Discrete Block Replacement Policies under Random Use Durations

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Abstract

This paper presents and compares two block replacement policies under random use durations. The units are put in service altogether and then idle for some time. The time durations during which units are put in service are random variables. Two block replacement policies, called N-policy and T-policy, are presented. Under N-policy, units are replaced altogether after the Nth use. Under T-policy, units are replaced altogether at the end of the use after cumulative use time T elapses. The failures during use durations are replaced by new ones individually. The cost rate expressions under the policies are derived for exponential use durations. Numerical examples are presented to compare the performances of the two policies.

1. Introduction

Maintenance policies have been a major concern for plant engineers and researchers. Many authors have treated maintenance policies such as age replacement, periodic replacement under minimal repair, and block replacement. Barlow and Proschan(1965) and Valdez-Flores and Feldman(1989) have summarized and discussed them in detail.

Most maintenance policies assume that the maintenance activity can be performed at any time. However, sometimes in practice, it may be economical and convenient to perform the maintenance work at the unit's idle time such as weekend or year-end.

Mine et al.(1981) proposed an age replacement policy in which the unit is preventively replaced when a number of use durations elapses. It is assumed that the use durations are constant and known. Nakagawa(1986a, 1986b, 1987) also considered this problem as discrete replacement policies: The unit is replaced only at periodic times, where the periodic times refers to as weekend, month-end, or year-end, and so forth.

When a number of units are put in service simultaneously, block replacement policy is commonly used. Under the policy, units are preventively replaced altogether at regular intervals. The failed unit is replaced by a new one individually.

In this paper, two block replacement policies under random use durations are presented and compared. Consider a job shop production system in which units are used intermittently. The units are put in service for a random duration when a production order is given, and then idle for some time duration. In such a situation, it is economical and convenient to perform the block replacement at the unit's idle time.

The presented policies, named as N-policy and T-policy, are as follows.

N-policy: Units are replaced altogether at the end of the Nth use duration. The failed unit during operation is individually replaced immediately.

T-policy: Units are replaced altogether at the end of the use after cumulative use time T elapses. The failures during use durations are replaced by new ones individually.

The next section derives the cost rate expressions under N-policy and T-policy. In section 3, numerical examples are presented to compare the performances of the two policies.

Notations

c_r replacement cost of a failed unit;

c_b block replacement cost of a unit;

$\mu, f(t), F(t)$ mean, pdf, cdf of the time-to-failure(TTF) of a unit;

$h(t)$ hazard rate function of a unit;

$H(t)$ cumulative hazard of a unit, where,

$$H(t) = \int_0^t h(x)dx;$$

$M(t), m(t)$ renewal function, renewal density of $F(t)$;

$\nu, G(t)$ mean, cdf of time duration of a use;

$G_n(t)$ n-fold convolution of $G(t)$ with itself.

Assumptions

(i) The time durations of uses are independent identically distributed (i.i.d.) random variables.

(ii) While the unit is not used, the aging of the unit does not occur.

(iii) The block replacement can be performed within the unit's idle time.

2. Cost Rate under the Policies

2.1 N-Policy

Under N-policy, all units are replaced after the Nth use, and each unit is replaced upon failure. The expected number of failures between scheduled replacements depends on the termination time point of the Nth use.

Let $X_i, i = 1, ..., N$, be the *i.i.d.* random variables denoting the duration of ith use, which is associated with $G(t)$. Then, assuming negligible idle time with Assumptions (ii) and (iii), $X_1 + X_2 + ... + X_N$ represents the total use duration, and $G_N(t)$ is its distribution function.

If the Nth use is terminated at time t whereas the 1st use starts at time 0, the conditional expected number of failures during (0,t] is $M(t)$(Barlow and Proschan 1965). Therefore the expected number of failures between two successive block replacements is

$$\int_0^\infty M(t) dG_N(t). \tag{1}$$

The expected time between two successive block replacement is simply $N\nu$. Thus the average cost rate over an infinite time span is, from the renewal reward theorem(Ross 1970),

$$C_1(N) = \frac{c_r \int_0^\infty M(t) dG_N(t) + c_b}{N\nu}. \tag{2}$$

Suppose that the use duration follows an exponential distribution with mean $1/\lambda$, that is, $G(t) = 1 - e^{-\lambda t}$. Then $G_N(t)$ becomes an N-stage Erlang distribution with mean N/λ:

$$G_N(t) = \int_0^t \lambda e^{-\lambda x} \frac{(\lambda x)^{N-1}}{\Gamma(N)} dx, \tag{3}$$

where $\Gamma(N) = \int_0^\infty y^{N-1} e^{-y} dy$.

Further suppose that the TTF of each unit follows a 2-stage Erlang distribution with parameter ρ, $F(t) = \rho^2 t e^{-\rho t}$. Then it follows that(Barlow and Proschan 1965)

$$M(t) = \frac{\rho t}{2} - \frac{1 - e^{-2\rho t}}{4}. \tag{4}$$

Substituting (3) and (4) into (1), we get, by the definition of mean of N-stage Erlang distribution,

$$\int_0^\infty M(t) dG_N(t) = \frac{\lambda^N \rho}{2\Gamma(N)} \int_0^\infty t^N e^{-\lambda t} dt + \frac{\lambda^N}{4\Gamma(N)} \int_0^\infty t^{N-1} e^{-(\lambda+2\rho)t} dt - \frac{1}{4}$$

$$= \frac{N\rho}{2\lambda} + \frac{\lambda^N}{4(\lambda + 2\rho)^N} - \frac{1}{4}. \tag{5}$$

Substituting (5) and $\nu = 1/\lambda$ into (2), the average cost rate under N-policy becomes

$$C_1(N) = c_r \left[\frac{\rho}{2} + \frac{\lambda^{N+1}}{4N(\lambda + 2\rho)^N} \right] + \frac{\lambda}{4N}(4c_b - c_r). \tag{6}$$

We seek N^*, the optimal value of N, which minimizes (6). Forming the inequality $C_1(N+1) > C_1(N)$, and $C_1(N) \leqq C_1(N-1)$, we get

$$L(N) < 1 - 4c_b/c_r \quad \text{and}$$
$$L(N-1) \geqq 1 - 4c_b/c_r\ ,\ N=1,\ 2,... \quad (7)$$

where $L(n) \equiv \dfrac{\lambda^n[\lambda+2\rho(n+1)]}{(\lambda+2\rho)^{n+1}}$.

Theorem 1. (a) If $c_r \leqq 4c_b$, then $N^* = \infty$ with the corresponding cost rate $c_r\rho/2$.

(b) If $c_r > 4c_b$, then N^* is the smallest N determined by (7). The smallest N is finite and unique.

Proof. (a) It is obvious from (6).

(b) First, notice that

$$L(0) = 1 \geqq 1 - 4c_b/c_r.$$

Since

$$L(N+1) - L(N) = \frac{\lambda^{N+1}[\lambda+2\rho(N+2)]}{(\lambda+2\rho)^{N+2}} - \frac{\lambda^N[\lambda+2\rho(N+1)]}{(\lambda+2\rho)^{N+1}}$$
$$= -\frac{4\lambda^N\rho^2(N+1)}{(\lambda+2\rho)^{N+2}} < 0,$$

$L(N)$ is decreasing in N. Further,

$$L(N) = \frac{\lambda^N[\lambda+2\rho(N+1)]}{(\lambda+2\rho)^{N+1}} = \frac{1+2\rho(N+1)/\lambda}{(1+2\rho/\lambda)^{N+1}} \to 0$$

as N goes to infinity since the denominator increases faster than the numerator. Thus the smallest N determined by (7) is finite and unique.

2.2 T-policy

Under T-policy, all units are replaced by new ones at the first idle time after the cumulative use time T elapses. Each unit is individually replaced by new one upon failure.

Let X be the residual use time beyond T, that is, time from T to the end of ongoing use duration. Regarding the end of each use as an event, the use process constitutes a renewal process with underlying distribution $G(t)$. From the theory of renewal process, the distribution function of the residual use time X, $R_T(x)$, is(Ross 1970)

$$R_T(x) = G(T+x) - \int_0^T \overline{G}(T+x-y)dM(y), \quad (8)$$

where $\overline{G}(\cdot) = 1 - G(\cdot)$.

If the residual use time X lasts x whereas the first use starts at time 0, the conditional expected number of failures during $(0,T+x]$ is $M(T+x)$(Barlow and Proschan 1965). Therefore the expected number of failures between two successive block replacements is

$$\int_0^\infty M(T+x)dR_T(x). \quad (9)$$

The expected one cycle time(time between two successive block replacements) is

$$T + \int_0^\infty \overline{R}_T(x)dx. \quad (10)$$

Dividing the total cost of failure

replacements and a block replacement by the expected length of a cycle (10), we get the average cost rate under T-policy over an infinite time span:

$$C_2(T) = \frac{c_r \int_0^{\infty} M(T+x)\, dR_T(x) + c_b}{T + \int_0^{\infty} \overline{R}_T(x)\, dx}. \tag{11}$$

Suppose that each use duration follows an exponential distribution with mean $1/\lambda$, $G(t) = 1 - e^{-\lambda t}$. Then, by the memoryless property of exponential distribution, $R_T(x)$ becomes also an exponential distribution with parameter λ. In this case, the cost rate function (11) becomes

$$C_2(T) = \frac{c_r \int_0^{\infty} M(T+x)\lambda e^{-\lambda x} dx + c_b}{T + 1/\lambda}. \tag{12}$$

We seek T^*, the optimum value of T, which minimizes (12). Differentiating (12) with respect to T and equating it to zero, we get the necessary condition for T being optimal:

$$\int_0^{\infty} [(\lambda T + 1) m(T+x) - \lambda M(T+x)] \times e^{-\lambda x} dx = \frac{c_b}{c_r}. \tag{13}$$

By rearranging (13), we get the corresponding cost rate at the optimal T^*:

$$c_r \int_0^{\infty} m(T^* + x)\lambda e^{-\lambda x} dx.$$

Suppose again that the TTF of each unit follows a 2-stage Erlang distribution with parameter ρ, $F(t) = \rho^2 t e^{-\rho t}$. Inserting $T+x$ in place of t in (4), the cost rate expression (12) becomes

$$C_2(T) = c_r \left[\frac{\rho}{2} + \frac{\lambda e^{-2\rho T}}{4(T+1/\lambda)(2\rho+\lambda)} \right] + \frac{4c_b - c_r}{4(T+1/\lambda)}. \tag{14}$$

Also condition (13) becomes, after some algebra,

$$\frac{\lambda e^{-2\rho T}}{2\rho + \lambda} [1 + 2\rho(T + 1/\lambda)] = 1 - \frac{4c_b}{c_r}. \tag{15}$$

Theorem 2. (a) If $c_r \leqq 4c_b$, then $T^* = \infty$ with the corresponding cost rate $c_r\rho/2$.

(b) If $c_r > 4c_b$, then the solution T of (15) is the optimal T^* which minimizes (14). The solution T is finite and unique.

Proof. (a) It is obvious from (14).

(b) It will be shown that (14) is unimodal in T. Let $Q(T)$ denote the left-hand-side of (15). Then $Q(T)$ is decreasing in T since

$$\frac{dQ(T)}{dT} = -\frac{4\lambda\rho^2 e^{-2\rho T}(T+1/\lambda)}{2\rho+\lambda} < 0.$$

It is easy to see $Q(0) = 1 > 1 - 4c_b/c_r$ and $Q(\infty) = 0$, which implies that (15) has an unique solution in T. Further it can be shown that

$$\lim_{T \to 0} \frac{dC_2(T)}{dT} = -\lambda^2 c_b < 0.$$

Therefore (14) is unimodal in T.

3. Numerical Examples and Conclusions

For some values of cost model parameters, a computer program written in BASIC was run on a personal computer to solve the equations (7) and (15). The following <Table 1> presents the optimal values of the decision variables, N^* and T^*, and the corresponding cost rates for a various values of λ.

<Table 1> The effect of λ on the optimal average cost rates ($\rho = 1$, $c_r = 50$, $c_b = 10$).

	λ	1.0	1.5	2.0	2.5
N-policy	N^*	3	4	4	5
	C_1^*	24.32	24.22	24.14	24.08
T-policy	T^*	1.02	1.09	1.14	1.17
	C_2^*	23.91	23.78	23.71	23.67

The table shows that the optimal average cost rate decreases as λ increases in both policies, the reason probably being that as λ increases (thus the mean of a use duration decreases) the policies would have more "discrete" chances to perform block replacement at the lower cost.

The table also shows that T-policy yields lower mean cost rate than N-policy. However, N-policy has more administrative conveniences than T-policy since no record-keeping for cumulative use time is needed.

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