THE TRAPEZOIDAL RULE WITH A NONLINEAR COORDINATE TRANSFORMATION FOR WEAKLY SINGULAR INTEGRALS

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ABSTRACT. It is well known that the application of the nonlinear coordinate transformations is useful for efficient numerical evaluation of weakly singular integrals. In this paper, we consider the trapezoidal rule combined with a nonlinear transformation $\Omega_m(b; x)$, containing a parameter b, proposed first by Yun [14]. It is shown that the trapezoidal rule with the transformation $\Omega_m(b; x)$, like the case of the Gauss-Legendre quadrature rule, can improve the asymptotic truncation error by using a moderately large b. By several examples, we compare the numerical results of the present method with those of some existing methods. This shows the superiority of the transformation $\Omega_m(b; x)$.

1. Introduction

We consider weakly singular integrals of the form

$$(1) \quad I(g; \alpha, \beta) := \int_{-1}^{1} (1 - x)^\alpha (1 + x)^\beta g(x) \, dx, \qquad \alpha, \ \beta > -1,$$

where $g(x)$ is a well-behaved functions with $g(1) \neq 0$, $g(-1) \neq 0$. For accurate numerical evaluation of these weakly singular integrals, many nonlinear coordinate transformation techniques have been developed by the literature [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15]. All of these techniques have a property that the Jacobian of the transformation is zero at the singular points, which weakens the order of the original singularity. Among the coordinate transformations, sigmoidal transformations are known to be prominent in the numerical fulfillment

[3, 4, 5, 6, 11, 12]. Elliott [4] and Johnston and Elliott [7] have derived the asymptotic truncation errors of the trapezoidal rule and the Gauss quadrature rule, respectively, both of which include the sigmoidal transformation.

Recently, Yun [14] has proposed a nonlinear transformation of order $m > 1$ such as

$$(2) \qquad \Omega_m(b; x) = \frac{e^{bx^m} - 1}{e^{bx^m} + e^{b(1-x)^m} - 2}, \qquad 0 \le x \le 1,$$

for an arbitrary $b \ne 0$, which looks like a sigmoidal transformation of algebraic type [4] except that its derivative is not strictly increasing on the interval $[0, \frac{1}{2}]$, in general. It has been shown that the Gauss quadrature rule using this transformation is very effective for accurate evaluation of weakly singular integrals by virtue of the auxiliary parameter b. On the other hand, Yun and Kim [15] proposed a sigmoidal transformation of integral type :

$$(3) \qquad \gamma_m(b; x) = \frac{1}{Q_m(b)} \int_0^x h_m(b; \xi) \, d\xi, \qquad 0 \le x \le 1,$$

where $Q_m(b) = \int_0^1 h_m(b; \xi) \, d\xi$ and $h_m(b; x)$ is defined as

$$(4) \qquad h_m(b; x) = \left(e^{bx(1-x)} - 1\right)^{m-1}, \qquad 0 \le x \le 1.$$

It has been shown that, in applying Gauss quadrature rule, the transformation $\gamma_m(b; x)$ as well as $\Omega_m(b; x)$ is available for accurate numerical evaluation of weakly singular integrals. However, the complete form of $\gamma_m(b; x)$ via analytical integration in (3) is not simple in view of numerical implementation.

In this paper, we consider the trapezoidal rule for accurate numerical evaluation of weakly singular integrals in (1) using the transformation $\Omega_m(b; x)$ rather than the complicated sigmoidal transformation $\gamma_m(b; x)$. From the formulae (13) and (14) in Section 3, we particularly note that the coefficients of the series expansion of the transformation near $x = 0$ are not negligible in the asymptotic truncation error analysis. This may be feasible because the number of subintervals, N for the trapezoidal rule should be limited in numerical implementation. Fortunately, it can be observed that the first two leading coefficients of the series expansion of $\Omega_m(b; x)$ are decreasing very fast as the parameter b goes large. This fact makes it possible that the trapezoidal rule using $\Omega_m(b; x)$ may address very accurate numerical evaluation of weakly singular integrals. For several numerical examples, we show that the present method dramatically improves the existing methods by choosing any value of b in a

proper range. Furthermore, it is observed that the numerical errors of the present method are consistent with the theoretical errors so long as N is large enough.

In the next section, we present general properties of $\Omega_m(b;x)$ including the geometrical behavior of the transformation near the singular point. It is noted that, for a fixed m, the first two leading coefficients of the expansion of $\Omega_m(b;x)$ near $x = 0$ are decreasing in the form of $O\left(b/e^b\right)$ and $O\left(b^2/e^b\right)$, respectively, as b becomes large enough. In Section 3, based on the error analysis of Elliott [4] for the trapezoidal rule, it is shown that one may expect further improvement in the asymptotic truncation error by increasing the value of b in $\Omega_m(b;x)$ with a fixed order m. In Section 4, comparing numerical results of the transformation $\Omega_m(b;x)$ with those of the Sidi- and elementary sigmoidal transformations, we show that the present method produces highly improved evaluation of weakly singular integrals according to the parameter b. In addition, we can find that the present method is also applicable to the logarithmic singular integrals appearing essentially in the two dimensional boundary element method.
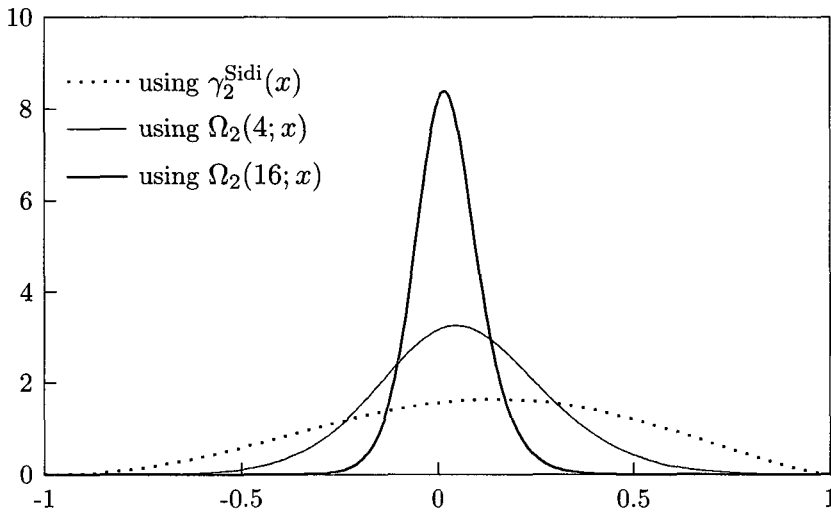
(a) $G_2(\xi; 0.2, 0.6)$ using $\gamma_2^{\text{Sidi}}(x)$ and $\Omega_2(b;x)$, $b = 4,\ 16$
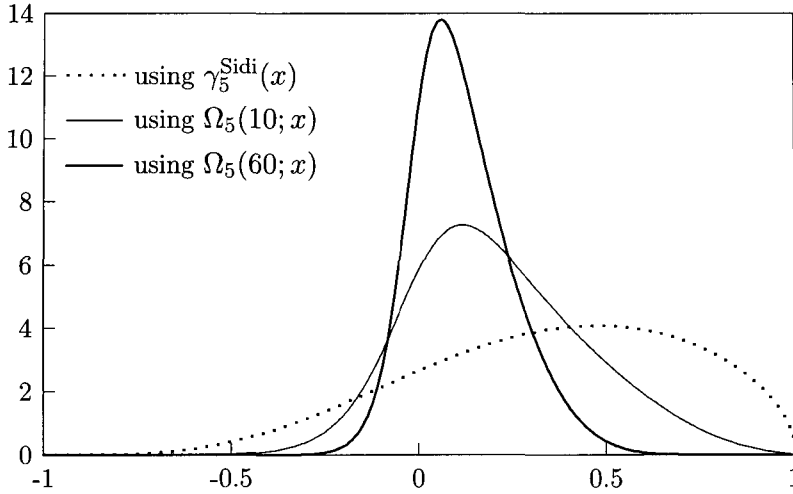
(b) $G_5(\xi; -0.7, 0)$ using $\gamma_5^{\text{Sidi}}(x)$ and $\Omega_5(b; x)$, $b = 10,\ 60$

FIGURE 1. Graphs of $G_m(\xi; \alpha, \beta)$ in (10) composed of sigmoidal transformations $\gamma_m^{\text{Sidi}}(x)$ and $\Omega_m(b; x)$, respectively.

2. Properties of a nonlinear transformation, $\Omega_m(b; x)$

From the definition in (2), we have the general properties of $\Omega_m(b; x)$ which almost accord with those of the traditional sigmoidal transformations [4].

THEOREM 1. *For any real $b \neq 0$ and $m > 1$, the transformation $\Omega_m(b; x)$ satisfies the followings.*

  (i) $\Omega_m(b; x) \in \mathbf{C}^1[0, 1] \cap \mathbf{C}^\infty(0, 1)$.

  (ii) $\Omega_m(b; x) + \Omega_m(b; 1 - x) = 1$, $0 \le x \le 1$.

 (iii) $\Omega_m(b; x)$ *is strictly increasing on* $[0, 1]$ *with* $\Omega_m(b; 0) = 0$ *and* $\Omega_m(b; 1) = 1$.

 (iv) *Near $x = 0$, $\Omega_m(b; x) = O(x^m)$.*

  (v) *The first two leading coefficients of the series expansion of $\Omega_m(b; x)$ near $x = 0$ behave as $O\left(b/e^b\right)$ and $O\left(b^2/e^b\right)$, respectively, for $b \gg 1$ with a fixed m.*

Proof. The properties (i)–(iv) have been shown in Yun [14] already. For the property (v), by tedious calculations, we have the Taylor series
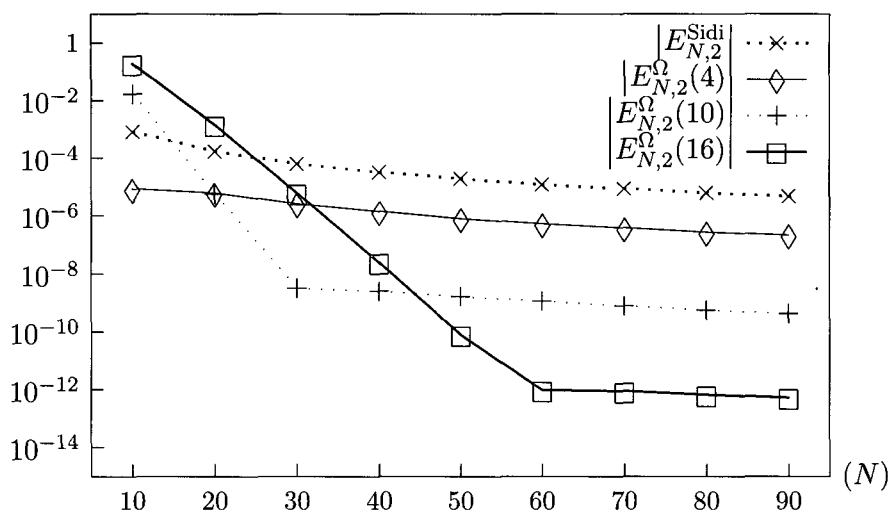
FIGURE 2. Numerical errors $\left|E_{N,2}^{\text{Sidi}}\right|$ and $\left|E_{N,2}^{\Omega}(b)\right|$, $b = 4, 10, 16$, for I_1 in the range $10 \leq N \leq 90$.

expansion of $\Omega_m(b;x)$ near $x = 0$ as

$$(5) \qquad \Omega_m(b;x) = C_0^{\Omega}(b;m)x^m \left\{ 1 + \sum_{k=1}^{\infty} D_k^{\Omega}(b;m)x^k \right\} ,$$

where

$$(6) \qquad C_0^{\Omega}(b;m) = C_0^{\Omega}(b) = \frac{b}{e^b - 1}, \qquad D_1^{\Omega}(b;m) = \frac{mbe^b}{e^b - 1}.$$

Therefore, we can find that the first two leading coefficients, $C_0^{\Omega}(b)$ and $C_0^{\Omega}(b) \cdot D_1^{\Omega}(b;m)$ in (5) imply the property (v).   □

From the property (ii) in Theorem 1, it follows that $\Omega_m'(b;x) = \Omega_m'(b;1-x)$, $0 \leq x \leq 1$. Moreover, one can see that the asymptotic behavior of the derivative, $\Omega_m'(b;x)$ at $x = 1/2$ is

$$(7) \qquad \Omega_m'\left(b;\tfrac{1}{2}\right) = \frac{mb\,e^{b/2^m}}{2^m(e^{b/2^m} - 1)} \sim O(b) ,$$

for a large value of the parameter b with a fixed m. On the other hand, for a fixed b, $\Omega_m'\left(b;\tfrac{1}{2}\right) \sim O(m)$ as m goes to the infinity. These relations indicate the rate of which the transformation $\Omega_m(b;x)$ spreads
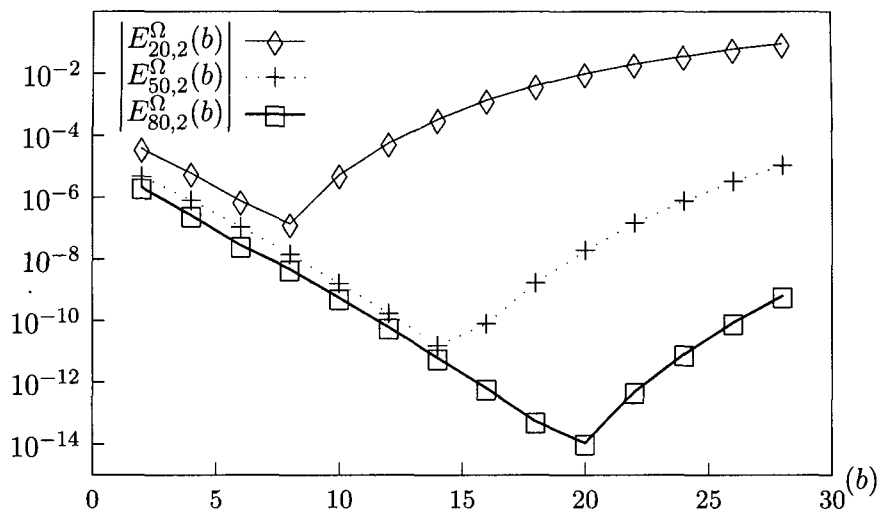
FIGURE 3. Numerical errors $\left|E_{N,2}^{\Omega}(b)\right|$, $N = 20, 50, 80$, for I_1 in the range $2 \leq b \leq 28$.

the central nodes of the numerical integration toward the end points of the integration interval.

In general, for any $m > 1$, let $\gamma_m(x)$ be a real valued function satisfying the properties (i)–(iv) such as the usual sigmoidal transformation, and let the series of $\gamma_m(x)$ take the form

$$(8) \qquad \gamma_m(x) = C_0(m)x^m \left\{ 1 + \sum_{k=1}^{\infty} D_k(m)x^k \right\}$$

near $x = 0$. For the integrand $f(x) = (1-x)^{\alpha}(1+x)^{\beta}g(x)$ in (1), we apply the coordinate transformation as

$$(9) \qquad x = 1 - 2\gamma_m\left(\frac{1-\xi}{2}\right), \qquad -1 \leq \xi \leq 1.$$
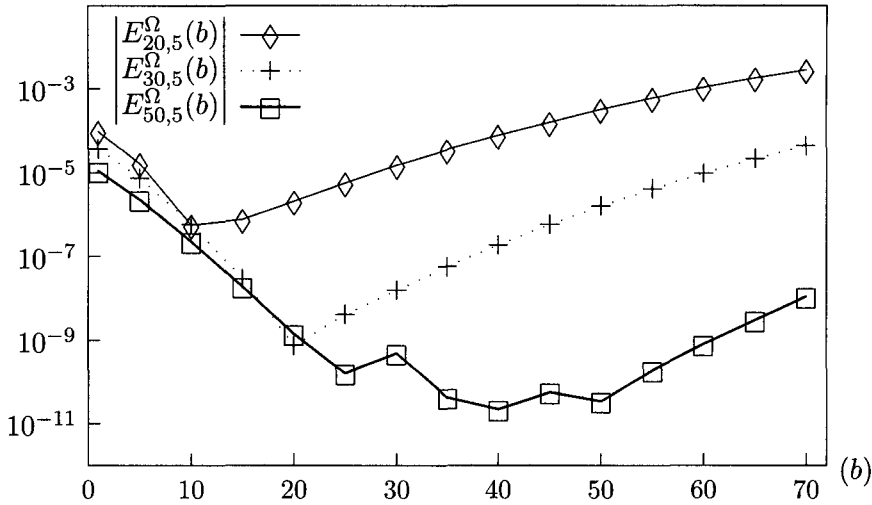
FIGURE 4. Numerical errors $\left|E_{N,5}^{\Omega}(b)\right|$, $N = 20, 30, 50$, for I_2 in the range $1 \leq b \leq 70$.

Then the integral in (1) becomes

$$
\begin{aligned}
I(g; \alpha, \beta) &= \int_{-1}^{1} f\left(1 - 2\gamma_m\left(\frac{1-\xi}{2}\right)\right) \gamma_m'\left(\frac{1-\xi}{2}\right) d\xi \\
&= 2^{\alpha+\beta} \int_{-1}^{1} \left[\gamma_m\left(\frac{1-\xi}{2}\right)\right]^{\alpha} \left[1 - \gamma_m\left(\frac{1-\xi}{2}\right)\right]^{\beta} \\
&\qquad \cdot g\left(1 - 2\gamma_m\left(\frac{1-\xi}{2}\right)\right) \gamma_m'\left(\frac{1-\xi}{2}\right) d\xi \\
&:= \int_{-1}^{1} G_m(\xi; \alpha, \beta) \, d\xi.
\end{aligned}
$$

(10)

It should be noted that $\gamma_m'(0) = \gamma_m'(1) = 0$ from Theorem 1 and, more precisely, $\gamma_m'(x) = O(x^{m-1})$ near $x = 0$ and $\gamma_m'(x) = O((1-x)^{m-1})$ near $x = 1$. Therefore, the singularities of $O((1-x)^{\alpha})$ near $x = 1$ and $O((1+x)^{\beta})$ near $x = -1$, in (1), have been translated into those of $O\left((1-\xi)^{m(1+\alpha)-1}\right)$ and $O\left((1+\xi)^{m(1+\beta)-1}\right)$, respectively, in (10). This fact is important in that the singularities at $x = -1$ and $x = 1$ are weakened enough by the transformation γ_m of large order m. Figure 1 shows the graphs of $G_2(\xi; 0.2, 0.6)$ and $G_5(\xi; -0.7, 0)$ with $g(x) = 1$
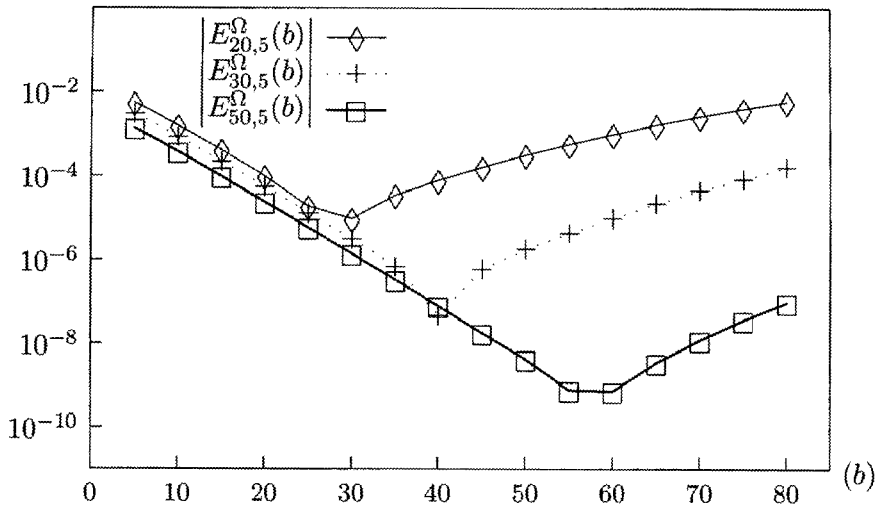
FIGURE 5. Numerical errors $\left|E^{\Omega}_{N,5}(b)\right|$, $N = 20, 30, 50$, for I_3 in the range $5 \leq b \leq 80$.

using the Sidi- transformation $\gamma^{\text{Sidi}}_m(x)$ defined in (16) and the present transformation $\Omega_m(b; x)$ for some integral values of b. Both $\gamma^{\text{Sidi}}_m(x)$ and $\Omega_m(b; x)$, for a proper value of m with respect to α and β, bring the values of $G_m(\xi, \alpha, \beta)$ at the end points into the zero. Particularly, in the graphs corresponding to $\Omega_m(b; x)$, it is observed that the flat ranges of $G_m(\xi, \alpha, \beta)$ near both end points become wider as the value of b goes large. Therefore, by intuition, we may expect much better numerical evaluation of the integral (1) by employing the transformation $\Omega_m(b; x)$.

3. The trapezoidal rule and asymptotic error analysis

It is well known that the truncation error of the trapezoidal rule is given by Euler-Maclaurin expansion which involves the values of the integrand and its derivatives at the end points. As mentioned in Elliott [4], a suitably chosen sigmoidal transformation of the variable of integration such as (9) will allow the derivatives at the end points to be zero and, thereby, it will improve the rate of convergence of the quadrature sum.
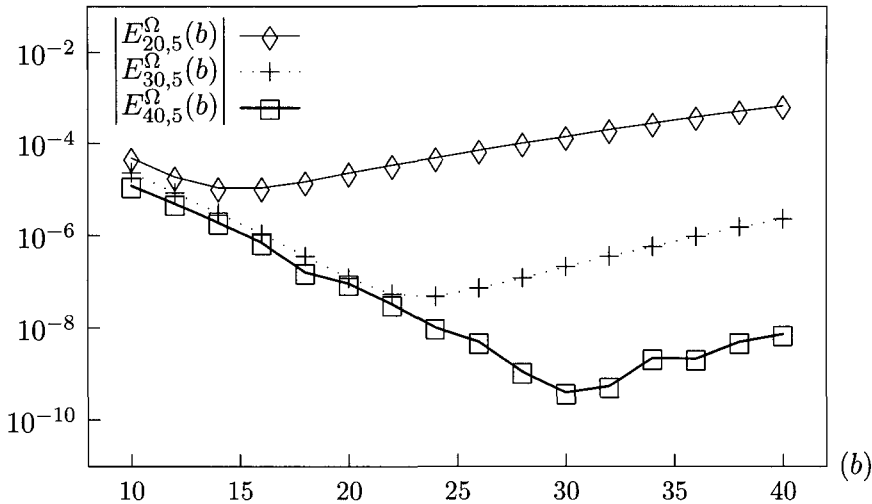
FIGURE 6. Numerical errors $\left|E^{\Omega}_{N,5}(b)\right|$, $N = 20$, 30, 40, for I_4 in the range $10 \le b \le 40$.

For numerical evaluation of the integral (10), let m satisfy $m(1+\alpha) - 1 > 0$ and $m(1 + \beta) - 1 > 0$ for given α, $\beta > -1$. Then, noting that $G_m(x; \alpha)$ vanishes at the end points $x = \pm 1$, we define a quadrature sum corresponding to the trapezoidal rule as follows :

$$(11) \qquad Q_N^{[m]}g := \frac{2}{N} \sum_{j=1}^{N-1} G_m\left(\xi_j; \alpha, \beta\right), \qquad \xi_j = -1 + 2\frac{j}{N},$$

where N is the number of subintervals of $[-1, 1]$. If we define the related error as

$$(12) \qquad E_{N,m}(g; \alpha, \beta) := I(g; \alpha, \beta) - Q_N^{[m]}g,$$

then we have the asymptotic truncation error for $E_{N,m}(g; \alpha, \beta)$ as the following theorem.

THEOREM 2. *Suppose* $g(z)$ *is holomorphic on the strip* $S = \{z \in \mathbf{C} \mid -1 \le \mathrm{Re}(z) \le 1\}$ *and real on the interval* $[-1, 1]$ *with* $g(1) \neq 0$, $g(-1) \neq 0$. *Let* $\gamma_m(x)$ *be an arbitrary transformation of order* $m > 1$ *satisfying the properties* (i)–(iv) *in Theorem 1 with the local behavior* (8) *near* $x = 0$. *Then for large enough* N, *we have*

$$(13) \qquad E_{N,m}(g; \alpha, \beta) \sim J(\alpha, m, N)g(1) + J(\beta, m, N)g(-1),$$

TABLE 1. Results of the numerical errors for the integral I_1 ($b = 4, 10, 16$)

N	b	$E_{N,2}^{\Omega}(b)$ $(\tilde{E}_{N,2}^{\Omega}(b))$	$E_{N,2}^{\text{Sidi}}$	$E_{N,2}^{\text{elem}}$
	4	-8.7×10^{-6} (2.7×10^{-5})		
10	10	1.7×10^{-2} (2.0×10^{-8})	2.7×10^{-3}	8.2×10^{-4}
	16	-1.9×10^{-1} (-2.9×10^{-11})		
	4	6.3×10^{-6} (6.5×10^{-6})		
20	10	-5.5×10^{-6} (1.1×10^{-8})	5.2×10^{-4}	1.7×10^{-4}
	16	-1.4×10^{-3} (8.6×10^{-12})		
	4	2.6×10^{-6} (2.6×10^{-6})		
30	10	3.2×10^{-9} (4.8×10^{-9})	2.0×10^{-4}	6.4×10^{-5}
	16	-6.3×10^{-6} (5.0×10^{-12})		
	4	1.4×10^{-6} (1.4×10^{-6})		
40	10	2.5×10^{-9} (2.6×10^{-9})	9.8×10^{-5}	3.3×10^{-5}
	16	-2.4×10^{-8} (3.0×10^{-12})		
	4	8.1×10^{-7} (8.2×10^{-7})		
50	10	1.6×10^{-9} (1.6×10^{-9})	5.8×10^{-5}	1.9×10^{-5}
	16	-7.9×10^{-11} (1.9×10^{-12})		
	4	5.3×10^{-7} (5.3×10^{-7})		
60	10	1.1×10^{-9} (1.1×10^{-9})	3.7×10^{-5}	1.2×10^{-5}
	16	9.5×10^{-13} (1.3×10^{-12})		
	4	3.7×10^{-7} (3.7×10^{-7})		
70	10	7.5×10^{-10} (7.6×10^{-10})	2.6×10^{-5}	8.6×10^{-6}
	16	8.8×10^{-13} (9.2×10^{-13})		

where, for $\omega = \alpha, \beta$,

$$
J(\omega, m, N)
$$

$$
(14) \quad = -2^{1+\alpha+\beta} m \left[C_0(m)\right]^{1+\omega} \left\{ \zeta\left(1 - m(1 + \omega), 1\right) \right.
$$

$$
\left. + \left(1 + \omega + \frac{1}{m}\right) D_1(m) \zeta\left(-m(1 + \omega), 1\right) \frac{1}{N} \right\} \frac{1}{N^{(1+\omega)m}}
$$

and ζ is a generalized Riemann zeta function.

TABLE 2. The optimal integer value b^* of b and the numerical error $E_{N,2}^{\Omega}(b^*)$ compared with $E_{N,2}^{\text{elem}}$ for the integral I_1.

N	b^*	$E_{N,2}^{\Omega}(b^*)$	$E_{N,2}^{\text{elem}}$
10	4	-8.7×10^{-6}	8.2×10^{-4}
20	8	-1.5×10^{-7}	1.7×10^{-4}
30	10	3.2×10^{-9}	6.4×10^{-5}
40	12	2.3×10^{-10}	3.3×10^{-5}
50	14	1.5×10^{-11}	1.9×10^{-5}
60	16	9.5×10^{-13}	1.2×10^{-5}
70	18	3.0×10^{-14}	8.6×10^{-6}

Proof. Referring to Theorem 4.7 in [4] with $t_\nu = 1$, if we employ the transformation (9) with the local behavior in (8) then, by straightforward algebra, we have (13) without difficulty.   □

Theorem 2 has been induced based on the assumption that, for large N, the major contribution to the asymptotic error comes from the neighborhood of the singular points $\xi = \pm 1$. Therefore, the resultant truncation error given in (13) contains only the local asymptotic behavior of the transformation $\gamma_m(x)$ such as

$$(15) \quad \begin{aligned} \gamma_m(x) &= C_0(m)\left\{x^m + D_1(m)x^{m+1}\right\} + O(x^{m+2}) \\ \gamma_m'(x) &= C_0(m)\left\{mx^{m-1} + D_1(m)(m+1)x^m\right\} + O(x^{m+1}), \end{aligned}$$

near $x = 0$ (i.e. $\xi = \pm 1$).

To compare the transformation $\Omega_m(b; x)$ defined in (2), in which we are interested, with well-known sigmoidal transformations, we introduce the elementary sigmoidal transformation [6] and the Sidi- transformation [11] of order $m > 1$ defined as

$$(16) \quad \begin{aligned} \gamma_m^{\text{elem}}(x) &= \frac{x^m}{x^m + (1-x)^m} \\ \gamma_m^{\text{Sidi}}(x) &= \frac{\sqrt{\pi}\,\Gamma((m+1)/2)}{\Gamma(m/2)} \int_0^x (\sin \pi\xi)^{m-1}\, d\xi, \end{aligned}$$

on the interval $0 \le x \le 1$. For large N, we denote $E_{N,m}^{\text{elem}}$, $E_{N,m}^{\text{Sidi}}$ as the asymptotic truncation errors of the trapezoidal rule (11) with $\gamma_m^{\text{elem}}(x)$ and $\gamma_m^{\text{Sidi}}(x)$, respectively. Similarly, we denote $E_{N,m}^{\Omega}(b)$ as the error corresponding to the transformation $\Omega_m(b; x)$.

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TABLE 3. Results of the numerical errors for the integral
I_2 ($b = 10, 20, 40$)

N	b	$E_{N,5}^{\Omega}(b)$ $(\tilde{E}_{N,5}^{\Omega}(b))$	$E_{N,5}^{\text{Sidi}}$	$E_{N,5}^{\text{elem}}$
	10	-1.7×10^{-3} (-2.9×10^{-6})		
10	20	-5.2×10^{-3} (-2.2×10^{-7})	5.9×10^{-3}	2.4×10^{-4}
	40	-3.1×10^{-2} (-3.9×10^{-11})		
	10	5.7×10^{-7} (1.3×10^{-6})		
20	20	-2.1×10^{-6} (-4.9×10^{-9})	1.0×10^{-3}	1.3×10^{-4}
	40	7.8×10^{-5} (-2.5×10^{-12})		
	10	5.9×10^{-7} (6.7×10^{-7})		
30	20	-7.5×10^{-10} (2.3×10^{-9})	3.7×10^{-4}	5.0×10^{-5}
	40	-1.9×10^{-7} (-3.8×10^{-13})		
	10	3.6×10^{-7} (3.8×10^{-7})		
40	20	1.5×10^{-9} (2.1×10^{-9})	1.8×10^{-4}	2.4×10^{-5}
	40	-4.7×10^{-10} (-5.6×10^{-14})		
	10	2.3×10^{-7} (2.4×10^{-7})		
50	20	1.4×10^{-9} (1.6×10^{-9})	1.0×10^{-4}	1.4×10^{-5}
	40	2.2×10^{-11} (1.2×10^{-14})		
	10	1.5×10^{-7} (1.6×10^{-7})		
60	20	1.9×10^{-9} (1.1×10^{-9})	6.6×10^{-5}	9.0×10^{-6}
	40	-1.8×10^{-11} (2.6×10^{-14})		
	10	1.1×10^{-7} (1.1×10^{-7})		
70	20	9.6×10^{-10} (8.4×10^{-10})	4.5×10^{-5}	6.1×10^{-6}
	40	9.1×10^{-11} (2.7×10^{-14})		

We can see that the leading coefficients of the series expansion of $\gamma_m^{\text{Sidi}}(x)$ near $x = 0$ are

$$(17) \qquad \left\{ C_0^{\text{Sidi}}(m) \right\}_{m=2}^{\infty} = \left\{ \frac{\pi^2}{4}, \frac{2\pi^2}{3}, \frac{3\pi^4}{16}, \frac{8\pi^4}{15}, \frac{5\pi^6}{32}, \frac{16\pi^6}{35}, \cdots \right\}.$$

The leading coefficients corresponding to most sigmoidal transformations of integral type such as Korobev [8]–, Sidi [11]– and Elliott [4]–

TABLE 4. The optimal integer value b^* of b and the numerical error $E_{N,5}^{\Omega}(b^*)$ compared with $E_{N,5}^{\text{elem}}$ for the integral I_2.

N	b^*	$E_{N,5}^{\Omega}(b^*)$	$E_{N,5}^{\text{elem}}$
10	1	4.1×10^{-5}	2.4×10^{-4}
20	11	6.1×10^{-8}	1.3×10^{-4}
30	19	1.7×10^{-10}	5.0×10^{-5}
40	25	8.5×10^{-13}	2.4×10^{-5}
50	40	2.2×10^{-11}	1.4×10^{-5}
60	40	-1.8×10^{-11}	9.0×10^{-6}
70	60	-9.1×10^{-12}	6.1×10^{-6}

transformations increase with respect to the order m while those of the transformations of algebraic type like $\gamma_m^{\text{elem}}(x)$ and $\Omega_m(b; x)$ are independent of m. That is, the leading coefficients of $\gamma_m^{\text{elem}}(x)$ are $C_0^{\text{elem}}(m) = 1$ and those of $\Omega_m(b; x)$ are $C_0^{\Omega}(b; m) = \frac{b}{e^b - 1}$ as shown in (6). In particular, it should be noted that the value of $C_0^{\Omega}(b; m)$ can be sufficiently reduced by increasing the parameter b. Moreover, in (8), $D_1^{\text{elem}}(m) = m$ for $\gamma_m^{\text{elem}}(x)$ and $D_1^{\Omega}(b; m) = mbe^b / (e^b - 1)$ for $\Omega_m(b; x)$ as given in (6). Thus, from Theorem 2, the asymptotic truncation error $E_{N,m}^{\Omega}(b)$ in using $\Omega_m(b; x)$ becomes

$$(18) \quad E_{N,m}^{\Omega}(b) \sim O\left(b\left[\frac{b}{e^b}\right]^{1+q}\right) \times \frac{1}{N^{(1+q)m+1}}, \quad q = \min\{\alpha, \beta\}$$

when b goes to the infinity, for a fixed m and a large N. From this, we can see that $E_{N,m}^{\Omega}(b)$ goes to zero very rapidly as b becomes large for any given $q > -1$.

If we consider the additional third term of the local behavior of $\Omega_m(b; x)$ near $x = 0$ then, since the corresponding coefficient in (8) is

(19)

$$D_2^{\Omega}(b; m) = \begin{cases} \dfrac{be^b \left\{3(e^{-b} - 1) + (1 - e^b) + 4b(1 + e^b)\right\}}{2\left(e^b - 1\right)^2}, & m = 2 \\[4ex] \dfrac{mbe^b \left\{(m - 1)(1 - e^b) + mb(1 + e^b)\right\}}{2\left(e^b - 1\right)^2}, & m \geq 3, \end{cases}$$

TABLE 5. Results of the numerical errors for the integral I_3 ($b = 10,\ 30,\ 60$)

N	b	$E_{N,5}^{\Omega}(b)\ (\tilde{E}_{N,5}^{\Omega}(b))$	$E_{N,5}^{\text{Sidi}}$	$E_{N,5}^{\text{elem}}$
	10	$3.3 \times 10^{-3}\ (5.3 \times 10^{-3})$		
10	30	$-1.2 \times 10^{-2}\ (2.7 \times 10^{-5})$	1.3×10^{-1}	4.1×10^{-2}
	60	$-9.6 \times 10^{-2}\ (6.0 \times 10^{-9})$		
	10	$2.6 \times 10^{-3}\ (1.6 \times 10^{-3})$		
20	30	$-9.7 \times 10^{-6}\ (5.2 \times 10^{-6})$	4.7×10^{-2}	1.5×10^{-2}
	60	$-1.0 \times 10^{-3}\ (1.4 \times 10^{-9})$		
	10	$8.5 \times 10^{-4}\ (8.5 \times 10^{-4})$		
30	30	$3.2 \times 10^{-6}\ (3.5 \times 10^{-6})$	2.5×10^{-2}	7.9×10^{-3}
	60	$-1.0 \times 10^{-5}\ (6.5 \times 10^{-10})$		
	10	$5.4 \times 10^{-4}\ (5.4 \times 10^{-4})$		
40	30	$2.0 \times 10^{-6}\ (2.1 \times 10^{-6})$	1.7×10^{-2}	5.1×10^{-3}
	60	$-1.0 \times 10^{-7}\ (3.8 \times 10^{-10})$		
	10	$3.8 \times 10^{-4}\ (3.8 \times 10^{-4})$		
50	30	$1.4 \times 10^{-6}\ (1.5 \times 10^{-6})$	1.1×10^{-2}	3.6×10^{-3}
	60	$-7.4 \times 10^{-10}\ (2.6 \times 10^{-10})$		
	10	$2.9 \times 10^{-4}\ (2.9 \times 10^{-4})$		
60	30	$1.1 \times 10^{-6}\ (1.1 \times 10^{-6})$	9.0×10^{-3}	2.8×10^{-3}
	60	$1.6 \times 10^{-10}\ (1.9 \times 10^{-10})$		
	10	$2.2 \times 10^{-4}\ (2.2 \times 10^{-4})$		
70	30	$8.3 \times 10^{-7}\ (8.5 \times 10^{-7})$	7.2×10^{-3}	2.2×10^{-3}
	60	$1.4 \times 10^{-10}\ (1.4 \times 10^{-10})$		

we may suspect that

$$(20) \qquad E_{N,m}^{\Omega}(b) \sim O\left(b^2 \left[\frac{b}{e^b} \right]^{1+q} \right) \times \frac{1}{N^{(1+q)m+2}}$$

when b goes to the infinity, for a fixed m and a large N.

TABLE 6. The optimal integer value b^* of b and the numerical error $E_{N,5}^{\Omega}(b^*)$ compared with $E_{N,5}^{\text{elem}}$ for the integral I_3

N	b^*	$E_{N,5}^{\Omega}(b^*)$	$E_{N,5}^{\text{elem}}$
10	13	-8.5×10^{-5}	4.1×10^{-2}
20	28	-4.2×10^{-7}	1.5×10^{-2}
30	40	-4.7×10^{-8}	7.9×10^{-3}
40	49	-9.9×10^{-10}	5.1×10^{-3}
50	58	-2.1×10^{-10}	3.6×10^{-3}
60	64	3.8×10^{-12}	2.8×10^{-3}
70	80	-6.1×10^{-12}	2.2×10^{-3}

In the result, using the transformation $\Omega_m(b; x)$, we may expect highly improved error of the trapezoidal rule for weakly singular integrals by selecting somewhat large b. This prominent advantage of $\Omega_m(b; x)$ persists also in applying the Gauss-Legendre quadrature rule as shown in [14].

4. Numerical examples

In this section, we examine several algebraic weakly singular integrals in the form of (1). In addition, a logarithmic singular integral is also considered to show the extensive availability of the present method. By applying the trapezoidal rule to the formula in (10), we compare numerical results of the present transformation $\gamma_m(x) = \Omega_m(b; x)$ with those of the existing sigmoidal transformations $\gamma_m(x) = \gamma_m^{\text{elem}}(x)$ and $\gamma_m(x) = \gamma_m^{\text{Sidi}}(x)$.

In the numerical implementation, we compare the results using the order $m = 2$ for the integral of α, $\beta > 0$ in (1) and the order $m = 5$ for the others. The value of the parameter b has been chosen within the range of integers for simplicity.

EXAMPLE 4.1. A case of $\alpha = 0.2$, $\beta = 0.6$ and $g(x) = 1$ in (1) :

$$(21) \qquad I_1 := I(1; 0.2, 0.6) = \int_{-1}^{1} (1 - x)^{0.2}(1 + x)^{0.6} \, dx,$$

of which the exact value is $1.704\,030\,414\,819\,117$ to 16 decimal digits.

In this case, we have chosen the order of the transformations as $m = 2$ which is sufficient to weaken the original singularity. Table 1 gives the results of the numerical errors comparing $E_{N,2}^{\Omega}(b)$, $b = 4$, 10, 16, with $E_{N,2}^{\text{Sidi}}$ and $E_{N,2}^{\text{elem}}$ for various numbers of subintervals, $N = 10(10)70$. Therein, $E_{N,2}^{\text{Sidi}}$ and $E_{N,2}^{\text{elem}}$ denote the numerical errors corresponding to Sidi- and elementary sigmoidal transformations, respectively. The results of $E_{N,2}^{\Omega}(4)$ and $E_{N,2}^{\Omega}(10)$ for all $N \geq 10$ and $N \geq 20$, respectively, are better than those of both $E_{N,2}^{\text{Sidi}}$ and $E_{N,2}^{\text{elem}}$. For all $N \geq 30$, the absolute values of $E_{N,2}^{\Omega}(10)$ are less than 10^{-8} while those of $E_{N,2}^{\text{Sidi}}$ and $E_{N,2}^{\text{elem}}$ are greater than 10^{-6}. In particular, the absolute values of $E_{N,2}^{\Omega}(16)$ are less than 10^{-10} for all $N \geq 50$, which is much improved than the cases of the lower values $b < 16$. Figure 2 shows that the rate of decrease of $|E_{N,2}^{\Omega}(b)|$ with respect to N grows rapid as the value of b becomes large.

In addition, Table 1 includes the theoretical values of the asymptotic truncation error $\tilde{E}_{N,2}^{\Omega}(b)$ calculated by the formula (13). One can find that the numerical errors $E_{N,2}^{\Omega}(4)$, $E_{N,2}^{\Omega}(10)$ and $E_{N,2}^{\Omega}(16)$ are almost compatible with their theoretical errors for $N \geq 20$, $N \geq 30$ and $N \geq 60$, respectively.

The optimal value of the parameter b within the range of integers, say, b^* which results in the minimum value of $|E_{N,2}^{\Omega}(b)|$ is given in Table 2 with respect to each $N = 10(10)70$.

In Figure 3, the numerical errors $|E_{N,2}^{\Omega}(b)|$ with $N = 20$, 50, 80 are compared in the range $2 \leq b \leq 28$, which shows the approximate optimal values of b and, at the same time, visualizes the global behavior of the errors with respect to b. That is, for a fixed N, the error of the trapezoidal rule using the transformation $\Omega_2(b; x)$ is decreasing so long as b is modestly large and increasing again when b becomes greater than the optimal value b^*.

EXAMPLE 4.2. Cases of $\alpha = -1/2$, $\beta = 1/2$ and $\alpha = -0.7$, $\beta = 0$ with $g(x) = 1$ in (1) :

$$(22) \qquad I_2 := I(1; -1/2, 1/2) = \int_{-1}^{1} (1 - x)^{-1/2}(1 + x)^{1/2} \, dx$$

$$(23) \qquad I_3 := I(1; -0.7, 0) = \int_{-1}^{1} (1 - x)^{-0.7} \, dx.$$

The exact values are $I_2 = \pi$ and $I_3 = 4.103\,814\,711\,149\,720$ to 16 decimal digits.

In these cases, we have used $m = 5$ as the order of the transformations. In Table 3 for I_2, including the theoretical values of the asymptotic error $\tilde{E}_{N,5}^{\Omega}(b)$ given in (13), the numerical errors $E_{N,5}^{\Omega}(b)$, $b = 10, 20, 40$, are compared with $E_{N,5}^{\text{Sidi}}$ and $E_{N,5}^{\text{elem}}$ for $N = 10(10)90$. It should be noted that larger values of b than those in the case of lower singularity such as Example 1 are required to obtain sufficient improvement. All the results of $E_{N,5}^{\Omega}(b)$, $b = 10, 20, 40$, are better than those both of $E_{N,5}^{\text{Sidi}}$ and $E_{N,5}^{\text{elem}}$ for all $N \geq 20$. It is remarkable that, for all $N \geq 40$, the absolute values of $E_{N,5}^{\Omega}(40)$ are less than 10^{-9} while those of $E_{N,5}^{\text{Sidi}}$ and $E_{N,5}^{\text{elem}}$ are greater than 10^{-6}. On the other hand, we can find that the numerical errors $E_{N,5}^{\Omega}(10)$ and $E_{N,5}^{\Omega}(20)$ are almost compatible with the asymptotic errors for $N \geq 30$ and $N \geq 40$, respectively. In Table 4 for I_2, optimal value b^* of b within the range of integers is given with respect to each $N = 10(10)70$.

Figure 4 shows the numerical errors $|E_{N,5}^{\Omega}(b)|$ for I_2 with $N = 20, 30, 50$ in the range $1 \leq b \leq 70$, which shows similar tendency to the case of I_1 in Example 1.

For the integral I_3, the numerical results of $E_{N,5}^{\Omega}(b)$ with $b = 10, 30, 60$, $E_{N,5}^{\text{Sidi}}$ and $E_{N,5}^{\text{elem}}$ are given in Table 5, and the optimal value b^* is given in Table 6 for $N = 10(10)70$. Figure 5 shows the behavior of $|E_{N,5}^{\Omega}(b)|$ with $N = 20, 30, 50$ in the range $5 \leq b \leq 80$.

EXAMPLE 4.3. An algebraic and logarithmic singular integral, that is, the case of $\alpha = -1/2$, $\beta = 0$ and $g(x) = \log(1 - x)$ in (1) :

$$(24) \quad I_4 := I(\log(1 - x); -1/2, 0) = \int_{-1}^{1} (1 - x)^{-1/2} \log(1 - x) \, dx$$

of which exact value is $-3.696\,337\,962\,555\,286$ to 16 decimal digits.

Since g does not satisfy the assumption of Theorem 2, we can not calculate the asymptotic truncation error by the form of (13). Nevertheless, similarly to the cases of the previous examples, the numerical errors for I_4 given in Table 7 and Figure 6 show the superiority of the transformation $\Omega_5(b; x)$ according to the values of b in a proper range.

5. Conclusions

In this paper, for weakly singular integrals in the form of (1), we have studied the trapezoidal rule using the transformation $\Omega_m(b, x)$ in (2). By various numerical examples, we have shown that the trapezoidal

TABLE 7. Results of the numerical errors for the integral
I_4 ($b = 12$, 24, 40)

N	b	$E_{N,5}^{\Omega}(b)$	$E_{N,5}^{\text{Sidi}}$	$E_{N,5}^{\text{elem}}$	b^*	$E_{N,5}^{\Omega}(b^*)$
	12	-1.4×10^{-2}				
10	24	-4.6×10^{-2}	-9.0×10^{-2}	-1.8×10^{-2}	6	-8.4×10^{-3}
	40	-1.6×10^{-1}				
	12	-1.9×10^{-5}				
20	24	-4.9×10^{-5}	-1.8×10^{-2}	-2.9×10^{-3}	15	-1.0×10^{-5}
	40	-6.8×10^{-4}				
	12	-8.5×10^{-6}				
30	24	-4.8×10^{-8}	-7.2×10^{-3}	-1.1×10^{-3}	23	-4.6×10^{-8}
	40	-2.3×10^{-6}				
	12	-4.9×10^{-6}				
40	24	-1.2×10^{-8}	-3.7×10^{-3}	-5.8×10^{-4}	30	-3.9×10^{-10}
	40	-7.2×10^{-9}				
	12	-3.1×10^{-6}				
50	24	-8.3×10^{-9}	-2.2×10^{-3}	-3.4×10^{-4}	40	-1.0×10^{-10}
	40	-1.0×10^{-10}				
	12	-2.0×10^{-6}				
60	24	-6.7×10^{-9}	-1.4×10^{-3}	-2.2×10^{-4}	40	4.2×10^{-12}
	40	4.2×10^{-12}				
	12	-1.5×10^{-6}				
70	24	-7.1×10^{-9}	-1.0×10^{-3}	-1.5×10^{-4}	60	1.0×10^{-11}
	40	-9.9×10^{-10}				

rule combined with $\Omega_m(b,x)$ dramatically improves the approximation errors. Furthermore, the numerical errors are almost consistent with the theoretical asymptotic truncation errors given in Theorem 2, for wide range of N, as long as b is not excessively large.

The superiority of the present technique using the transformation $\Omega_m(b,x)$ over the other transformation techniques is that the former

requires very small number of the subintervals, N to obtain highly accurate approximation by taking a proper value of the parameter b which depends mainly on the singularities, α and β and partially on N. It can be found in numerical experiment that, for any arbitrary $g(x)$ in (1), the accuracy is maintained by a properly chosen b according to α and β. In addition, it has been shown that the efficiency of present method persists for the logarithmic singular integrals as well as the algebraic weakly singular integrals.

Although we have mainly used the integer value of b for simplicity, one can observe that particular real values of b would result in better errors. It is left for the further work to search theoretical procedure on the optimal value of b for given singularities of the integrands.

References

[1] M. Cerrolaza and E. Alarcon, *A bi-cubic transformation for the numerical evaluation of the Cauchy principal value integrals in boundary elements*, Internat. J. Numer. Methods Engrg. **28** (1989), 987–999.

[2] M. Doblare and L. Gracia, *On non-linear transformations for the integration of weakly-singular and Cauchy principal value integrals*, Internat. J. Numer. Methods Engrg. **40** (1997), 3325–3358.

[3] D. Elliott, *The cruciform crack problem and sigmoidal transformations*, Math. Methods Appl. Sci. **20** (1997), 121–132.

[4] _____, *Sigmoidal transformations and the trapezoidal rule*, J. Aust. Math. Soc. Ser. **B 40(E)** (1998), E77–E137.

[5] P. R. Johnston, *Application of sigmoidal transformations to weakly singular and neer singular boundary element integrals*, Internat. J. Numer. Methods Engrg. **45** (1999), 1333–1348.

[6] _____, *Semi-sigmoidal transformations for evaluating weakly singular boundary element integrals*, Internat. J. Numer. Methods Engrg. **47** (2000), 1709–1730.

[7] P. R. Johnston and D. Elliott, *Error estimation of quadrature rules for evaluating singular integrals in boundary element*, Internat. J. Numer. Methods Engrg. **48** (2000), 949–962.

[8] N. M. Korobev, *Number – Theoretic Method of Approximate Analysis*, GIFL, Moscow, 1963.

[9] J. M. Sanz Serna, M. Doblaré, and E. Alarcon, *Remarks on methods for the computation of boundary-element integrals by co-ordinate transformation*, Commun. Appl. Numer. Methods **6** (1990), 121–123.

[10] M. Sato, S. Yoshiyoka, K. Tsukui and R. Yuuki, *Accurate numerical integration of singular kernels in the two-dimensional boundary element method*, in: C.A. Brebbia(ed.), Boundary Elements X, **1** (1988), Springer, Berlin, 279–296.

[11] A. Sidi, *A new variable transformation for numerical integration*, in: H. Brass and G. Hämmerlin(ed.), numerical Integration IV, ISNM Vol. 112, Birkhaiser-Verlag, Berlin, 1993, 359–373.

[12] K. M. Singh and M. Tanaka, *On non – linear transformations for accurate numerical evaluation of weakly singular boundary integrals*, Internat. J. Numer. Methods Engrg. **50** (2001), 2007–2030.

[13] J. C. F. Telles, *A self-adaptive co-ordinate transformation for efficient numerical evaluation of general boundary element integrals*, Internat. J. Numer. Methods Engrg. **24** (1987), 959–973.

[14] B. I. Yun, *An efficient transformation with Gauss quadrature rule for weakly singular integrals*, Comm. Numer. Methods Eng. **17** (2001), 881–891.

[15] B. I. Yun and P. Kim, *A new sigmoidal transformation for weakly singular integrals in the boundary element method*, SIAM J. Sci. Comput. **24** (2003), 1203–1217.

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