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Electromagnetic Field Analysis Using the Point Collocation Method Based on the FMLSRK Approximation

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Abstract - This paper presents a description of the point collocation method and its application to the electromagnetic field computation. The interpolation scheme is based on the fast moving least square reproducing kernel approximation. In the method, the integration cell is not required and the essential boundary conditions can be enforced directly. Numerical simulations on 1-D and 2-D problems are carried out to validate the method. It is found that computational efficiency is higher than the general mesh-free methods.

Keywords: mesh-free method, moving least square, point collocation method

1. Introduction

Various types of mesh-free methods have been successfully applied to the analysis of electromagnetic field problems [1, 2]. The shape functions for most mesh-free methods are derived from the moving least square approximation [2, 3]. Because the shape functions for conventional mesh-free methods do not contain the Kronecker delta function property, special treatment for imposing the essential boundary conditions are required [4]. In this paper, the point collocation method based on the fast moving least square reproducing kernel (FMLSRK) approximation is presented. The method does not necessitate integration cells for numerical integration and the computation of derivatives of the shape function is much faster than the conventional moving least square approximations. Through the numerical simulations, the rate of convergence and the accuracy of the method are shown.

2. Fast Moving Least Square Reproducing Kernel Approximation

Let Ω be a bounded domain in $\mathbf{R}^n$ and $u(\mathbf{x})$ be a continuous function defined in $\Omega \subset \mathbf{R}^n$. We also let $\Lambda = \{x_I \in \overline{\Omega} \mid I = 1, \ldots, NP\}$ be a set of distributed nodes both in Ω and on its boundary. Throughout the paper, multi-index notations are employed. When $\alpha = (\alpha_1, \ldots, \alpha_n)$ is an n-tuple of non-negative integers and $x = (x_1, x_2, \ldots, x_n) \in \mathbf{R}^n$, we define

$$\mathrm{x}^{\alpha} \equiv x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}, \tag{1}$$

and write the α-th derivative of a smooth function as

$$D_{\mathrm{x}}^{\alpha} \equiv \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \ldots \partial_{x_n}^{\alpha_n} . \tag{2}$$

Now let us introduce the vector of complete basis polynomials in $\mathbf{R}^n$ of order less than or equal to m such that

$$\mathbf{P}_m(\mathbf{x}) = (\mathbf{x}^{\beta_1}, \mathbf{x}^{\beta_2}, \ldots, \mathbf{x}^{\beta_L})^T , \quad L = \frac{(n+m)!}{n!m!} \tag{3}$$

where β_k 's are all multi-indices of n-tuples in lexicographical order and $|\beta_k| \le m$. For example, if $n = 2$ and m=2, then the multi-indices are arranged in order of (0,0), (1,0), (0,1), (2,0), (1,1) and (0,2) and thus $\mathbf{P}_m(\mathrm{x})$ is $(1, x, y, x^2, xy, y^2)^T$.

What we want to do first is to find the best local approximation of $u(\mathbf{x})$ at $\bar{\mathbf{x}} \in \overline{\Omega}$ of the following form

$$\mathbf{U}_m(\mathbf{x}, \bar{\mathbf{x}}) = \mathbf{P}_m\left(\frac{\mathbf{x} - \bar{\mathbf{x}}}{\rho}\right) \cdot \mathbf{a}(\bar{\mathbf{x}}) = \mathbf{P}_m^T\left(\frac{\mathbf{x} - \bar{\mathbf{x}}}{\rho}\right)\mathbf{a}(\bar{\mathbf{x}}) , \tag{4}$$

in the manner of minimizing the locally weighted square functional

$$J(\mathbf{a}(\bar{\mathbf{x}})) \equiv \sum_{\mathbf{x}_I \in \Lambda} |u(\mathbf{x}_I) - \mathbf{U}_m(\mathbf{x}_I, \bar{\mathbf{x}})|^2 \, \Phi\left(\frac{\mathbf{x}_I - \bar{\mathbf{x}}}{\rho}\right) \tag{5}$$

where $\Phi(\mathbf{y})$ is a non-negative and compactly supported continuous function in $\mathbf{R}^n$ referred to as the window function.

The minimizing procedure yields the best local approximation of $u(\mathbf{x})$ at $\bar{\mathbf{x}}$

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$$\mathbf{U}_m(\mathbf{x},\bar{\mathbf{x}}) = \mathbf{P}_m^T(\frac{\mathbf{x}-\bar{\mathbf{x}}}{\rho}) \cdot$$

$$\sum_{\mathbf{x}_I\in\Lambda} M^{-1}(\bar{\mathbf{x}})\mathbf{P}_m(\frac{\mathbf{x}_I-\bar{\mathbf{x}}}{\rho})\Phi(\frac{\mathbf{x}_I-\bar{\mathbf{x}}}{\rho})u(\mathbf{x}_I) \quad (6)$$

where $M(\bar{\mathbf{x}})$ is the moment matrix defined by

$$M(\bar{\mathbf{x}}) \equiv \sum_{\mathbf{x}_I\in\Lambda} \mathbf{P}_m(\frac{\mathbf{x}_I-\bar{\mathbf{x}}}{\rho})\,\mathbf{P}_m^T(\frac{\mathbf{x}_I-\bar{\mathbf{x}}}{\rho})\,\Phi(\frac{\mathbf{x}_I-\bar{\mathbf{x}}}{\rho}) \quad (7)$$

In this study, we use the window function of $\Phi(\mathbf{y}) = (1-\|\mathbf{y}\|)^4$ for $\|\mathbf{y}\|<1$.

Based on the best local approximation $\mathbf{U}_m(\mathbf{x},\bar{\mathbf{x}})$, the locally approximated derivatives of $u(\mathbf{x})$ are defined by

$$D_{\mathbf{x}}^{\alpha}\mathbf{U}_m(\mathbf{x},\bar{\mathbf{x}}) = [D^{\alpha}\,\mathbf{P}_m^T(\frac{\mathbf{x}-\bar{\mathbf{x}}}{\rho})] \cdot$$

$$\sum_{\mathbf{x}_I\in\Lambda} M^{-1}(\bar{\mathbf{x}})\mathbf{P}_m(\frac{\mathbf{x}_I-\bar{\mathbf{x}}}{\rho})\Phi(\frac{\mathbf{x}_I-\bar{\mathbf{x}}}{\rho})u(\mathbf{x}_I), \quad (8)$$

where $|\alpha| \le m$.

We arrive at the position where the global approximations of all derivatives of $u(\mathbf{x})$ are resultant from the local ones of (8) simply by taking the limit as $\bar{\mathbf{x}}$ goes to $\mathbf{x}$. Thus we define notations of the global approximations of $u(\mathbf{x})$ and obtain the identities

$$D_m^{h,\alpha}u(\mathbf{x}) \equiv \lim_{\bar{\mathbf{x}}\to\mathbf{x}} D_{\mathbf{x}}^{\alpha}U_m(\mathbf{x},\bar{\mathbf{x}}) = \sum_{\mathbf{x}_I\in\Lambda}\Psi_I^{[\alpha]}(\mathbf{x})\,u(\mathbf{x}_I) \quad (9)$$

where $|\alpha| \le m$ and the function of $\Psi_I^{[\alpha]}(\mathbf{x})$ are defined as the solution of the following matrix equation

$$M(\mathbf{x})\begin{pmatrix} \frac{\rho^{|\beta_1|}}{\beta_1!}\Psi_I^{[\beta_1]}(\mathbf{x}) \\ \frac{\rho^{|\beta_2|}}{\beta_2!}\Psi_I^{[\beta_2]}(\mathbf{x}) \\ \vdots \\ \frac{\rho^{|\beta_L|}}{\beta_L!}\Psi_I^{[\beta_L]}(\mathbf{x}) \end{pmatrix} = \mathbf{P}_m(\frac{\mathbf{x}_I-\mathbf{x}}{\rho})\;\Phi(\frac{\mathbf{x}_I-\mathbf{x}}{\rho}). \quad (10)$$

We call $\Psi_I^{[\alpha]}(\mathbf{x})$ *the α-th shape function* associated with the window function Φ. Note that $\Psi_I^{[0]}(\mathbf{x})$'s are standard shape functions of mesh-free methods.

Using a compactly supported continuous non-negative window function $\Phi(\mathbf{y})$, the resulting shape functions of FMLSRK are defined as follows [5]:

$$\Psi_I^{[\alpha]}(\mathbf{x}) = \frac{\alpha!}{\rho^{|\alpha|}}\mathbf{e}_{\alpha}^{\mathrm{T}}\mathbf{M}^{-1}(\mathbf{x})\mathbf{P}_m(\frac{\mathbf{x}_I-\mathbf{x}}{\rho})\Phi(\frac{\mathbf{x}_I-\mathbf{x}}{\rho}) \quad (11)$$

where $|\alpha| \le m$, $\mathbf{e}_{\alpha}^{\mathrm{T}}$ is the α-th unit vector in $\mathbf{R}^{\frac{(n+m)!}{n!m!}}$.

3. Point Collocation Scheme Based on FMLSRK Approximation

We will propose a general point collocation scheme. In order to obtain the mesh-free numerical solution of a partial differential equation (PDE), we first interpolate the solution $u(\mathbf{x})$ of the PDE such that

$$u(\mathbf{x}) \approx \mathrm{U}(\mathbf{x}) = \sum_{\mathbf{x}_I\in\Lambda} u_I\Psi_I^{[0]}(\mathbf{x}) \quad (12)$$

where the nodal values u_I should be determined later from the governing equations. The derivatives of $u(\mathbf{x})$ in the PDE and on the boundary conditions are replaced with the following approximated derivatives

$$\mathrm{D}^{\beta}u(\mathbf{x}) \approx \mathrm{D}_m^{\mathrm{h},\beta}\mathrm{U}(\mathbf{x}) = \sum_{\mathbf{x}_I\in\Lambda} u_I\Psi_I^{[\beta]}(\mathbf{x}) \quad \text{for } 0 < |\beta| \le m. \quad (13)$$

In order to expose a point collocation scheme, we consider the following Poisson problem.

$$-\Delta u = f \quad \text{in } \Omega \quad (14)$$

$$u = g \quad \text{on } \Gamma_D \text{ and } \frac{\partial u}{\partial n} = h \text{ on } \Gamma_N \quad (15)$$

We propose the point collocation discretization of the Poisson problem using the approximations (12) and (13) as follows:

$$u(\mathbf{x}) = \sum_{\mathbf{x}_J\in\Lambda} u_J\,\Psi_J^{[(0,0)]}(\mathbf{x}) \quad (16)$$

$$-\sum_{\mathbf{x}_J\in\Lambda} u_J\,(\Psi_J^{[(2,0)]}(\mathbf{x}^i)+\Psi_J^{[(0,2)]}(\mathbf{x}^i)) = f(\mathbf{x}^i) \quad (17)$$

$$\sum_{\mathbf{x}_J\in\Lambda} u_I\,\Psi^{[(0,0)]}(\mathbf{x}^d) = g(\mathbf{x}^d) \quad (18)$$

$$\sum_{\mathbf{x}_J\in\Lambda} u_J\,(\Psi_J^{[(1,0)]}(\mathbf{x}^n),\Psi_J^{[(0,1)]}(\mathbf{x}^n))\cdot\mathbf{n}(\mathbf{x}^n) = h(\mathbf{x}^n) \quad (19)$$

$$\mathbf{x}^i \in \Lambda_i, \mathbf{x}^d \in \Lambda_d, \mathbf{x}^n \in \Lambda_n \quad (20)$$

where Λ_i, Λ_d and Λ_n are sets of interior nodes, Dirichlet boundary nodes and Neumann boundary nodes, respectively. Here $\mathbf{n}(\mathbf{x}^n)$ is the outward unit normal vector at $\mathbf{x}^n \in \Lambda_n$.

In this case, the second order-approximated derivatives are needed. Thus we choose the order m of basis polynomial $\mathbf{P}_m$ greater than or equal to 2.

It is worth noting that the extension of the above method using the dilation function is considerably natural. The constant dilation parameter ρ can be replaced with the continuous dilation function ρ_x. It is almost impossible to introduce ρ_x with the generalized m-th order consistency condition preserved except FMLSRK approximation.

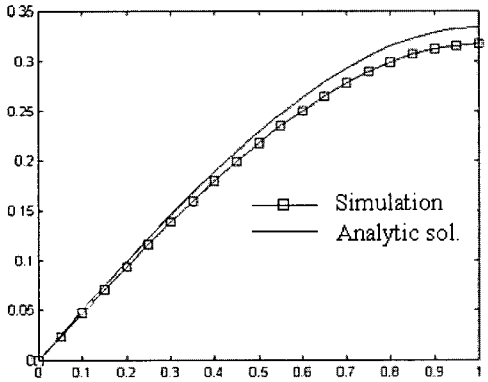
4. Numerical Simulations

To validate the method, a one-dimensional problem with an analytical solution is solved at first. The model problem is

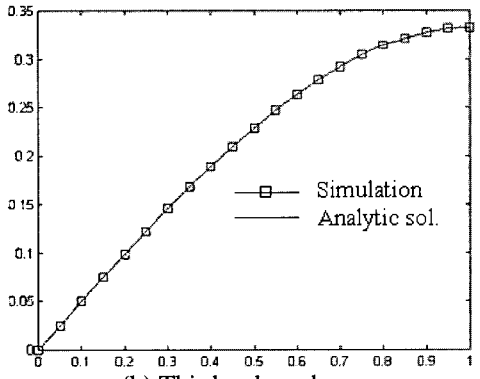
$$-\frac{d^2u}{dx^2} = x \qquad (0 < x < 1) \tag{21}$$

$$u(0) = 0 \quad \text{and} \quad u'(1) = 0. \tag{22}$$

Fig. 1 shows the comparison between the numerical and analytic solutions. The numerical results are obtained using the second and third order scheme (m=2 and 3 respectively) with five nodes. For the third order case, the numerical result is nearly the same as the analytic solution only with five nodes.



(a) Second order scheme



(b) Third order scheme

Fig. 1 Comparison between numerical and analytic solutions.

The next numerical test problem is for the study of the convergence rate of the method. The governing equations are

$$\Delta u = f \quad \text{in} \quad \Omega = [0,1] \times [0,1] \tag{23}$$

$$u = g \quad \text{on} \quad \partial\Omega, \tag{24}$$

where f and g are obtained from the solution of $u(x, y) = (x + 2y)^5$.

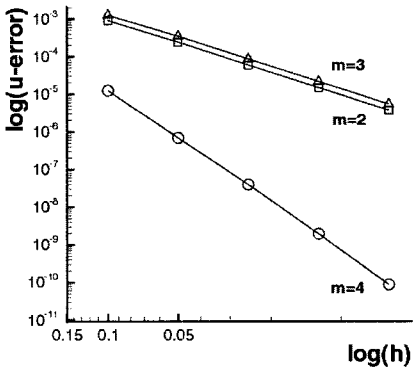
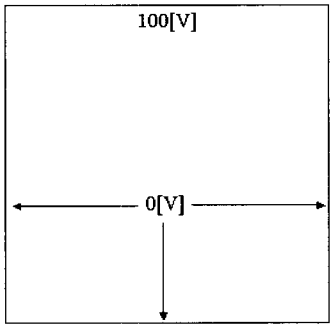
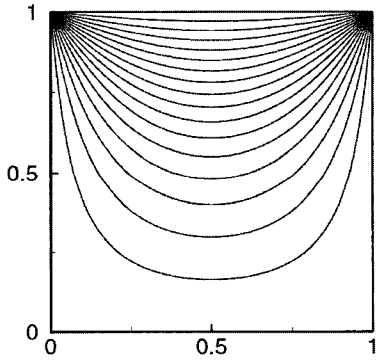


Fig. 2 Relative L^∞ error for u with respect to nodal distance h.



(a) Boundary condition



(b) Plot of equi-potential lines

Fig. 3 Analysis of 2-D electrostatic problem

Fig. 2 indicates the plot of the relative error in L^∞-norm when a uniform h-refinement is used. For the case m = 4, the convergence rate is observed to be better than that of the case m = 2 or m = 3.

The proposed method is also applied to the electrostatic problem. Fig. 3(a) shows the analysis domain and boundary condition and (b) shows the plot of computed equi-potential lines. A reasonably accurate solution can be obtained for this problem.

5. Conclusion

The point collocation method proposed in this paper is a very promising mesh-free method and has many merits compared with other mesh-free methods. Therefore, the proposed method is applicable to many applications such as optimization problems, 3-D problems, moving boundary problems and so on.

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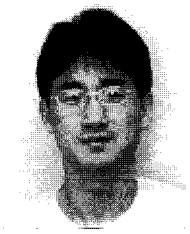

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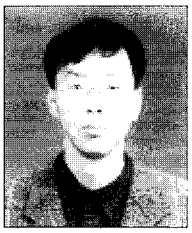

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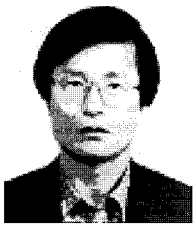

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