# 수준 연속인 퍼지 랜덤 변수의 가중 합에 대한 약 수렴성

# Weak convergence for weighted sums of level-continuous fuzzy random variables

김윤경

Yun Kyong Kim

동신대학교 정보통신공학과

Dept. of Information & Communication Engineering,

Dongshin University

E-mail: ykkim@dsu.ac.kr

## 요 약

이 논문에서는 퍼지 랜덤 변수의 합에 대한 약한 대수의 법칙을 일반화로서, 컴팩트 일양 적분 가능한 수준 연속 퍼지 랜덤 변수의 가중 합이 약 수렴하기 위한 동치 조건을 구하였다.

## Abstract

The present paper establishes a necessary and sufficient condition for weak convergence for weighted sums of compactly uniformly integrable level-continuous fuzzy random variables as a generalization of weak laws of large numbers for sums of fuzzy random variables.

## 1. 서 론

Since Puri and Ralescu [14] introduced the concept of a fuzzy random variable, there has been increasing interests in limit theorems for fuzzy random variables. Statistical inference for fuzzy probability models led to the need for laws of large numbers in order to ensure consistence in estimation problems. Strong laws of large numbers for sums of independent fuzzy random variables have been studied by several researchers. For example, Inoue [2], Joo [4], Joo and Kim [7], Kim [10, 11], Klement et al. [12], Molchanov [13], Uemura [17] and so on. On the other hand, weak laws of large numbers for sums of fuzzy random variables have been studied by Joo [5], Taylor et al. [16].

It is one of significant problems how we can generalize the above results for sums of fuzzy random variables to the case of weighted sums.

Related to this problem, Joo et al. [8] obtained strong convergence for weighted sums of fuzzy random variables under stochastic geometric condition. But as far as I know, there is no results for weak convergence for

weighted sums of fuzzy random variables as yet.

The purpose of this paper is to obtain some results on weak convergence for weighted sums of level-continuous fuzzy random variables. Section 2 is devoted to describe basic facts for fuzzy numbers. The main results are given in section 3.

## 2. Preliminaries

Let $K(R^p)$ be the family of all non-empty compact and convex subsets of $R^p$. Then $K(R^p)$ is metrizable by the Hausdorff metric $h$ defined by

$$h(A, B) = \max \{ \sup_{a \in A} \inf_{b \in B} |a - b|,$$
$$\sup_{b \in B} \inf_{a \in A} |a - b| \},$$

where $| \cdot |$ is the usual norm in $R^p$.

A norm of $A \in K(R^p)$ is defined by

$$\| A \| = h(A, \{0\}) = \sup_{a \in A} | a | .$$

It is well known that the metric $(K(R^p), h)$ is complete and separable (See Debreu [1]). The addition and scalar multiplication on $K(R^p)$ are defined as usual;

$$A \oplus B = \{a + b \mid a \in A, b \in B\}, \quad \lambda A = \{\lambda a \mid a \in A\}.$$

Let $F(R^p)$ denote the space of fuzzy numbers in $R^p$, i.e., the family of all normal, fuzzy convex and upper-semicontinuous fuzzy sets $u$ in $R^p$ such that

$$supp \ u \ = \ cl \ \{x \in R^p \ : \ u(x) \ > \ 0\}$$

is compact. For a fuzzy set $u$ in $R^p$, we define the $\alpha$ - level set of $u$ by

$$L_\alpha u \ = \ \begin{cases} \{x \ : \ u(x) \ \geq \ \alpha\}, & 0 < \alpha \leq 1, \\ supp \ u, & \alpha = 0. \end{cases}$$

Then it follows that $u \in F(R^p)$ if and only if $L_\alpha u \in K(R^p)$ for each $\alpha \in [0,1]$.

**Lemma 2.1.** For $u \in F(R^p)$, let us define $f_u : [0,1] \to ((K(R^p), h)$ by $f_u(\alpha) = L_\alpha u$.
Then the followings hold;

(1) $f_u$ is non-increasing; i.e., $\alpha \leq \beta$ implies $f_u(\alpha) \supset f_u(\beta)$.

(2) $f_u$ is left-continuous on $(0,1]$.

(3) $f_u$ has right-limits on $[0,1)$ and is right continuous at $0$.

Conversely, if $g : [0,1] \to ((K(R^p), h)$ is a function satisfying the above conditions (1) - (3), then there exists a unique $v \in F(R^p)$ such that

$$g(\alpha) = L_\alpha v \ \text{for all} \ \alpha \in [0,1].$$

**Proof:** See Joo and Kim [6].

If we denote the right-limit of $f_u$ at $\alpha \in [0,1)$ by $L_{\alpha^+} u$, then

$$L_{\alpha^+} u = cl\{x \in R^p \ : \ u(x) > \alpha\}.$$

If $f_u$ is continuous on $[0,1]$, then $u \in F(R^p)$ is called level-continuous We denote by $F_C(R^p)$ the family of all level-continuous $u \in F(R^p)$.

The addition and scalar multiplication on $F(R^p)$ are defined as usual;

$$(u \oplus v)(x) = \sup_{y + z = x} \min(u(y), v(z)),$$

$$(\lambda u)(x) = \begin{cases} u(x/\lambda), & \text{if } \lambda \neq 0 \\ 0(x), & \text{if } \lambda = 0 \end{cases}$$

where $0 = I_{\{0\}}$ is the indicator function of $\{0\}$.
Then it is well-known that for each $\alpha \in [0,1]$,

$$L_\alpha(u \oplus v) = L_\alpha u \oplus L_\alpha v$$

and

$$L_\alpha(\lambda u) = \lambda L_\alpha u.$$

Now, the uniform metric $d_\infty$ on $F(R^p)$ is defined by

$$d_\infty(u, v) \ = \ \sup_{0 \leq \alpha \leq 1} h(L_\alpha u, L_\alpha v).$$

Also, the norm of $u \in F(R^p)$ is defined as

$$\| u \| \ = d_\infty(u, I_{\{0\}}) = \sup_{x \in L_0 u} | x | .$$

Then it is well-known that $(F(R^p), d_\infty)$ is complete, but is not separable. (See Klement et al. [12]). However, $(F_C(R^p), d_\infty)$ is complete and separable (See Choi et al. [3]).

## 3. Main Results

Let $(\Omega, \mathcal{F}, P)$ be a probability space. A set-valued function $X : \Omega \to (K(R^p), h)$ is called a random set if it is measurable.

A random set $X$ is called integrably bounded if $E \| X \| < \infty$. The expectation of integrably bounded random set $X$ is defined by

$$E(X) = \{E(\xi) \mid \xi \in L(\Omega, R^p) \text{ and}$$
$$\xi(\omega) \in X(\omega) \text{ a.s.}\},$$

where $L(\Omega, R^p)$ denotes the class of all $R^p$- valued random variables $\xi$ such that $E|\xi| < \infty$.

A fuzzy number valued function $X : \Omega \to F(R^p)$ is called a fuzzy random variable if for each $\alpha \in [0,1]$, $L_\alpha X$ is a random set.

If $X : \Omega \to (F(R^p), d_\infty)$ is measurable, then it is fuzzy random variable. But the converse is not true (For details, see Kim [9]). Nevertheless, if we restrict our concern to $F_C(R^p)$-valued case, then these conditions are equivalent, i.e., $X : \Omega \to F_C(R^p)$ is a fuzzy random variable if and only if $X : \Omega \to (F_C(R^p), d_\infty)$ is measurable.

A fuzzy random variable $X$ is called integrably bounded if $E \| X \| < \infty$. The expectation of integrably bounded fuzzy random variable $X$ is a fuzzy number defined by

$$E(X)(x) = \sup\{\alpha \in [0,1] \mid x \in E(L_\alpha X)\}.$$

It is well-known that if $X, Y$ are integrably bounded, then

(1) $L_\alpha E(X) = E(L_\alpha X)$ for all $\alpha \in [0,1]$.

(2) $E(X \oplus Y) = E(X) \oplus E(Y)$.

(3) $E(\lambda X) = \lambda E(X)$.

(4) If $X \in F_C(R^p)$ a.s., then $E(X) \in F_C(R^p)$.

In this paper, we restrict our concerns to level-continuous fuzzy random variables, i.e., $F_C(R^p)$-valued fuzzy random variables.

Let $\{X_n\}$ be a sequence of level-continuous fuzzy random variables and let $\{\lambda_{ni}\}$ be a double array of real numbers satisfying $\sum_{i=1}^{\infty} \mid \lambda_{ni} \mid \leq C$ for each $n$ and for some constant $C$. We consider the following three statements;

(A): $d_{\infty}(\oplus_{i=1}^{n}\lambda_{ni}X_i, \oplus_{i=1}^{n}\lambda_{ni}E(X_i)) \to 0$
   in probability as $n\to\infty$.

(B): For each $\alpha \in [0,1]$,
   $h(\oplus_{i=1}^{n}\lambda_{ni}L_{\alpha}X_i, \oplus_{i=1}^{n}\lambda_{ni}E(L_{\alpha}X_i)) \to 0$
   in probability $n\to\infty$.

(C): For each rational $\alpha \in [0,1]$,
   $h(\oplus_{i=1}^{n}\lambda_{ni}L_{\alpha}X_i, \oplus_{i=1}^{n}\lambda_{ni}E(L_{\alpha}X_i)) \to 0$
   in probability $n\to\infty$.

It is obvious that (A) implies (B) and (B) implies (C). Thus, the purpose of this paper is to find the condition guaranteeing that (C) implies (A). To this end, we need the following concepts.

Definition 3.1. Let $\{X_n\}$ be a sequence of level-continuous fuzzy random variables.

(1) $\{X_n\}$ is said to be tight if for each $\varepsilon > 0$, there exists a compact subset $A$ of $(F_C(R^p), d_{\infty})$ such that $P(X_n \notin A) < \varepsilon$ for all $n$.

(2) $\{X_n\}$ is said to be compactly uniformly integrable if for each $\varepsilon > 0$, there exists a compact subset $A$ of $(F_C(R^p), d_{\infty})$ such that
$$\int_{\{X_n \notin A\}} \| X_n \| \ dP < \varepsilon \text{ for all } n.$$

Our first main result is as follows;

Theorem 3.2. Let $\{X_n\}$ be a sequence of level-continuous fuzzy random variables. If $\{X_n\}$ is compactly uniformly integrable, then (C) implies (A).

To prove this, we need some lemmas.

Lemma 3.3. Let $K$ be a subset of $(F_C(R^p), d_{\infty})$. Then $K$ is relatively compact if and only if
$$sup_{u \in K} \mid u \mid \ < \ \infty$$
and
$$\lim_{\delta \to 0} sup_{u \in K} \phi(u, \delta) = 0$$
where $\phi(u, \delta) = sup_{\mid \alpha - \beta \mid \leq \delta} h(L_{\alpha}u, L_{\beta}u)$.

Proof: See Theorem 3.7 of Choi et al. [3].

Now, for $u \in F_C(R^p)$ and a positive integer $m$, if we define
$$g_m(u) = sup\{\alpha \mid \alpha \in \lambda L_{k/m}u \oplus (1-\lambda)L_{(k-1)/m}u, \ 0 \leq \lambda \leq 1 \text{ and } k=1,...,m\},$$
then for $(k-1)/m \leq \alpha \leq k/m$,
$$L_{\alpha}g_m(u) = (1-k+m\alpha)L_{k/m}u \oplus (k-m\alpha)L_{(k-1)/m}u.$$
It follows that
$$g_m(u \oplus v) = g_m(u) \oplus g_m(v)$$
and
$$g_m(\lambda u) = \lambda g_m(u).$$

Lemma 3.4. If $K$ is a relatively compact subset of $(F_C(R^p), d_{\infty})$, then
$$\lim_{m\to\infty} sup_{u \in K} d_{\infty}(u, g_m(u)) = 0.$$

Proof: For each $m$ and $(k-1)/m \leq \alpha \leq k/m$,
$$h(L_{\alpha}u, L_{\alpha}g_m(u)) \leq h(L_{(k-1)/m}u, L_{k/m}u)$$
$$\leq \phi(u, 1/m).$$
Thus, by Lemma 3.3,
$$\lim_{m\to\infty} sup_{u \in K} d_{\infty}(u, g_m(u))$$
$$\leq \lim_{m\to\infty} sup_{u \in K} \phi(u, 1/m) = 0. \text{ Q.E.D.}$$

Proof of Theorem 3.2: It suffices to prove the theorem for $C = 1$. Let $\epsilon > 0$ and $0 < \delta < 1$ be given. By compact uniform integrability of $\{X_n\}$, we can choose a compact subset $A$ of $(F_C(R^p), d_{\infty})$ such that

(3.1) $\int_{\{X_n \notin A\}} \| X_n \| \ dP < \varepsilon\delta/12$ for all $n$.

We may assume that $0 \in A$. If we let
$$K = A \oplus (-1)A = \{u \oplus (-1)v \mid u, v \in A\},$$
then $K$ is compact and $0 \in K$. Since the convex hull of $K$ is relatively compact by Lemma 3.3, we may assume that $K$ is convex without loss of generality. Also, (3,1) implies that

(3.2) $\int_{\{X_n \notin K\}} \| X_n \| \ dP < \varepsilon\delta/12$ for all $n$.

By Lemma 3.4, we can choose $m$ such that

(3.3) $d_{\infty}(u, g_m(u)) < \epsilon/6$ for all $u \in K$.

Let us denote
$$Y_i = I_{\{X_i \in K\}}X_i \text{ and } Z_i = I_{\{X_i \notin K\}}X_i.$$
Then $X_i = Y_i \oplus Z_i$.

We note that by construction of $K$, we have
$$E(Y_i) \in K \text{ and } \oplus_{i=1}^{n}\lambda_{ni}E(Y_i) \in K.$$
Thus, by (3.3),

(3.4) $d_{\infty}(\oplus_{i=1}^{n}\lambda_{ni}EY_i, g_m(\oplus_{i=1}^{n}\lambda_{ni}EY_i)) < \varepsilon/6$.

Then by (3.1) and (3.3), we have
$$d_{\infty}(\oplus_{i=1}^{n}\lambda_{ni}E(X_i), g_m(\oplus_{i=1}^{n}\lambda_{ni}E(X_i)))$$

$$\leq d_\infty(\oplus^n_{i=1}\lambda_{ni}E(Z_i), g_m(\oplus^n_{i=1}\lambda_{ni}E(Z_i))) + \varepsilon/6$$

$$\leq 2\sum^n_{i=1} |\lambda_{ni}| \| E(Z_i) \| + \varepsilon/6$$

$$\leq \varepsilon\delta/6 + \varepsilon/6 \leq \varepsilon/3$$

Hence we obtain

$$d_\infty(\oplus^n_{i=1}\lambda_{ni}X_i, \oplus^n_{i=1}\lambda_{ni}E(X_i))$$

$$\leq d_\infty(\oplus^n_{i=1}\lambda_{ni}X_i, \oplus^n_{i=1}\lambda_{ni}g_m(X_i))$$

$$+ d_\infty(\oplus^n_{i=1}\lambda_{ni}g_m(X_i), \oplus^n_{i=1}\lambda_{ni}g_m(E(X_i))) + \varepsilon/3.$$

This implies that

$$P\{d_\infty(\oplus^n_{i=1}\lambda_{ni}X_i, \oplus^n_{i=1}\lambda_{ni}E(X_i))\rangle \varepsilon\}$$

$$\leq P\{d_\infty(\oplus^n_{i=1}\lambda_{ni}X_i, \oplus^n_{i=1}\lambda_{ni}g_m(X_i))\rangle \varepsilon/3\} +$$

$$P\{d_\infty(\oplus^n_{i=1}\lambda_{ni}g_m(X_i), \oplus^n_{i=1}\lambda_{ni}g_m(EX_i))\rangle \varepsilon/3\}$$

$$= (I) + (II).$$

For ( I ). by using (3.2), (3.3) and (3.4), we have

$$(I) \leq P\{d_\infty(\oplus^n_{i=1}\lambda_{ni}Z_i, g_m(\oplus^n_{i=1}\lambda_{ni}Z_i))\rangle \varepsilon/6\}$$

$$\leq P\{2 \| \oplus^n_{i=1}\lambda_{ni}Z_i \| \rangle \varepsilon/6\}$$

$$\leq \frac{12}{\varepsilon} E \| \oplus^n_{i=1}\lambda_{ni}Z_i \|$$

$$\leq \frac{12}{\varepsilon} \sum^n_{i=1} |\lambda_{ni}| E \| Z_i \| \leq \delta.$$

Now for ( II ). since $g_m(E(X)) = E(g_m(X))$, we have

$$(II)$$

$$= P\{\sup_{0\leq k\leq m} h(\oplus^n_{i=1}\lambda_{ni}L_{k/m}X_i,$$

$$\oplus^n_{i=1}\lambda_{ni}E(L_{k/m}X_i))\rangle \varepsilon/3\}$$

$$\leq \sum^m_{k=0} P\{h(\oplus^n_{i=1}\lambda_{ni}L_{k/m}X_i,$$

$$\oplus^n_{i=1}\lambda_{ni}E(L_{k/m}X_i))\rangle \frac{\varepsilon}{3(m+1)}\}$$

$$\leq \delta$$

for sufficiently large $n$ from the assumption (C).
This completes the proof. Q.E.D.

**Corollary 3.5.** Let $\{X_n\}$ be a sequence of level-continuous fuzzy random variables. If $\{X_n\}$ is tight and $sup_n E | X_n |^r < \infty$ for some $r > 1$, then (C) implies (A).

**Proof:** This follows from the fact that tightness and $r$-th $(r > 1)$ moment condition implies compact uniform integrability. Q.E.D.

The next example shows that tightness and 1-st moment condition do not imply compact uniform integrability.

**Example.** Let $u\in F_C(R^p)$ be a fixed fuzzy set with $| u | = 1$. If $\{X_n\}$ is a sequence of fuzzy random

variables such that

$$P(X_n = nu) = 1/n \quad \text{and} \quad P(X_n = 0) = 1 - 1/n,$$

then $\{X_n\}$ is tight and $E\| X_n \| = 1$.

But for each compact subset $K$ of $(F_C(R^p), d_\infty)$, there exists a $n$ such that $nu \not\in K$. Hence,

$$\int_{\{X_n \not\in A\}} \| X_n \| dP = 1 \text{ for all } n.$$

However, we can obtain similar results by requiring identical distribution.

**Corollary 3.6.** Let $\{X_n\}$ be a sequence of level-continuous fuzzy random variables. If $\{X_n\}$ is identically distributed and $E | X_1 | < \infty$, then (C) implies (A).

**Proof:** Note that if $\{X_n\}$ is identically distributed, then it is tight. Thus for each $n$, we can choose a compact subset $K_n$ of $(F_C(R^p), d_\infty)$ such that

$$P(X_1 \not\in K_n) < 1/n.$$

Then $I_{\{X_1 \not\in K_n\}} \to 0$ almost uniformly as $n \to \infty$. By Lebesgue dominated convergence, we have

$$\int_{\{X_1 \not\in K_n\}} \| X_1 \| dP \to 0 \text{ as } n \to \infty.$$

Hence, for each $\epsilon > 0$, there exists a compact subset $K$ of $(F_C(R^p), d_\infty)$ such that for all $n$,

$$\int_{\{X_n \not\in K\}} \| X_1 \| dP = \int_{\{X_1 \not\in K\}} \| X_1 \| dP < \varepsilon.$$

which implies compact uniform integrability of $\{X_n\}$. This completes the proof. Q.E.D.

## References

[1]. G. Debreu, Integration of correspondences, Proc. Fifth Berkley Symp. on Math. Statist. Prob. Vol. 2 (1966), 251-372.

[2]. H. Inoue, A strong law of large numbers for fuzzy random sets, Fuzzy Sets and Systems 41 (1991), 285-291.

[3] .G. S. Choi, S. Y. Joo, Y. K. Kim and J. S. Kwon, Convergence in distribution for level-continuous fuzzy random sets, submitted.

[4]. S. Y. Joo, Strong law of large numbers for tight fuzzy random variables, J. Korean Statist. Soc. 31 (2002), 129-140.

[5]. S. Y. Joo, Weak laws of large numbers for fuzzy random variables, Fuzzy Sets and Systems 147 (2004), 453-464.

[6]. S. Y. Joo and Y. K. Kim, Topological properties

on the space of fuzzy sets, J. Math. Anal. Appl. 246 (2000), 576-590.

[7]. S. Y. Joo and Y. K. Kim, Kolmogorov's strong law of large numbers for fuzzy random variables, Fuzzy Sets and Systems 120 (2001), 499-503.

[8]. S. Y. Joo, Y. K. Kim and J. S. Kwon, Strong convergence for weighted sums of fuzzy random sets, submitted.

[9]. Y. K. Kim, Measurability for fuzzy valued functions, Fuzzy Sets and Systems 129 (2002), 105-109.

[10]. Y. K. Kim, Strong laws of large numbers for random upper semi-continuous fuzzy sets, Bull. Korean Math. Soc. 39 (2002), 511-526.

[11]. Y. K. Kim, Strong laws of large numbers for tight fuzzy random variables, Journal of fuzzy logic and intelligence 13 (2003, 754-758.

[12]. E. P. Klement, M. L. Puri and D. A. Ralescu, Limit theorems for fuzzy random variables, Proc. Roy. Soc. London Ser. A 407 (1986), 171-182.

[13]. I. S. Molchanov, On strong laws of large numbers for random upper semi-continuous functions, J. Math. Anal. Appl. 235(1999), 349-355.

[14]. M. L. Puri and D. A. Ralescu, Fuzzy random variables, J. Math. Anal. Appl 114 (1986), 402-422.

[15]. R. L. Taylor and H. Inoue, A strong law of large numbers for random sets in Banach spaces, Bull. Inst. Math. Academia Sinica 13(1985), 403-409.

[16]. R. L. Taylor, L. Seymour and Y. Chen, Weak laws of large numbers for fuzzy random sets, Nonlinear Analysis 47(2001), 1245-1256.

[17]. T. Uemura, A law of large numbers for random sets, Fuzzy Sets and Systems 59(1983), 181-188.

## 저 자 소 개



김윤경(YunKyongKim)

He received the B.S. degree in Mathematics Education from Kangwon National University in 1978.

And then, he received the M.S. and Ph. D degrees in Mathematics from Korea University in 1983 and 1988, respectively. From 1989 to 2002, he was a professor in Dept. of Mathematics, Dongshin University. Since 2002, he has been a professor in Dept. of Information & Communication Engineering, Dongshin University. His research interests center on Fuzzy Probability Theory, Fuzzy Measure Theory and related fields.

Phone    : 061-330-3312
Fax      : 061-330-3190
E-mail   : ykkim@dsu.ac.kr