, x_2 + Q, n) \le S, \quad (2)$$

$$\Delta_1 f(x_1, 0, n) \ge \Delta 1 f(x_1, Q, n), \quad (3)$$

$$\Delta_1 f(x_1+1, x_2+1, n) \ge \Delta_1 f(x_1, x_2, n), \quad (4)$$

$$f(x_1+1, x_2+1, n) - f(x_1, x_2, n) \ge 0, \quad (5)$$

$$\Delta_3 f(x_1, 0, n) \ge \Delta_3 f(x_1, Q, n), \quad (6)$$

$$\Delta_3 f(x_1+1, x_2+1, n) \ge \Delta_3 f(x_1, x_2, n), \quad (7)$$

$$f(x_1+1, 0, 0) - f(x_1, 0, K-1) \ge f(x_1+1, Q, 0) - f(x_1, Q, K-1), \quad (8)$$

$$f(x_1+2, x_2+1, 0) - f(x_1+1, x_2+1, K-1) \ge f(x_1+1, x_2, 0) - f(x_1, x_2, K-1). \quad (9)$$

Equation (3) says that the marginal cost incurred by holding one more demand in queue is smaller when inventory has Q units than when it is empty. This result is straightforward because keeping more inventory means less production delay and thus less holding cost for demands in queue. Equation (4) can be rewritten as $f(x_1+2, x_2+1, n) + f(x_1, x_2, n) \ge f(x_1+1, x_2+1, n) + f(x_1+1, x_2, n)$, which is interpreted as diagonal dominance property (see Ha [13] for the terminology).

Equation (6) states that the marginal cost incurred by having one more phase transition for the arriving demand is smaller when inventory has Q units than when it is empty. Equation (8) implies that the cost right after the completion of the current demand process relative to the cost prior to its completion is smaller when inventory has Q units than when it is empty.

In the following lemma, we characterize an optimal property for the replenishment when inventory is not available in the facility.

Lemma 1 : *It cannot be optimal to place a replenishment order when the inventory level is positive.*

Proof : See the Appendix.

The following lemma characterizes the optimal property when the system is empty.

Lemma 2 : *When no order is in production and both production queue and inventory are empty, the optimal policy never places a replenishment order.*

Proof : See the Appendix.

The results presented in Lemma 1~2 are straightforward because lead time is assumed to be negligible in the model. We now establish a threshold property of the optimal replenishment policy when inventory is depleted.

Lemma 3 : *Suppose that inventory is empty. If it is optimal to replenish with x_1 demands in production queue and phase n in arrival process, then it is also optimal to replenish with x_1+1 demands in production queue and phase n in arrival process.*

Proof : See the Appendix.

To prove Lemma 3, it is sufficient to show that

$$\Delta_1 T_u f(x_1, 0, n) \ge \Delta_1 T_p f(x_1, 0, n), \ f \in F, \quad (10)$$

which says that the marginal cost incurred by holding an additional demand in production queue is smaller for the action *Replenish* than for the action *Do not replenish*. Therefore, it guarantees a threshold property of the optimal replenishment policy when the inventory level is zero. To see this, we first note that it is optimal to replenish if and only if $T_p f(x_1, 0, n) - T_u f(x_1, 0, n) < 0$. Then, Equation (10) gives

$$T_p f(x_1+1, 0, n) - T_u f(x_1+10, n) \\ \le T_p f(x_1, 0, n) - T_u f(x_1, 0, n) < 0,$$

which supports Lemma 3.

In the following lemma, we establish the dependency of the optimal replenishment policy on n, the phase status of the current arrival process.

Lemma 4 : *Suppose that inventory is empty. If it is optimal to place a replenishment order when the arrival process is in phase n and x_1 demands are in production queue, then it is also optimal to place a replenishment order when it is in phase $n+1$ and x_1 demands are in production queue.*

Proof : See the Appendix.

The proof of Lemma 4 requires that

$$\Delta_3 T_u f(x_1, 0, n) \ge \Delta_3 T_p f(x_1, 0, n), \ f \in F. \quad (11)$$

Equation (11) states that the marginal cost incurred by having one more phase transition on the current arrival process is larger for the action *Do not replenish* than for the action *Replenish*, which establishes a monotonicity of the optimal policy on n. To see this, suppose that $T_u f(x_1, 0, n) - T_p f(x_1, 0, n) > 0$. Then, Lemma 4 implies

$$T_u f(x_1, 0, n+1) - T_p f(x_1, 0, n+1) \\ \ge T_u f(x_1, 0, n) - T_p f(x_1, 0, n) > 0.$$

That is, it establishes that the threshold function is monotonically decreasing in n.

The following lemma guarantees that Equation (2)~(9) are preserved under the value iteration operator T.

Lemma 5 : If $f \in F$, then $Tf \in F$.

Proof : See the Appendix.

From Lemma 1~5, the optimal replenishment policy under the discounted cost criterion can be characterized as follows. Since the proof directly follows from Lemma 1~5, we omit it.

Theorem 1

(*i*) *The optimal value function J satisfies Equation (2)~(9), that is, $J \in F$.*

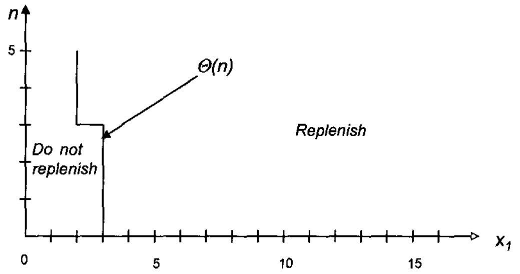
(*ii*) *Let $\Theta(n) = \min\{x_1 : \Delta_1 f(x_1, 0, n) \ge 0\}$. The optimal replenishment policy is defined by a reorder point curve $\Theta(n)$ such that when inventory is empty, it is optimal to place a replenishment order if $x_1 \ge \Theta(n)$.*

(*iii*) *Furthermore, the reorder point curve $\Theta(n)$ is decreasing in arrival phase n.*

As defined at the beginning of Section 3, the first difference of f in x_1, $\Delta_1 f(x_1, 0, n)$, is the marginal cost incurred by holding one more demand in queue when the system is in state in $(x_1, 0, n)$. If it is negative, then it is beneficial to increase the demand in queue by one unit.

The optimal replenishment policy is given by a monotonic curve $\Theta(n)$. Part (*ii*) states that $\Theta(n)$ separates the replenishment and non-replenishment regions in the state space. Part (*iii*) shows that the timing of replenishment can be varied according to the arrival phase status even though the number of demands in production queue remains the same.

[Figure 1] graphically represents the structure of the optimal replenishment policy in a typical problem.



[Figure 1] Optimal reorder point as a function of n

4. Sensitivity analysis for the optimal replenishment policy

In the previous section we have characterized the structure of the optimal replenishment policy. We next proceed to establish a monotonicity of the optimal reorder point $\Theta(n)$, defined in Theorem 1, with respect to some system costs. Consider two instances of the replenishment problem described by (1). To differentiate the second instance from the first one, we use the prime symbol for system parameters, optimal cost function, and optimal reorder point curve in the second case.

We first show a monotonicity of the optimal reorder point curve with respect to the replenishment setup cost. Suppose that $\lambda=\lambda'$, $\mu=\mu'$, $c_1=c'_1$, $c_2=c'_2$, and $S<S'$. Denote the optimal total discount costs corresponding to two instances by $J(x_1,x_2,n)$ and $J'(x_1,x_2,n)$, respectively. Consider the following set of functional properties established by J and J' :

$$J'(x_1,0,n)-J'(x_1,Q,n)\le J(x_1,0,n)-J(x_1,Q,n)+S'-S \quad (12)$$

$$J'(x_1+1,x_2+1,n)-J'(x_1,x_2,n)\le J(x_1+1,x_2+1,n)-J(x_1,x_2,n) \quad (13)$$

We first prove the following lemma :

Lemma 6 : *Suppose* (12)~(13) *hold. If it is optimal not to replenish in state* $(x_1,0,n)$ *under the first instance, it is also optimal not to replenish in state* $(x_1,0,n)$ *under the second instance.*

Proof : See the Appendix.

To prove Lemma 6, it is sufficient to show that

$$T_uJ'(x_1,0,n)-T_pJ'(x_1,0,n)\le T_uJ(x_1,0,n)-T_pJ(x_1,0,n). \quad (14)$$

Suppose that it is optimal not to replenish in state $(x_1,0,n)$ under the first instance, that is, $T_uJ(x_1,0,n)-T_pJ(x_1,0,n)<0$. Then, by (14), $T_uJ'(x_1,0,n)-T_pJ'(x_1,0,n)<0$. Therefore, it is also optimal not to replenish in state $(x_1,0,n)$ under the second instance. This result follows since the second instance has a larger replenishment setup cost than the first instance.

Next, we show that (12)~(13) are preserved under the value iteration operator T.

Lemma 7

(i) $TJ'(x_1,0,n)-TJ'(x_1,Q,n)\le TJ(x_1,0,n)-TJ(x_1,Q,n)+S'-S$

(ii) $TJ'(x_1+1,x_2+1,n)-TJ'(x_1,x_2,n)\le TJ(x_1+1,x_2+1,n)-TJ(x_1,x_2,n)$

Proof : See the Appendix.

By Lemma 6 and 7, we can present the monotonicity of the optimal replenishment policy with respect to the replenishment setup cost S as follows :

Theorem 2 : *For each phase of the arrival process, n, the optimal reorder point $\Theta(n)$ is monotonically increasing as the replenishment setup cost increases provided that other parameters remain the same.*

Proof : See the Appendix.

In a similar way, we can establish the monotonicity of the optimal reorder point $\Theta(n)$ with respect to the inventory holding cost c_2. Since the proof is similar to that of Theorem 2, we omit the details.

Theorem 3 : *For each phase of the arrival process, n, the optimal reorder point $\Theta(n)$ is monotonically increasing as the inventory holding cost increases provided that other parameters remain the same.*

5. Numerical results

In this section, we numerically evaluate the optimal performance with respect to system parameters and compare it to the model with exponential arrival process. To this end, we compute the optimal average costs using value iteration [9]. Even though we present the results under the discounted cost criterion, we note that Theorems 1-3 continue to be true for the average cost problem. The production queue is truncated to 40 because of the magnitude of the state space. The stopping rule given by Proposition 7, Ch 7 of Bertsekas [9] is used and the termination criterion ϵ is set to 10^{-2}.

Test examples and computational results are reported in <Table 1>. In this table, Q^* and $\bar{J}^*$ represent the optimal order quantity and optimal average cost corresponding to Q^*. To find Q^* and $\bar{J}^*$, we first compute the optimal average cost $\bar{J}(Q)$ using value iteration. Then, varying Q, we find Q^* which minimizes $\bar{J}(Q)$. Numerical investigation for a variety of examples indicates that $\bar{J}(Q)$ is convex in Q even though we could not prove it.

Examples 1~12 in <Table 1> are grouped into 3 sets by K=2, 3, 4. Each group has 4 different arrival rates : λ = 0.3, 0.5, 0.7, 0.9. Examples 13~24 and 25~36 in <Table 1> are the same as Example 1~12 except that c_2 and K are increased twice, respectively.

Through numerical study, we observe the following monotonic characteristics of the optimal performance with respect to system parameters, which can be explained intuitively. Assuming all other parameters are held constant :

- The optimal order quantity Q^* and the optimal cost $\bar{J}^*$ are increasing in λ.
- Q^* is non-decreasing and $\bar{J}^*$ is decreasing in K.
- Q^* is decreasing but $\bar{J}^*$ is increasing in c_2.
- Q^* and $\bar{J}^*$ are increasing in S.

In <Table 2>, we compare a 2-Erlang arrival process with an exponential arrival process with rate λ. For the comparison, Examples 1~4, 13~16, and 25~28 in <Table 1> are selected. *Difference* in <Table 2> is defined as the change in percent of the optimal cost under the exponential arrival process to that of the 2-Erlang arrival process. The average difference of 12 test examples is 7.1\%, which means that the facility can cut the total cost by 7.1\% on the average if it reduces the variance of the arrival process by half.

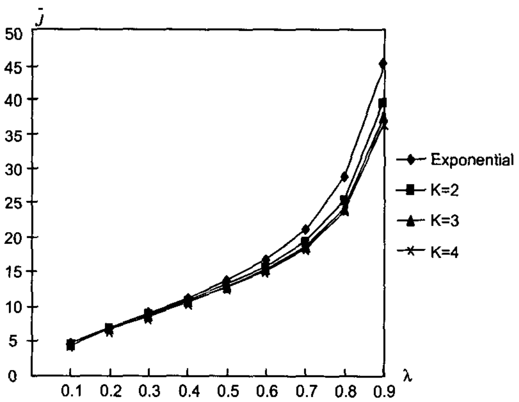
<Table 1> Test examples and optimal performance.

Ex.	S	c_1	c_2	K	λ	μ	Q^*	$\bar{J}^*$
1	100	4	1	2	0.3	1	9	8.70
2					0.5		11	12.92
3					0.7		13	19.17
4					0.9		14	39.43
5				3	0.3		9	8.63
6					0.5		11	12.72
7					0.7		13	18.55
8					0.9		14	37.27
9				4	0.3		9	8.61
10					0.5		11	12.64
11					0.7		13	18.24
12					0.9		14	36.18
13	100	4	2	2	0.3		6	11.11
14					0.5		8	16.51
15					0.7		9	23.84
16					0.9		10	45.12
17				3	0.3		6	11.07
18					0.5		8	16.33
19					0.7		9	23.26
20					0.9		10	43.07
21				4	0.3		6	11.07
22					0.5		8	16.27
23					0.7		9	23.01
24					0.9		10	42.01
25	200	4	1	2	0.3		12	11.68
26					0.5		15	16.87
27					0.7		17	23.87
28					0.9		19	44.81
29				3	0.3		12	11.61
30					0.5		15	16.64
31					0.7		17	23.24
32					0.9		19	42.65
33				4	0.3		12	11.60
34					0.5		15	16.55
35					0.7		17	22.94
36					0.9		19	41.51

<Table 2> also shows that the cost difference between exponential and 2-Erlang arrival processes becomes larger as the arrival rate λ increases. In other words, as the capacity utilization becomes higher, the larger variability in the arrival process incurs more cost. [Figure 2] displays $\bar{J}^*$ as a function of the arrival rate λ for different values of K for the example with μ=1, $c_1 = 4$, $c_2 = 1$, and $S = 100$. These results imply the beneficial effect of decreasing demand process variability.

<Table 2> Performance evaluation of exponential arrival processes

Ex.	Q^*_{exp}	$\bar{J}^*_{\text{exp}}$	Difference(%)
1	9	8.93	2.6
2	11	13.63	5.5
3	13	21.11	10.1
4	14	45.17	14.6
5	6	11.32	1.9
6	8	17.16	3.9
7	9	25.66	7.6
8	10	50.68	12.3
9	12	11.91	2.0
10	15	17.58	4.2
11	17	25.82	8.2
12	19	50.56	12.8



[Figure 2] Optimal cost as a function of λ for different values of K

6. Extension to phase-type arrival distributions

In the previous sections, we assumed that the demand arrival distribution is Erlang. This section extends our formulation to handle phase-type distributions for inter-arrival times. Suppose that each arriving demand goes through K phases before joining the production queue. The time to complete phase $n(=0, 1, \cdots, K-1)$ is exponentially distributed with a mean of $1/\lambda_n$. On completion of phase n, an arriving demand enters phase $n+1$. Let $\lambda^* = \max\{\lambda_0, \lambda_1, \cdots, \lambda_K\}$. Then, the model with phase-type arrival processes can be formulated with a discrete time MDP with a transition rate $\gamma' \triangleq \lambda^* + \mu$ and a discount factor $\frac{\gamma'}{\beta+\gamma'}$. Because it is always possible to redefine the time scale, we assume that $\beta+\gamma'=1$ without any loss of generality.

We let $J(x_1, x_2, n)$ be the optimal expected discounted cost function when the initial state is given by (x_1, x_2, n). Then, we can write

$$J(x_1, x_2, n) = \min\{T_u J(x_1, x_2, n), T_p J(x_1, x_2, n)\}$$

where

$$T_u J(x_1, x_2, n) = \sum_{i=1}^{2} c_i x_i + \lambda_n J(A(x_1, x_2, n)) + \mu J(D(x_1, x_2, n)) + (\lambda^* - \lambda_n) J(x_1, x_2, n),$$

$$T_p J(x_1, x_2, n) = S + T_u J(x_1, x_2 + Q, n).$$

The operator T_u, T_p, $A(\cdot)$, and $D(\cdot)$ are as defined in Section 2. The transition corresponding to $\lambda^* - \lambda_n$ represents a self-loop transition due to the uniformization. With this formulation, we can present the results similar to the ones in Section 3~4 but we omit the detailed procedures here.

7. Conclusions

In this paper, we considered a two-echelon supply chain model in which a facility produces a single class of product based on customer order and raw material used in production is purchased from an outside supplier. We studied the optimal raw material replenishment policy under the Erlang demand and exponential production processes and investigated the impact of the variability in demand arrival process on it.

The main results are summarized as follows. We showed that when inventory is zero, i) there exists a reorder point which is a function of the phase transition in the demand process, ii) it is optimal to replenish if and only if the number of demands in production queue reaches this reorder point, and iii) it is monotonically decreasing as the phase transition evolves. Further, through a sensitivity analysis, we showed that the optimal reorder point is monotonically increasing as either the setup cost or inventory holding cost increases.

Numerical test performed for a variety of examples suggested that the optimal cost function may be convex with respect to the replenishment order quantity. Hence, it is conjectured that there may exist an unique quantity which minimizes the optimal cost function even though we could not prove it. The comparison of the Erlang demand process with the exponential one supported our intuition that less demand variability contributes to the reduction in the inventory management cost.

The results obtained under the Erlang demand arrival process are extended to the one with more generalized phase-type distributions. With it, we are capable of modeling a demand arrival process that is composed of multiple stages in sequence. In the future research, we will investigate the model that order failure is allowed in each stage of the arrival process, that is, when each arriving order does not pass any of stages, it is regarded failed and hence immediately removed from the order process.

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Appendix

Proof of Lemma 1 : To prove Lemma 1, it is sufficient to show that if $f \in F$, $T_u f(x_1, x_2, n) < T_p f(x_1, x_2, n), x_2 > 0$.
$T_u f(x_1, x_2, n) - T_p f(x_1, x_2, n) = -c_2 Q - S + K\lambda[f(A(x_1, x_2, n)) - f(A(x_1, x_2 + Q, n))]$
$+ \mu[f(D(x_1, x_2, n)) - f(D(x_1, x_2 + Q, n))] \leq -c_2 Q - \gamma S$ (by(2)) < 0 (since $\gamma < 1$). □

Proof of Lemma 2 : To prove Lemma 2, it is sufficient to show that if $f \in F$, $T_u f(0,0,n) < T_p f(0,0,n)$.
$T_u f(0,0,n) - T_p f(0,0,n) = -c_2 Q - S + K\lambda[f(A(0,0,n)) - f(A(0,0,Q,n))]$
$+ \mu[f(0,0,n) - f(0,0,Q,n)] \leq -c_2 Q - S + \gamma S$ (by(2)) < 0 (since $\gamma < 1$). □

Proof of Lemma 3 : $\Delta_1 T_u f(x_1, 0, n) - \Delta_1 T_p f(x_1, 0, n) = K\lambda[\Delta_1 f(A(x_1, 0, n)) - \Delta_1 f(A(x_1, Q, n))]$ $+ \mu[\Delta_1 f(x_1, 0, n) - \Delta_1 f(D(x_1, Q, n))] \geq 0$. The inequality of $K\lambda$ term follows by (3). When $x_1 > 0$, the μ term becomes
$\Delta_1 f(x_1, 0, n) - \Delta_1 f(x_1 - 1, Q-1, n) \geq \Delta_1 f(x_1, 0, n) - \Delta_1 f(x_1, Q, n)$ (by (4))≥ 0 (by(3)).
If $x_1 = 0$, it becomes $\Delta_1 f(0,0,n) - (f(0, Q-1, n) - f(0, Q, n)) \geq \Delta_1 f(0,0,n) - (f(1, Q, n) - f(0, Q, n))$ (by (5)) ≥ 0 (by(3)). □

Proof of Lemma 4 : Suppose that $n < K-2$. $\Delta_3 T_u f(x_1, 0, n) \geq \Delta_3 T_p f(x_1, 0, n) = K\lambda[\Delta_3 f(x_1, 0, n+1)$ $- \Delta_3 f(x_1, Q, n+1)] + \mu[\Delta_3 f(x_1, 0, n) - \Delta_3 f(D(x_1, Q, n))] \geq 0$. The inequality of $K\lambda$ term follows by (6). The inequality of μ term follows by (6) when $x_1 = 0$. When $x_1 > 0$, the μ term becomes
$\Delta_3 f(x_1, 0, n) - \Delta_3 f(x_1 - 1, Q-1, n) \geq \Delta_3 f(x_1, 0, n) - \Delta_3 f(x_1, Q, n)$ (by(7) ≥ 0 (by(6)).
Suppose that $n = K-2$. $\Delta_3 T_u f(x_1, 0, K-2) - \Delta_3 T_p f(x_1, 0, K-2) = K\lambda[f(x_1+1, 0, 0) - f(x_1, 0, K-1)] -$ $(f(x_1+1, Q, 0) - f(x_1, Q, K-1))] + \mu[\Delta_3 f(D(x_1, Q, K-2) - \Delta_3 f(x_1, 0, K-2)] \geq 0$. The inequality of $K\lambda$ term follows by (8). The inequality of μ term can be shown using the same argument as in case $n < K-2$. □

Proof of Lemma 5 : Denote by (u/p) the optimal action in state $(x_1, 0, n)$ where u and p represent *Do not replenish* and *Replenish* actions, respectively.
(i) $Tf(x_1, x_2, n) - Tf(x_1, x_2 + Q, n) \leq S$:
$Tf(x_1, x_2, n) \leq T_p f(x_1, x_2, n) = S + T_u f(x_1, x_2 + Q, n)$ (by the definition of value function). Since $Tf(x_1, x_2 + Q, n) = T_u f(x_1, x_2 + Q, n)$ by Lemma 1, the result follows.

(ii) $\Delta_1 Tf(x_1, 0, n) \geq \Delta_1 Tf(x_1, Q, n)$: By Lemma 1, $\Delta_1 Tf(x_1, Q, n) = \Delta_1 T_u f(x_1, Q, n)$. Therefore, we focus on combinations of actions admissible in $(x_1 + 1, 0, n)$ and $(x_1, 0, n)$. Case (u, p) is excluded by Lemma 3. For (p, p), by the definition of value function,

$\Delta_1 T_p f(x_1, 0, n) - \Delta_1 T_u f(x_1, Q, n) = 0$. For (u, u), $\Delta_1 T_u f(x_1, 0, n) \geq \Delta_1 T_p f(x_1, 0, n)$ (by Lemma 3) $= \Delta_1 T_u f(x_1, Q, n)$. For (p, u), it can be shown using the result of (p, p) because $T_p f(x_1+1, 0, n) - T_u f(x_1, 0, n) > T_p f(x_1+1, 0, n) - T_p f(x_1, 0, n) = \Delta_1 T_u f(x_1, Q)$.

(iii) $\Delta_1 Tf(x_1+1, x_2+1, n) \geq \Delta_1 Tf(x_1, x_2, n)$:
By Lemma 1, $\Delta_1 Tf(x_1+1, x_2+1, n) = \Delta_1 T_u f\ (x_1+1, x_2+1, n)$. Hence, we focus on combinations of actions admissible in (x_1+1, x_2, n) and (x_1, x_2, n). By Lemma 3, case (u, p) is excluded. For (u, u),

$$\Delta_1 T_u f(x_1+1, x_2+1, n) - \Delta_1 T_u f(x_1, x_2, n) = K\lambda[\Delta_1 f(A(x_1+1, x_2+1, n)) - \Delta_1 f(A(x_1, x_2, n))] + \mu[\Delta_1 f(x_1, x_2, n) - \Delta_1 f(D(x_1, x_2, n))] \geq 0.$$

The inequality of $K\lambda$ term follows by (4). The inequality of μ term follows by (4) when $x_1 > 0$ and $x_2 > 0$. When $x_1 = 0$ and $x_2 > 0$, the μ term becomes

$$\Delta_1 f(0, x_2, n) - (f(0, x_2-1, n) - f(0, x_2, n)) = f(1, x_2, n) - f(0, x_2-1, n) \geq 0 \text{ (by (5))}.$$

When $x_1 \geq 0$ and $x_2 = 0$, it becomes zero. For (p, p), $x_2 = 0$ and $x_1 > 0$ by Lemma 1 and 2 and $\Delta_1 T_u f(x_1+1, 1, n) - \Delta_1 T_p f(x_1, 0, n) \geq \Delta_1 T_u f(x_1+1, 1, n) - \Delta_1 T_u f(x_1, 0, n)$ (by Lemma 3) ≥ 0 (by case (u, u)). For (p, u), it can be shown using the result of (u, u).

(iv) $Tf(x_1+1, x_2+1, n) \geq Tf(x_1, x_2, n)$: By Lemma 1, the optimal action in (x_1+1, x_2+1, n) is *Do not replenish*. If the optimal action in $(x_1,\ x_2, n)$ is *Do not replenish*,

$$T_u f(x_1+1, x_2+1, n) - T_u f(x_1, x_2, n) = c_1 + c_2 + K\lambda[f(A(x_1+1, x_2+1, n)) - f(A(x_1+1, x_2, n))] + \mu[f(x_1, x_2, n) - f(D(x_1, x_2))]\, 1\{x_1 > 0, x_2 > 0\} \geq 0 \quad \text{(by } c_1, c_2 \geq 0 \text{ and (5))}.$$

Otherwise, the inequality can be shown using the above result because

$$T_u f(x_1+1, 1, n) - T_p f(x_1, 0, n) > T_u f(x_1+1, 1, n) - T_u f(x_1, 0, n) \geq 0.$$

(v) $\Delta_3 Tf(x_1, 0, n) \geq \Delta_3 Tf(x_1, 0, n)$: By Lemma 1, $\Delta_3 Tf(x_1, Q, n) = \Delta_3 T_u f(x_1, Q, n)$. Hence, we focus on combinations of actions admissible in $(x_1, 0, n+1)$ and $(x_1, 0, n)$. Case (u, p) is excluded by Lemma 4. For (p, p), by the definition of value function,
$\Delta_3 T_p f(x_1, 0, n) = \Delta_3 Tf(x_1, Q, n)$. For (u, u), $\Delta_3 T_u f(x_1, 0, n) \geq \Delta_3 T_p f(x_1, 0, n)$ (by Lemma 4) $= \Delta_3 Tf(x_1, 0, n)$. For (p, u), the inequality can be shown using the result of (p, p).

(vi) $\Delta_3 Tf(x_1+1, x_2+1, n) \geq \Delta_3 Tf(x_1, x_2, n)$: By Lemma 1, $\Delta_3 Tf(x_1+1, x_2+1, n) = \Delta_3 T_u f$ (x_1+1, x_2+1, n). We focus on combinations of actions admissible in $(x_1, x_2, n+1)$ and (x_1, x_2, n). Case (u, p) is excluded by Lemma 4. For (u, u),

$$\Delta_3 T_u f(x_1+1, x_2+1, n) - \Delta_3 T_u f(x_1, x_2, n) = K\lambda\ [\Delta_3 f(A(x_1+1, x_2+1, n)) - \Delta_3 f(A(x_1, x_2, n))] + \mu[\Delta_3 f(x_1, x_2, n) - \Delta_3 f(D(x_1, x_2, n))] \geq 0.$$

The inequality of $K\lambda$ term follows by (7) when $n < K-2$. When $n = K-2$, the $K\lambda$ term becomes $f(x_1+2, x_2+1, 0) - f(x_1+1, x_2+1, K-1) - (f(x_1 + 1, x_2, 0) - f(x_1, x_2, K-1)) \geq 0$ (by (9)). The inequality of μ term follows by (7) when $x_1>0$ and $x_2>0$. Otherwise, the μ term becomes zero. For (p, p), $x_1>0$ and $x_2=0$ by Lemma 1 and 2. The inequality follows by (u, u) because $\Delta_3 T_p f(x_1, 0, n) \leq \Delta_3 T_u f(x_1, 0, n)$ (by Lemma 4) $\leq \Delta_3 T_u f(x_1+1, 1, n)$. For (p, u), the inequality can be shown using the result of (u, u).

(vii) $Tf(x_1+1, 0, 0) - Tf(x_1, 0, K-1) \geq Tf(x_1+1, Q, 0) - Tf(x_1, Q, K-1)$: By Lemma 1, the optimal action admissible in $(x_1+1, Q, 0)$ and $(x_1, Q, K-1)$ is *Do not replenish.* We focus on combinations of actions admissible in $(x_1+1, 0, 0)$ and $(x_1, 0, K-1)$. For (u, u),

$$T_u f(x_1+1, 0, 0) - T_u f(x_1, 0, K-1) - (T_u f(x_1+1, Q, 0) - T_u f(x_1, Q, K-1))$$
$$= K\lambda[f(x_1+1, 0, 1) - f(x_1+1, 0, 0) - (f(x_1+1, Q, 1) - f(x_1+1, Q, 0))]$$
$$+ \mu[f(x_1+1, 0, 0) - f(x_1, 0, K-1) - (f(x_1, Q-1, 0) - f(D(x_1, Q, K-1)))] \geq 0.$$

The inequality of $K\lambda$ term follows by (6). When $x_1>0$, the μ term becomes

$$f(x_1+1, 0, 0) - f(x_1, 0, K-1) - (f(x_1, Q-1, 0) - f(x_1-1, Q-1, K-1)) \geq f(x_1+1, 0, 0)$$
$$- f(x_1, 0, K-1) - (f(x_1+1, Q, 0) - f(x_1, Q, K-1)) \text{ (by (9))} \geq \text{(by (8))}.$$

When $x_1=0$, it becomes

$$f(x_1+1, 0, 0) - f(x_1, 0, K-1) - (f(x_1, Q-1, 0) - f(x_1, Q, K-1))$$
$$\geq f(x_1+1, 0, 0) - f(x_1, 0, K-1) - (f(x_1+1, Q, 0) - f(x_1, Q, K-1)) \text{ (by (5))} \geq \text{(by (8))}.$$

For (p, p), it follows by the definition of value function that
$T_p f(x_1+1, 0, 0) - T_p f(x_1, 0, K-1) = K + Tf(x_1+1, Q, 0) - (K + Tf(x_1, Q, K-1))$. For (u, p) and (p, u), the inequality can be shown using the result of (u, u) and (p, p), respectively.

(viii) $Tf(x_1+2, x_2+1, 0) - Tf(x_1+1, x_2+1, K-1) \geq Tf(x_1+1, x_2, 0) - Tf(x_1, x_2, K-1)$: By Lemma 1, the action admissible in $(x_1+2, x_2+1, 0)$ and $(x_1+1, x_2+1, K-1)$ is *Do not replenish.* We focus on combinations of actions admissible in $(x_1+1, x_2, 0)$ and $(x_1, x_2, K-1)$. For (u, u),

$$T_u f(x_1+2, x_2+1, 0) - T_u f(x_1+1, x_2+1, K-1) - (T_u f(x_1+1, x_2, 0) - T_u f(x_1, x_2, K-1))$$
$$= K\lambda[f(x_1+2, x_2+1, 1) - f(x_1+2, x_2+1, 0) - (f(x_1+1, x_2, 1) - f(x_1+1, x_2, 0))]$$
$$+ \mu[f(x_1+1, x_2, 0) - f(x_1, x_2, K-1) - (f(D(x_1+1, x_2, 0) - (f(D(x_1, x_2, K-1)))]$$

The inequality of $K\lambda$ term follows by (7). When $x_1>0$ and $x_2>0$, the inequality of μ term follows by (9). When $x_1 \geq 0$ and $x_2=0$, the μ term becomes zero. When $x_1=0$ and $x_2>0$, it becomes $f(1, x_2, 0) - f(0, x_2, K-1) - (f(0, x_2-1, 0) - f(0, x_2, K-1)) = f(1, x_2, 0) - f(0, x_2-1, 0) \geq 0$ (by (5)).

For (p, p), $x_1>0$ and $x_2=0$ by Lemma 1 and Lemma 2 and

$$T_u f(x_1+2, 1, 0) - T_u f(x_1+1, 1, K-1) - (T_p f((x_1+1, 0, 0) - T_p f(x_1, 0, K-1))$$
$$= K\lambda[f(x_1+2, 1, 1) - f(x_1+2, 1, 0) - (f(x_1+1, Q, 1) - f(x_1+1, Q, 0))]$$
$$+ \mu[f(x_1+1, 0, 0) - f(x_1, 0, K-1) - (f(x_1, Q-1, 0)) - (f(x_1, Q, K-1)))] \geq 0.$$

The inequality of $K\lambda$ term follows because

$\Delta_3 f(x_1+2,1,0) - \Delta_3 f(x_1+1,Q,0) \geq \Delta_3 f(x_1+1,0,0) - \Delta_3 f(x_1+1,Q,0)$ (by (7)) $\geq$ (by (6)) The inequality of μ term follows because

$$f(x_1+1,0,0)-f(x_1,0,K-1)-(f(x_1,Q-1,0)-f(x_1-1,Q-1,K-1))$$
$$\geq f(x_1+1,0,0)-f(x_1,0,K-1)-(f(x_1+1,Q,0)-f(x_1,Q,K-1)) \text{ (by (9))} \geq \text{(by (8))}.$$

For (u,p) and (p,u), the inequality can be shown using the result of (p,p) and (u,u), respectively. □

Proof of Lemma 6 : It is sufficient to show

$$T_u J'(x_1,0,n) - T_p J'(x_1,0,n) \leq T_u J(x_1,0,n) - T_p J(x_1,0,n). \quad (14)$$

$$T_u J'(x_1,0,n) - T_p J'(x_1,0,n) - (T_u J(x_1,0,n) - T_p J(x_1,0,n))$$
$$= -S' + S + K\lambda[J'(A(x_1,0,n)) - J'(A(x_1,Q,n)) - (J(A(x_1,0,n)) - J(A(x_1,Q,n)))]$$
$$+\mu[J'(x_1,0,n) - J'(D(x_1,Q,n)) - (J(x_1,0,n) - J(D(x_1,Q,n)))]$$

(12) is applied to the $K\lambda$ term. (12) is applied to the μ term when $x_1=0$. If $x_1>0$, the μ term becomes

$$J'(x_1,0,n) - J'(x_1-1,Q-1,K-1)) - (J(x_1,0,n) - J(x_1-1,Q-1,K-1))$$
$$\leq J'(x_1,0,n) - J'(x_1,Q,n) - (J(x_1,0,n) - J(x_1,Q,n)) \text{ (by (13))} \leq S' - S. \ \square$$

Proof of Lemma 7 :

(i) $TJ'(x_1,0,n) - TJ'(x_1,Q,n) \leq TJ(x_1,0,n) - TJ(x_1,Q,n) + S' - S$: By Lemma 1, the optimal action admissible in $(x_1,Q,n)'$ and (x_1,Q,n) is *Do not replenish*. Therefore, we focus on combinations of actions admissible in $(x_1,0,n)'$ and $(x_1,0,n)$. By Lemma 6, case (p',u) is excluded. For (p',p), $T_p J'(x_1,0,n) - TJ'(x_1,Q,n) - (T_p J(x_1,0,n) - TJ(x_1,Q,n)) = S' - S$. For (u',u),

$$T_u J'(x_1,0,n) - TJ'(x_1,Q,n) - (T_u J(x_1,0,n) - TJ(x_1,Q,n))$$
$$\leq T_p J'(x_1,0,n) - T_j'(x_1,Q,n) - (T_p J(x_1,0,n) - TJ(x_1,Q,n)) \text{(by Lemma 6)} = S' - S.$$

Case (u',p) can be shown using the result of case (u',u).

(ii) $TJ'(x_1+1,x_2+1,n) - TJ'(x_1,x_2,n) \leq TJ(x_1+1,x_2+1,n) - TJ(x_1,x_2,n)$: By Lemma 1, the optimal action admissible in $(x_1+1,x_2+1,n)'$ and (x_1+1,x_2+1,n) is *Do not replenish*. Therefore, we focus on combinations of actions admissible in $(x_1,x_2,n)'$ and (x_1,x_2,n). By Lemma 6, case (p',u) is excluded. For (u',u),

$$T_u J'(x_1+1,x_2+1,n) - T_u J'(x_1,x_2,n) - (T_u J(x_1+1,x_2+1,n) - T_u J(x_1,x_2,n))$$
$$= K\lambda[J'(A(x_1+1,x_2+1,n)) - J'(A(x_1,x_2,n)) - (J(A(x_1+1,x_2+1,n)) - J(A(x_1,x_2,n)))]$$
$$+\mu[J'(D(x_1+1,x_2+1,n)) - J'(D(x_1,x_2,n)) - (J(D(x_1+1,x_2+1,n) - J(D(x_1,x_2,n)))] \leq 0.$$

The inequality of $K\lambda$ term follows by (13). The inequality of μ term follows by (13) when $x_1>0$ and $x_2>0$. Otherwise, the μ term becomes zero. For (p',p), $x_2=0$ and $x_1>0$ by Lemma 1 and 3 and

$T_u J'(x_1+1,1,n) - T_p J'(x_1,0,n) - (T_u J(x_1+1,1,n) - T_p J(x_1,0,n))$
$\le T_u J'(x_1+1,1,n) - T_u J'(x_1,0,n) - (T_u J(x_1+1,1,n) - T_u J(x_1,0,n))$ (by Lemma 6)
≤ 0 (*by case* (u',u)). Case (u',p) can be shown using the result of case (u',u). □

Proof of Theorem 2 : We now prove $\Theta(n) \le \Theta'(n)$ using contradiction. Suppose $\Theta(n) > \Theta'(n)$. Then, we have $T_u J(\Theta'(n),0,n) \le T_p J(\Theta'(n), 0,n)$ and $T_u J'(\Theta'(n),0,n) > T_p J'(\Theta'(n),0,n)$. It follows that

$$T_u J'(\Theta'(n),0,n) - T_u J(\Theta'(n),0,n) > T_p J'(\Theta'(n),0,n) - T_p J(\Theta'(n),0,n),$$

which is a contradiction by (14) of Lemma 6. □