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Controller Design for Fuzzy Systems via Piecewise Quadratic Value Functions

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Abstract

This paper concerns controller design for the Takagi-Sugeno (TS) fuzzy systems. The design method proposed in this paper is derived in the framework of the optimal control theory utilizing the piecewise quadratic optimal value functions. The major part of the proposed design procedure consists of solving linear matrix inequalities (LMIs). Since LMIs can be solved efficiently within a given tolerance by the recently developed interior point methods, the design procedure of this paper is useful in practice. A design example is given to illustrate the applicability of the proposed method.

Key words : Fuzzy control, Optimal control, Takagi-Sugeno fuzzy model, Optimal value function, Linear matrix inequality

I. Introduction

Providing systematic controller design procedure is a very important research topic in the area of fuzzy control [1-3]. In this paper, we present a new design procedure yielding robust and stabilizing controllers for the nonlinear systems described by the TS (Takagi-Sugeno) fuzzy model [4-6]. Based on the well-known fact that the robustness achieved as a result of the optimality is largely independent of the particular choice of a certain part of the cost function, we address the problem of designing robust and stabilizing controllers for the TS fuzzy systems in the framework of the optimal control theory. Also, it is shown that with the optimal value functions which are piecewise quadratic and continuous, the problem of finding the parameters of the optimal controllers can be represented as an LMI (linear matrix inequality) problem. Formulation of the controller synthesis problems with LMIs is generally considered to be a practical solution to the problem since they can be solved by reliable and efficient convex optimization tools [7], e.g., the LMI control Toolbox for use with Matlab [8].

Throughout this paper, we use the following definitions and notation, in which R^n denotes the normed linear space of real n-vectors. A symmetric matrix $A \in R^{n \times n}$ is positive definite if $x^T A x > 0$ for any $x \neq 0$, and $A > 0$ denotes this. Also, the inequality $A < 0$ means that the symmetric matrix A is negative definite, i.e., $x^T A x < 0$ for any $x \neq 0$. $L_f h(x)$ denotes the Lie derivative of a scalar function $h: R^n \to R$ with respect to a vector field $f: R^n \to R^n$ i.e., $L_f h(x) \triangleq (\partial h / \partial x) f(x)$. By I, we denote the identity matrix.

The rest of this paper is organized as follows: In Section 2, preliminaries are provided regarding the TS fuzzy model and optimal control theory. Our main results on the design of optimal controllers for the TS fuzzy systems are presented in Section 3. In Section 4, controllers are designed for the inverted pendulum system to illustrate the proposed method. Finally, concluding remarks are given in Section 5.

2. Preliminaries

In this paper, we are concerned with the design of optimal controllers for the systems described by the TS fuzzy model. The **IF-THEN** rules of the TS fuzzy model are given in the following form [4-6]:

Plant Rule l:

IF $x_1(t)$ is M_1^l and $\cdots$ $x_n(t)$ is M_n^l

THEN

$$\dot{x}(t) = A_l x(t) + B_l u(t) + a_l \tag{1}$$

$l = 1, \cdots, m.$

Here, $x_i(t)$, $i = 1, \cdots$, n and M_i^l, $i = 1, \cdots$, n, $l = 1, \cdots$,m are state variables and fuzzy sets, respectively, $x(t) \in D \subset R^n$ and $u(t) \in R^p$ are the state and input vector, respectively, and m is the number of **IF-THEN** rules. Constant matrices A_l, B_l, and a_l of compatible dimensions represent the l-th local model of the system. Following the usual inference method utilizing the singleton fuzzifier, product inference, and center-average defuzzifier, we have the following state equation for the TS fuzzy system:

$$\dot{x}(t) = \frac{\sum_{l=1}^{m} w_l(x(t)) \{A_l x(t) + B_l u(t) + a_l\}}{\sum_{l=1}^{m} w_l(x(t))} \tag{2}$$

In (2), the w_l are defined as $w_l(x(t)) = \Pi_{i=1}^{n} M_i^l(x_i(t))$, where $M_i^l(x_i(t))$ is the grade of membership of $x_i(t)$ in the fuzzy set M_i^l. Each weight function $w_l(x)$ takes a nonnegative value for each $x \in R^n$, and usually satisfy

$$\sum_{l=1}^{m} w_l(x) > 0 \text{ for any } x \in D. \tag{3}$$

In this paper, we will assume that (3) holds always, and

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that the state vector $x(t)$ is available in real time. With the normalization of weight functions

$$\mu_l(x) \triangleq w_l(x) / \sum_{i=1}^{m} w_i(x) \tag{4}$$

the state equation (2) can be written in the following form :

$$\dot{x}(t) = \sum_{l=1}^{m} \mu_l(x(t)) \{A_l x(t) + B_l u(t) + a_l\}, \tag{5}$$

where the normalized weight function μ_l satisfy $\mu_l(x) \geq 0, \forall l \in \{1, \cdots, m\}$, and $\sum_{l=1}^{m} \mu_l(x) = 1$ for any $x \in D$. For simplicity, we will denote the normalized weight function $\mu_l(x(t))$ by μ_l from now on.

Now, let $L \triangleq \{1, \cdots, m\}$ be the index set for the local models (1), and let S_l be the cell where the l-th local model plays a dominants role, i.e., for $l = 1, \cdots, m$,

$$S_l \triangleq \{x \in R^n \mid \mu_l(x) \geq \mu_i(x) \ \text{ for} \forall i \in L\} \tag{6}$$

In general, the cell $S_l \subset R^n$ are n-dimensional convex polyhedra, which will be assumed to be true throughout this paper. Since the stability of the origin is of primal importance, we let $L_0 \subset L$ be the index set for cells which do not contain the origin, and let L_1 be the complement set of L_0 in L. For notational convenience, we also introduce

$$\overline{A}_l \triangleq \begin{bmatrix} A_l & a_l \\ 0 & 0 \end{bmatrix}, \ \overline{B}_l \triangleq \begin{bmatrix} B_l \\ 0 \end{bmatrix}, \text{ and } \ \overline{x} \triangleq \begin{bmatrix} x \\ 1 \end{bmatrix}, \tag{7}$$

where it is assumed that $a_l = 0$ for all $l \in L_0$. Using this notation, the TS fuzzy system (5) can also be written as

$$\frac{d\overline{x}}{dt}(t) = (\sum_{i=1}^{m} \mu_i \overline{A}_i)\overline{x}(t) + (\sum_{i=1}^{m} \mu_i \overline{B}_i) u(t) \tag{8}$$

for $x(t) \in S_l$, $l \in L$. For future reference , we also define $N_l \triangleq \{i \in L \mid \mu_i(x) > 0 \text{ for } some\ x \in S_l\}$.

Note that in S_l, both $\sum_{i=1}^{m} \mu_i$ and $\sum_{i \in N_l} \mu_i$ have the same meaning.

The TS fuzzy controller for the TS fuzzy system of (1) is described by the following fuzzy **IF-THEN** rule [6,9]:

Controller Rule l:

IF $x_1(t)$ is M_1^l and $\cdots$ and $x_n(t)$ is M_n^l,

THEN

$$u(t) = K_l x(t) + k_l \ , \tag{9}$$

l=1, $\cdots$, m.

In general, the value of k_l in (9) is zero for $l \in L_0$ to meet the stability of $x = 0$. Note that the IF part of the above controller rule shares the same fuzzy set with that of the plant rule of (1). Also, note that the usual inference method for the TS fuzzy model yields the following representation for the TS fuzzy controller:

$$u(t) = \sum_{l=1}^{m} \mu_l \{K_l\, x(t) + k_l\}. \tag{10}$$

Here, the μ_l are the same as in (5), and the controller parameters need to be found so that design goals such as stability and robustness may be met. Our main strategy for finding a satisfactory set of the K_l and k_l is to utilize the optimal control theory (see [10], for example).

One of the most important problems in the area of optimal control is to find an optimal feedback control law for the nonlinear system described by

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t). \tag{11}$$

In the problem, we wish to find a control law $u(t) = u(x(t))$ which can achieve the following: (1) Asymptotic stability of the equilibrium point x=0. (2) Minimization of the cost function

$$J = \int_0^{\infty} \{l(x(t)) + u(t)^T R(x(t)) u(t)\} dt, \tag{12}$$

where $l(x)$ is a positive definite function and $R(x) = R(x)^T$ is a positive definite matrix for any $x \in R^n$. For a given feedback control $u(x)$, the value of cost J depends on the initial state $x(0)$. Thus, we write the value of J as $J(x(0))$, or simply $J(x)$. When J is at its minimum, $J(x)$ is called the optimal value function. As is shown in the next lemma [10], the above optimal control problem can be reduced to solving the **HJB** (Hamilton-Jacobi-Bellman) equation.

Lemma 1[10]. Suppose that there exists a positive definite function $V(x) \in C^1(R^n)$ which satisfies the ***HJB*** equation

$$l(x) + L_f V(x) - \frac{1}{4}(L_g V(x)) R^{-1}(x) (L_g V(x))^T = 0,$$
$$V(0) = 0 \tag{13}$$

and the control $u^* = -\frac{1}{2} R^{-1}(x)(L_g V(x))^T$ *achieves the asymptotic stability of the equilibrium point* x*=0 for the system (11). Then* u^* *is the optimal stabilizing control which minimizes the cost function (12) over all* $u(t)$ *guaranteeing* $\lim_{t \to \infty} x(t) = 0$, *and* $V(x)$ *is the optimal value function*

Many optimal control problems deal with the fixed cost functions. However, to solve their corresponding HJB equation is not a feasible task in general. On the other hand, it is well-known that the robustness achieved as a result of the optimality is largely independent of the particular choice of $l(x)$ when R(x)=I. Hence, it is motivated to pursue the strategy in which the positive definite function $l(x)$ is a posteriori determined at the stage of controller design. More precisely, we use the following lemma [10]:

Lemma 2[10]. A stabilizing control u^* solves an optimal control problem for system (11) if it is of the form

$$u^* = -\frac{1}{2}(L_g V(x))^T, \tag{14}$$

where $V(x) \in C^1(R^n)$ *is a positive definite function such* *that*

$$l(x) \triangleq -L_f V(x) + \frac{1}{4}(L_g V(x))(L_g V(x))^T > 0 \qquad (15)$$

for any $x \neq 0$.

In this paper, Lemma 2 is utilized in the following manner. First, we find a positive definite function $V(x) \in C^1(R^n)$ satisfying the inequality (15) for any $x \neq 0$. Then, it is obvious that the function $V(x)$ satisfies the HJB equation (13) with $l(x)$ defined as in (15) and $R(x) = I$. Moreover, under the condition that $V(x) \in C^1(R^n)$ is a positive definite function satisfying (15), the following holds true for any $x \neq 0$:

$$\begin{aligned}\frac{dV(x)}{dt}\Big|_{u=u^*} &= L_f V(x) - \frac{1}{2}(L_g V(x))(L_g V(x))^T \\ &= (L_f V(x) - \frac{1}{4}(L_g V(x))(L_g V(x))^T) \\ &\quad - (L_g V(x))(L_g V(x))^T/4 \\ &= -l(x) - \|u^*\|^2 < 0.\end{aligned}$$

Thus, by the Lyapunov stability theorem [11], u^* is a stabilizing control. Therefore, we can conclude that under the condition of Lemma 2, it is guaranteed that u^* obtained via (14) is the optimal stabilizing feedback minimizing $J = \int_0^\infty (l(x) + u^T u)dt$.

Finally, in this paper, the problem of finding optimal controllers for the TS fuzzy system will be formulated as the LMI(linear matrix inequality) problem. An LMI is any constraint of the form

$$A(z) \triangleq A_0 + z_1 A_1 + \cdots + z_N A_N < 0, \qquad (16)$$

where $z \triangleq (z_1, \cdots, z_N)$ is the variable, $A_0, \cdots, A_N$ are the given symmetric matrices, and "<" stands for negative definiteness. Since $A(y) < 0$ and $A(z) < 0$ imply $A(\lambda y + (1-\lambda)z) < 0$ for any $\lambda \in [0,1]$, the LMI (16) is a convex constraint on the variable z. It is well known that the LMI feasibility problem which finds a solution z satisfying (16) or determines that there does not exist such z can be solved efficiently [7], and a toolbox of Matlab for convex problems involving LMIs is now available [8].

3.LMI-based design procedure

In this section, we establish a design procedure for the optimal control of the TS fuzzy systems. First, note that the TS fuzzy system (5) is an example of the class represented by the canonical form (11) with $f(x) = \sum_{i=1}^{m} \mu_i \{A_i x + a_i\}$ and $g(x) = \sum_{i=1}^{m} \mu_i B_i$; thus Lemma 2 is applicable to our problem. Next, for the sake of convenience in controller design, we restrict focus only on the cases that the optimal value function $V(x)$ can be expressed in a piecewise quadratic function which is continuous across the cell boundaries. As shown in [5]. a convenient way to express such piecewise quadratic functions is via constructing matrices $\overline{F}_l = [F_l \ f_l]$ with $f_l = 0$ for $l \in L_0$ satisfying

$$\overline{F}_l \begin{bmatrix} x \\ 1 \end{bmatrix} = \overline{F}_j \begin{bmatrix} x \\ 1 \end{bmatrix} \text{ for } x \in \overline{S}_l \cap \overline{S}_j, \ l, j \in L. \qquad (17)$$

Here, it is demanded that the F_l and $\overline{F}_l$ satisfy the following rank condition: F_{l_0} has full column rank for $l_0 \in L_0$, and $\overline{F}_{l_1}$ has the full column rank for $l_1 \in L_1$. A systematic procedure for finding such F_{l_0} and $\overline{F}_{l_1}$ is given in [5]. Once the F_{l_0} and $\overline{F}_{l_1}$ are constructed, a class of the piecewise quadratic function that are continuous across the cell boundaries can be parametrized as

$$V(x) = \begin{cases} x^T P_{l_0} x & \text{for } x \in \overline{S}_{l_0}, l_0 \in L_0 \\ \begin{bmatrix} x \\ 1 \end{bmatrix}^T \overline{P}_{l_1} \begin{bmatrix} x \\ 1 \end{bmatrix} & \text{for } x \in \overline{S}_{l_1}, l_1 \in L_1 \end{cases} \qquad (18)$$

with

$$P_{l_0} \triangleq F_{l_0}^T \, T \, F_{l_0}, \quad l_0 \in L_0, \qquad (19)$$

$$\overline{P}_{l_1} \triangleq \overline{F}_{l_1}^T \, T \, \overline{F}_{l_1}, \quad l_1 \in L_1, \qquad (20)$$

where the free parameters of $V(x)$ are collected in the symmetric matrix T. Note that with the parametrization (18), the controller u^* of (14) can be reduced to

$$u^* = \begin{cases} \sum_{i=1}^{m} \mu_i (-B_i^T P_i) x & \text{for } x \in \overline{S}_{l_0}, l_0 \in L_0 \\ \sum_{i=1}^{m} \mu_i (-\overline{B}_i^T \overline{P}_i) \begin{bmatrix} x \\ 1 \end{bmatrix} & \text{for } x \in \overline{S}_{l_1}, l_1 \in L_1 \end{cases} \qquad (21)$$

Also, note that the optimal controller (21), which is continuous since the F_l and $\overline{F}_l$ satisfy the continuity of (17), takes the form of the TS fuzzy controller (10). For related details on this observation, please refer to [12].

Applying Lemma 2 to the case with quadratic value functions (18), we see that the TS fuzzy controller u^* of (21) becomes an optimal stabilizing controller for the TS fuzzy system (5) if $T > 0$ and the following holds for any $x \in D - \{0\}$:

$$l(x) \triangleq -L_f V(x) + (L_g V(x))(L_g V(x))^T/4 =$$

$$\begin{cases} -x^T\Big\{(\sum_{i=1}^{m}\mu_i A_i)^T P_{l_0} + P_{l_0}(\sum_{i=1}^{m}\mu_i A_i) \\ \quad - P_{l_0}(\sum_{i=1}^{m}\mu_i B_i)(\sum_{i=1}^{m}\mu_i B_i)^T P_{l_0}\Big\} x \ \text{ for } x \in \overline{S}_{l_0}, \ l_0 \in L_0 \\ -\overline{x}^T\Big\{(\sum_{i=1}^{m}\mu_i \overline{A}_i)^T \overline{P}_{l_1} + \overline{P}_{l_1}(\sum_{i=1}^{m}\mu_i \overline{A}_i) \\ \quad - \overline{P}_{l_1}(\sum_{i=1}^{m}\mu_i \overline{B}_i)(\sum_{i=1}^{m}\mu_i \overline{B}_i)^T \overline{P}_{l_1}\Big\}\overline{x} \ \text{ for } x \in \overline{S}_{l_1}, \ l_1 \in L_1 \end{cases}$$

$$> 0 \qquad , \qquad (22)$$

where P_{l_0} and $\overline{P}_{l_1}$ and matrices shown in (19) and (20), Hence, we have the following :

Theorem 1. Consider the TS fuzzy system (5), and matrices F_l and $\overline{F}_l$ satisfying (17). If there exists a positive definite matrix T such that

$$P_{l_0} \triangleq F_{l_0}^T T F_{l_0}, \quad l_0 \in L_0$$
$$\overline{P}_{l_1} \triangleq \overline{F}_{l_1}^T T \overline{F}_{l_1}, \quad l_1 \in L_1$$

satisfy (22), then the feedback (21) is the optimal stabilizing control which minimizes the cost function $J=\int_0^{\infty}(l(x(t))+\|u(t)\|^2)dt$ *with* $l(x)$ *of (22).*

Theorem 1 can be extended toward directions that can cover various controller design goals. In this paper, we consider the case with a further design goal on the decay rate. The decay rate of a given systems is defined to be the largest β such that

$$\lim_{t\to\infty} \exp(\beta t)\|x(t)\| = 0 \tag{23}$$

holds for each trajectory $x(t)$ of the system [7]. Utilizing the fact that the existence of positive definite quadratic function $V(x)$ satisfying $dV(x)/dt \le -2\beta V(x)$ for all trajectories guarantees the decay rate of the system greater than β [7], we have the following conjecture: *Consider the TS fuzzy system (5), a desirable lower bound* $\beta > 0$ *for the decay rate, and the matrices* F_l and f_l *satisfying (17). If there exists a positive definite matrix* T *such that*

$$P_{l_0} \triangleq F_{l_0}^T T F_{l_0}, \quad l_0 \in L_0$$
$$\overline{P}_{l_1} \triangleq \overline{F}_{l_1}^T T \overline{F}_{l_1} \quad l_1 \in L_1$$

satisfy

$$(\sum_{i=1}^{m}\mu_i A_i)^T P_{l_0} + P_{l_0}(\sum_{i=1}^{m}\mu_i A_i) + 2\beta P_{l_0} - P_{l_0}(\sum_{i=1}^{m}\mu_i B_i)(\sum_{i=1}^{m}\mu_i B_i)^T P_{l_0} < 0, \; x \in \overline{S}_{l_0}, \; l_0 \in L_0 \tag{24}$$

and

$$(\sum_{i=1}^{m}\mu_i \overline{A}_i)^T \overline{P}_{l_1} + \overline{P}_{l_1}(\sum_{i=1}^{m}\mu_i \overline{A}_i) + 2\beta \overline{P}_{l_1} - \overline{P}_{l_1}(\sum_{i=1}^{m}\mu_i \overline{B}_i)(\sum_{i=1}^{m}\mu_i \overline{B}_i)^T \overline{P}_{l_1} < 0, \; x \in \overline{S}_{l_1} \; l_1 \in L_1 \tag{25}$$

then the feedback (21) satisfies the following :

- It is the optimal stabilizing control which minimizes the cost function $J=\int_0^{\infty}(l(x(t))+\|u(t)\|^2)dt$ with $l(x)$ of (22).
- It makes the decay rate of the closed-loop greater than $\beta > 0$.

In order to find a positive definite matrix T satisfying the above conditions, the method of this paper proceeds as follows: First, note that by the congruence transformation [7], the inequalities (24) and (25) are equivalent to (26) and (27) below, respectively:

$$\Gamma_{l_0}(x) \triangleq Q_{l_0}(\sum_{i=1}^{m}\mu_i A_i)^T + (\sum_{i=1}^{m}\mu_i A_i)Q_{l_0} + 2\beta Q_{l_0} - (\sum_{i=1}^{m}\mu_i B_i)(\sum_{i=1}^{m} m_i B_i)^T < 0, \; x \in \overline{S}_{l_0}, \; l_0 \in L_0 \tag{26}$$

$$\overline{\Gamma}_{l_1}(x) \triangleq \overline{Q}_{l_1}(\sum_{i=1}^{m}\mu_i \overline{A}_i)^T + (\sum_{i=1}^{m}\mu_i \overline{A}_i)\overline{Q}_{l_1} + 2\beta \overline{Q}_{l_1} - (\sum_{i=1}^{m}\mu_i \overline{B}_i)(\sum_{i=1}^{m} m_i \overline{B}_i)^T < 0, \; x \in \overline{S}_{l_1} \; l_1 \in L_1 \tag{27}$$

where

$$Q_{l_0} \triangleq P_{l_0}^{-1} = (F_{l_0}^T F_{l_0})^{-1} F_{l_0}^T T^{-1} F_{l_0} (F_{l_0}^T F_{l_0})^{-1} \text{ for } l_0 \in L_0 \tag{28}$$

and

$$\overline{Q}_{l_1} \triangleq \overline{P}_{l_1}^{-1} = (\overline{F}_{l_1}^T \overline{F}_{l_1})^{-1} \overline{F}_{l_1}^T T^{-1} \overline{F}_{l_1} (\overline{F}_{l_1}^T \overline{F}_{l_1})^{-1} \text{ for } l_1 \in L_1 \tag{29}$$

Note that here $(F_{l_0}^T F_{l_0})$ and $(\overline{F}_{l_1}^T \overline{F}_{l_1})$ are all invertible since F_{l_0} and $\overline{F}_{l_1}$ have full column rank for $l_0 \in L_0$ and $l_1 \in L_1$, respectively.

Next, note that we can find the following bounds in each cell $\overline{S}_l$:

$$-(\sum_{i=1}^{m}\mu_i B_i)(\sum_{i=1}^{m}\mu_i B_i)^T \le -E_{lB}E_{lB}^T. \tag{30}$$

Since the μ_i and B_i are given precisely, the above bounds can be easily found. For a hint on how to get these bounds, see [13], for example. Also, (30) together with (7) give the following bounds :

$$-(\sum_{i=1}^{m}\mu_i \overline{B}_i)(\sum_{i=1}^{m}\mu_i \overline{B}_i)^T \le -E_{l\overline{B}}E_{l\overline{B}}^T \triangleq \begin{bmatrix} -E_{lB}E_{lB}^T & 0 \\ 0 & 0 \end{bmatrix} \tag{31}$$

Thus, we have the following upper bounds in each cell $\overline{S}_l$:

$$\Gamma_{l_0}(x) \le (\sum_{i=1}^{m}\mu_i A_i)Q_{l_0} + Q_{l_0}(\sum_{i=1}^{m}\mu_i A_i)^T + 2\beta Q_{l_0} - E_{l_0 B}E_{l_0 B}^T < 0 \quad \text{for } x \in \overline{S}_{l_0}, \; l_0 \in L_0 \tag{32}$$

$$\overline{\Gamma}_{l_1}(x) \le (\sum_{i=1}^{m}\mu_i \overline{A}_i)\overline{Q}_{l_1} + \overline{Q}_{l_1}(\sum_{i=1}^{m}\mu_i \overline{A}_i)^T + 2\beta \overline{Q}_{l_1} - E_{l_1\overline{B}}E_{l_1\overline{B}}^T \quad \text{for } x \in \overline{S}_{l_1}, \; l_1 \in L_1 \tag{33}$$

Hence, the following guarantee that $\Gamma_{l_0}(x) < 0$ in each $\overline{S}_{l_0}$, $l_0 \in L_0$, and $\overline{\Gamma}_{l_1}(x) < 0$ in each $\overline{S}_{l_1}$, $l_1 \in L_1$:

$$\Gamma^i_{l_0} \triangleq A_i Q_{l_0} + Q_{l_0} A_i^T + 2\beta Q_{l_0} - E_{l_0 B}E_{l_0 B}^T < 0 \quad \text{for } i \in N_{l_0}, \quad l_0 \in L_0. \tag{34}$$

$$\Gamma^i_{l_1} \triangleq \overline{A}_i \overline{Q}_{l_1} + \overline{Q}_{l_1} \overline{A}_i^T + 2\beta \overline{Q}_{l_1} - E_{l_1\overline{B}}E_{l_1\overline{B}}^T < 0 \quad \text{for } i \in N_{l_1}, \; l_1 \in L_1. \tag{35}$$

Therefore, we have the following LMI-based design procedure:

[Design procedure]

Step 1: Given a TS fuzzy system (5), a desirable lower bound

$\beta > 0$ for the decay rate, and the matrices F_{l_0} and $\overline{F}_{l_1}$ satisfying the continuity condition (17), solve the following LMIs to obtain a symmetric matrix T:

$$\begin{array}{l} T>0 \\ A_i Q_{lo} + Q_{lo} A_i^T + 2\beta Q_{lo} - E_{loB} E_{loB}^T < 0 \\ \text{for } l_o \in L_o, i \in N_{lo} \\ \overline{A}_i \overline{Q}_{l1} + \overline{Q}_{l1} \overline{A}_i^T + 2\beta \overline{Q}_{l1} - E_{l1\overline{B}} E_{l_1\overline{B}}^T < 0 \\ \text{for } l_1 \in L_1, i \in N_{l1} \end{array} \qquad (36)$$

where Q_{l_0} and $\overline{Q}_{l_1}$ are defined as in (28) and (29), respectively

Step 2: Compute

$$K_{lo} = -B_{lo}^T Q_{lo}^{-1} = -B_{lo}^T F_{lo}^T T F_{lo} \text{ for } l_o \in L_o,$$
$$K_{l1} = -\overline{B_{l1}}^T \overline{Q_{l1}}^{-1} = -\overline{B_{l1}}^T \overline{F_{l1}}^T T \overline{F_{l1}} \text{ for } l_1 \in L_1, \quad (37)$$

and set

$$u^* = \begin{cases} \sum_{i=1}^{m} \mu_i K_i x & \text{for } x \in \overline{S}_{l_0},\ l_0 \in L_0 \\ \sum_{i=1}^{m} \mu_i \overline{K}_i \begin{bmatrix} x \\ 1 \end{bmatrix} & \text{for } x \in \overline{S}_{l_1},\ l_1 \in L_1 \end{cases} \qquad (38)$$

4. A numerical example

In this section, we applied the proposed design procedure to the problem of balancing an inverted pendulum on a cart [9], The state equation for the pendulum system are

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \frac{g\sin(x_1) - amlx_2^2\sin(2x_1)/2 - a\cos(x_1)u}{4l/3 - aml\cos^2(x_1)}$$

where $x_1 \in (-\pi/2, \pi/2)$ is the angle of the pendulum from the vertical, x_2 is the angular velocity of the pendulum, $g = 9.8m/s^2$ is the gravity constant, m is the mass of the pendulum, M is the mass of the cart, a is the constant $1/(m+M)$, $2l$ is the length of the pendulum, and u is the force applied to the cart. In this numerical example, the system with the following parameters was chosen: $m = 2[kg]$, $M = 8[kg]$, $2l = 1.0[m]$. According to the usual fuzzy approximation scheme, this system can be approximated by a TS fuzzy model with the following four rules :

Rule 1 : **IF** x_1 is about 0° (where $x_1 < 0$)

THEN $\dot{x} = A_1 x + B_1 u$

Rule 2 : **IF** x_1 is about 0° (where $x_1 > 0$)

THEN $\dot{x} = A_2 x + B_2 u$

Rule 3: **IF** x_1 is about - 88°

THEN $\dot{x} = A_3 x + B_3 u + a_3$

Rule 4 : **IF** x_1 is about + 88°

THEN $\dot{x} = A_4 x + B_4 u + a_4$

where the local models are represented by

$$A_1 = \begin{bmatrix} 0 & 1 \\ 17.29410 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ -0.1765 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} 0 & 1 \\ 17.29410 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ -0.1765 \end{bmatrix},$$
$$A_3 = \begin{bmatrix} 0 & 1 \\ 9.360 & 0 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 0 \\ -0.0052 \end{bmatrix}, \quad a_3 = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$
$$A_4 = \begin{bmatrix} 0 & 1 \\ 9.360 & 0 \end{bmatrix}, \quad B_4 = \begin{bmatrix} 0 \\ -0.0052 \end{bmatrix}, \quad a_4 = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

and the normalized membership function are

$$\mu_1 = \begin{cases} 1 + \frac{2x_1}{\pi} & \text{for } -\frac{\pi}{2} < x_1 \le 0 \\ 0 & \text{for } 0 < x_1 < \frac{\pi}{2} \end{cases},$$
$$\mu_2 = \begin{cases} 0 & \text{for } -\frac{\pi}{2} < x_1 < 0 \\ 1 - \frac{2x_1}{\pi} & \text{for } 0 \le x_1 < \frac{\pi}{2} \end{cases},$$
$$\mu_3 = \begin{cases} \frac{-2x_1}{\pi} & \text{for } -\frac{\pi}{2} < x_1 \le 0 \\ 0 & \text{for } 0 < x_1 < \frac{\pi}{2} \end{cases},$$
$$\mu_1 = \begin{cases} 0 & \text{for } -\frac{\pi}{2} < x_1 < 0 \\ +\frac{2x_1}{\pi} & \text{for } 0 \le x_1 < \frac{\pi}{2} \end{cases}.$$

Based on the method of [5], the F_l and $\overline{F}_l$ satisfying (17) were chosen as follows :

$$F_1 = \begin{bmatrix} 0 & 0 \\ -4/\pi & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad F_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 4/\pi & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix},$$
$$\overline{F}_3 = \begin{bmatrix} -4/\pi & 0 & -1 \\ 4/\pi & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \overline{F}_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -4/\pi & 0 & 2 \\ 4/\pi & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}.$$

For the bounds E_{lB} of (30), the following were used:

$$E_{1B} = E_{2B} = (B_1 + B_2)/2,$$
$$E_{3B} = E_{4B} = B_3.$$

The lower bound for the decay rate was set $\beta = 0.5$. By solving the corresponding LMIs (36) with the LMI Control Toolbox [8], we obtained a solution for T. The resulting controller

$$u^* = \begin{cases} \sum_{i=1}^{m} \mu_i K_i x & \text{for } x \in \overline{S}_{l_0},\ l_0 = 1, 2 \\ \sum_{i=1}^{m} \mu_i \overline{K}_i \begin{bmatrix} x \\ 1 \end{bmatrix} & \text{for } x \in \overline{S}_{l_1},\ l_1 = 3, 4 \end{cases}$$

yielded the responses shown in Fig. 1 for the initial conditions $x(0) = [x_1(0)\ 0]^T$, where $x_1(0) = 35°, 70°$.

5. Concluding remarks

In this paper, we considered the problem of designing optimal controllers for the TS fuzzy systems in the framework of the optimal control theory. Utilizing the optimal value functions which are piecewise quadratic and continuous, we derived an LMI-based design procedure. Since LMIs can be solved efficiently within a given tolerance by the interior point methods, LMI-based controller design is useful in practice. A numerical example of balancing the inverted pendulum on the cart was considered for an illustration. Works yet to be done include further studies for theoretical completeness and extensive simulation studies to identify the strength and weakness of the proposed method.

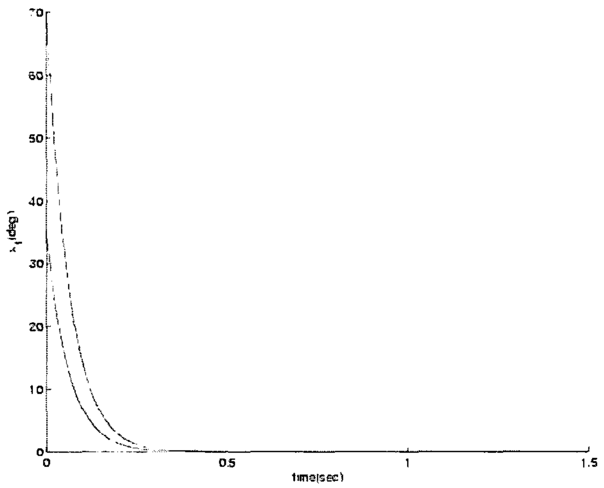

Fig. 1 Responses of the controlled system for the β = 0.5 case

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