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A NUMBER SYSTEM IN $\mathbb{R}^n$

EUI-CHAI JEONG

ABSTRACT. In this paper, we establish a number system in $\mathbb{R}^n$ which arises from a Haar wavelet basis in connection with decompositions of certain Cuntz algebra representations on $L^2(\mathbb{R}^n)$. Number systems in $\mathbb{R}^n$ are also of independent interest [9]. We study radix-representations of $x \in \mathbb{R}^n$:

$$x := a_l a_{l-1} \cdots a_1 a_0 \cdot a_{-1} a_{-2} \cdots$$

as

$$x = M^l a_l + \cdots + M a_1 + a_0 + M^{-1} a_{-1} + M^{-2} a_{-2} + \cdots$$

where each $a_k \in D$, and D is some specified digit set. Our analysis uses iteration techniques of a number-theoretic flavor. The viewpoint is a dual one which we term "fractals in the large vs. fractals in the small," illustrating the number theory of integral lattice points vs. "fractions".

1. Introduction

D. E. Knuth [12] had raised the question of describing, for a given positive integer M which will be called the radix, or base, those finite sets D of real numbers with the property that every real number r can be represented in the form

$$r = \pm \sum_{i=-N(r)}^{\infty} a_i M^{-i}, a_i = d_i(r) \in D. \tag{1.1}$$

If every real number has a representation then the set D will be called feasible for radix M. The most common representation is the one we are using with $M = 10$ and the set $D = \{0, 1, 2, \ldots, 8, 9\}$. If we defined the set $\mathbf{T} = \{ r \mid r = \sum_{i=1}^{\infty} a_i 10^{-1}, a_i \in \{0, 1, 2, \ldots, 8, 9\}\}$, then the set $\mathbf{T}$ is the closed interval [0,1]. On the other hand, this kind of radix

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representation in $\mathbb{R}^n, n \geq 2$, is not much known yet, but the set $\mathbf{T}$ is well known as a fractal tile, see Figure 2 in this paper. Fractal tiles are used for wavelet base in digital theory. We study the simplest number system in $\mathbb{R}^n, n \geq 2$ which is similar to the number system in $\mathbb{R}$ which we are using. In section 2, we construct a radix-representation in $\mathbb{Z}^n$ using fractals in large. In section 3, we use a fractal tile in the small to extend a radix-representation of $\mathbb{Z}^n$ to $\mathbb{R}^n$. In section 4, we show an example of radix-representation of $\mathbb{R}^2$, and as an application of the radix-representation $\mathbb{Z}^n$, we introduce decomposition of representations of certain C^*-algebras, so called Cuntz algebra. We give some remarks about uniqueness of the radix-representations of x in $\mathbb{R}^n$.

2. A radix-representation in $\mathbb{Z}^n$

We start with elementary facts. If M is a fixed positive integer with $1 < M$, then for any given positive integer x, there exists a nonnegative integer l and a set of $l+1$ integers $a_0, a_1, \ldots, a_l$ such that x may be represented uniquely in the following form:

$$x = M^l a_l + \cdots + M a_1 + a_0 \tag{2.1}$$

with $0 \leq a_i < M$ for $i \neq l$ and $0 < a_l < M$.

See [13] Representation Theorem for the proof.

The following questions arise immediately:

1. What are the conditions for uniqueness in (2.1) when we consider $x \in \mathbb{Z}$ without the restriction $x > 0$? How do we apply this in $\mathbb{Z}^n$?
2. Is there a radix-representation in $\mathbb{Z}^n$, $n = 2, 3, \ldots$, with a single base M in the sense of (2.1)?
3. How do we extend a radix-representation in $\mathbb{Z}^n$ to $\mathbb{R}^n$?

LEMMA 2.1. *Let $M \geq 3$ be an integer and*

$$D := \left(-\frac{M}{2}, \frac{M}{2}\right] \cap \mathbb{Z}$$

a digit set. Then there exists a unique radix-representation for each integer x, i.e., for any $x \in \mathbb{Z}$, there exists a unique nonnegative integer l and unique integers $a_0, a_1, \ldots, a_l$ where $a_j \in D$ for $0 \leq j < l$ and $a_l \in D - \{\mathbf{0}\}$, such that

$$x = M^l a_l + M^{l-1} a_{l-1} + \cdots + M a_1 + a_0.$$

Proof. Let $D_0 = D$ and $D_n = \{Mn + d \mid d \in D_0\}$ for $n = \dots, -2, -1, 0, 1, 2, \dots$. Let x be an arbitrary integer. Since $\mathbb{Z} = \bigcup_{n \in \mathbb{Z}} D_n$ and $D_n \cap D_m = \varnothing$ unless $n = m$, there exists a unique integer n_1 such that $x \in D_{n_1}$. If $n_1 = 0$, then $x = a_0$ for some $a_0 \in D$; otherwise, we have

$$x = Mn_1 + a_0$$

for some $a_0 \in D$. If $-\frac{M}{2} < n_1 \leq \frac{M}{2}$, then replace n_1 by $a_1 \in D$ so that we have

$$x = Ma_1 + a_0.$$

If not, we repeat the above step for n_1 to get

$$\begin{aligned} x &= Mx_1 + a_0 \\ &= M(Mn_2 + a_1) + a_0 \\ &= M^2 n_2 + Ma_1 + a_0 \end{aligned}$$

with $a_0, a_1 \in D$, Since $|n_1| > |n_2| > \cdots$, there exists a smallest positive integer l such that $-\frac{M}{2} < n_l \leq \frac{M}{2}$. Put $a_l = n_l \in D$ so that we have

$$x = M^l a_l + \cdots + Ma_1 + a_0.$$

This completes the proof. □

For an integer $M \geq 3$, define

$$\begin{aligned} P_M = \{M^l a_l + \cdots + Ma_1 + a_0 \mid a_0, \dots, a_{l-1} \in D,& \\ a_l \in D - \{\mathbf{0}\} \text{ for some positive integer } l\},& \end{aligned}$$

where the digit set $D := \left(-\frac{M}{2}, \frac{M}{2}\right] \cap \mathbb{Z}$. There then exists an one-to-one and onto natural correspondence between the integer set $\mathbb{Z}$ and the set P_M of polynomials by Lemma 1. Since the set

$$I := \left\{ \sum_{k=1}^{\infty} M^{-k} a_k \;\middle|\; a_k \in D \right\}$$

is an interval of length 1, every real number $r \in \mathbb{R}$ can be decomposed into

$$r = p + q$$

where $p \in P_M$ and $q \in I$. In the following, we shall establish a number system in $\mathbb{R}^n$, $n = 2, 3, \dots$, in the manner described for $\mathbb{R}$.

The choice of a base M and a digit set D for construction of a radix-representation in $\mathbb{Z}^n$. Let M be an $n \times n$ matrix with integer entries. Then $|\det M| = N$ is a positive integer. The set U

denotes an open-closed hypercube $\prod_{i=1}^{n}\left(-\frac{1}{2},\frac{1}{2}\right]$ in $\mathbb{R}^n$. By elementary linear algebra the set

$$MU := \{Mx \mid x \in U\}$$

has Lebesgue measure N, and the set $MU \cap \mathbb{Z}^n$ has exactly N elements which consists a residue set modulo $M\mathbb{Z}^n$. In particular, for $x, y \in MU \cap \mathbb{Z}^n$, $x - y \notin M\mathbb{Z}^n$ unless $x = y$.

Define C_0 to be the set

$$C_0 := \{x = (x_1, \dots, x_n) \in \mathbb{Z}^n \mid x_i = 0 \quad \text{for all but one } i,\ 1 \le i \le n,\ i = i_0,\ x_{i_0} = 1 \text{ or } -1\}$$

and let

$$C = C_0 \cup \{\mathbf{0}\}.$$

THEOREM 2.2. *Let M be an $n \times n$ matrix with integer entries and $U := \prod_{i=1}^{n}\left(-\frac{1}{2},\frac{1}{2}\right]$. If*

(1) $C \subset MU$ (or equivalently $C \subset MU \cap \mathbb{Z}^n$),

(2) $\lim_{k\to\infty} M^k U = \mathbb{R}^n$,

then for every $x \in \mathbb{Z}^n$, a nonnegative integer l and points $a_0, a_1, \dots, a_{l-1} \in D$ and $a_l \in D - \{\mathbf{0}\}$ are uniquely determined by the formula

$$x = M^l a_l + \cdots + M a_1 + a_0.$$

REMARK 2.3. It seems likely that the condition $\lim_{k\to\infty} M^k U = \mathbb{R}^n$ is equivalent to M satisfying the condition that every eigenvalue of M has modules greater than 1, but this remains as an open question. However, it can be shown that $\lim_{k\to\infty} M^k U = \mathbb{R}^n$ by diagonalization of the matrix with the diagonal entries being those real eigenvalues if M has only real eigenvalue greater than 1.

Proof of Theorem 2.2. Let $D_0 = D := MU \cap \mathbb{Z}^n$ be the digit set. Define the sequence of sets D_k, $k = 0, 1, 2, \dots$, in $\mathbb{Z}^n$ inductively as

$$D_k = \bigcup_{x \in D_{k-1}} \{Mx + d \mid d \in D\}.$$

Then we have $D_0 \subsetneq D_1 \subsetneq D_2 \subsetneq \cdots$. To show $\lim_{k\to\infty} D_k = \mathbb{Z}^n$, observe that

$$D_{k+1} = \mathbb{Z}^n \cap \{M(x + U) \mid x \in D_k\}$$

for each $k = 0, 1, 2, \dots$. See figure 1 on the next page.

By induction, suppose

$$M^k U \cap \mathbb{Z}^n \subset D_{k+1}.$$

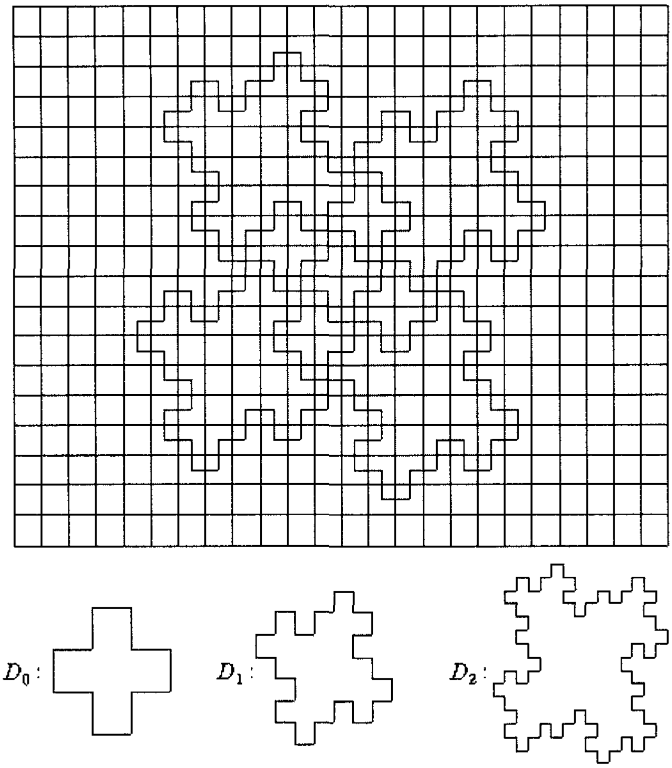

Figure 1. $M = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$

Then

$$M^k U \subset \bigcup_{x \in D_{k+1}} (x + U)$$

and

$$M^{k+1} U \subset \bigcup_{x \in D_{k+1}} M\,(x + U)\,.$$

Thus

$$M^{k+1} U \cap \mathbb{Z}^n \subset \left(\bigcup_{x \in D_{k+1}} M\,(x + U) \right) \cap \mathbb{Z}^n.$$

By (2.3) we have

$$M^{k+1} U \cap \mathbb{Z}^n \subset D_{k+2}.$$

Since, by our initial hypotheses, $\lim_{k \to \infty} \left(M^{k+1} U \cap \mathbb{Z}^n \right) = \mathbb{Z}^n$, we have $\lim_{k \to \infty} D_k = \mathbb{Z}^n$. Now consider any $x \in \mathbb{Z}^n$. If $x \in D_0$, we are done.

Otherwise, $x \notin D_o$, and so there exists an unique nonnegative integer l such that $x \in D_l$ and $x \notin D_{l-1}$, which ensures that there exists a unique $x_1 \in D_{l-1}$ and $a_0 \in D$ such that

$$x = Mx_1 + a_0.$$

If $l > 1$, by repeating this step l times we get

$$x = M^l x_l + M^{l-1} a_{l-1} + \cdots + Ma_1 + a_0$$

where $a_0, a_1, \ldots, a_{l-1} \in D$ and $x_l \in D_{l-l} = D_0 = D$. Since $x \notin D_{l-1}$, we have $x_l \neq (0, \ldots, 0)$ in $\mathbb{Z}^n$. We complete the proof by setting $a_l = x_l \in D - \{\mathbf{0}\}$. □

REMARK 2.4. Since the cardinality of C is $2n+1$, C is a subset of D and the cardinality of the digit set D is the Lebesgue measure of MU which is $|\det M| = N$, we need an integer $n \times n$ matrix satisfying $|\det M| \geq 2n+1$ in Theorem 2.2.

3. A radix-representation in $\mathbb{R}^n$

We say that an $n \times n$ integer matrix M_1 is *integrally similar* to another $n \times n$ matrix M_2 if there exists some $Q \in \mathrm{GL}(n, \mathbb{Z})$ such that $M_2 = QM_1Q^{-1}$. We say that an $n \times n$ matrix is *integrally reducible* if M is integrally similar to a matrix

$$\begin{pmatrix} A_1 & 0 \\ B & A_2 \end{pmatrix}$$

where A_1, A_2 are $r \times r$ and $(n-r) \times (n-r)$ matrices respectively for some $1 \leq r \leq n-1$ such that $|\lambda_{1i}| > 1$ and $|\lambda_{2i}| > 1$ for all eigenvalues λ_{1i} of A_1 and λ_{2i} of A_2. We call M *integrally irreducible* if it is not integrally reducible.

THEOREM 3.1. *Let M be an $n \times n$ integer matrix such that*
(1) $C \subset MU$,
(2) $\lim_{k\to\infty} M^k U = \mathbb{R}^n$,
(3) *every eigenvalue of M has modulus greater than* 1,
(4) *M is integrally irreducible.*

Then we have a radix-representation of $x \in \mathbb{R}^n$,

$$x := a_l a_{l-1} \cdots a_2 a_1 a_0 \cdot a_{-1} a_{-2} \cdots,$$

in the sense that $x = M^l a_l + \cdots + Ma_1 + a_0 + M^{-1}a_{-1} + M^{-2}a_{-2} + \cdots$ where $a_k \in D$ for $k = l-1, l-2, \ldots, 1, 0, -1, \ldots$ and $a_l \in D - \{\mathbf{0}\}$ for some nonnegative integer l.

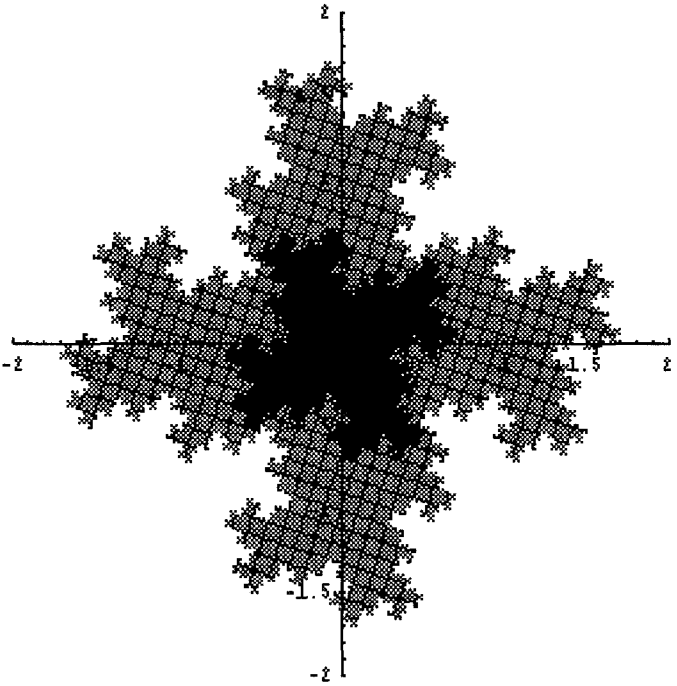

Figure 2. $\bigcup\{\, T + d \mid d \in MU \cap \mathbb{Z}^2\}$ with $M = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$

COROLLARY 3.2. *Suppose that an $n \times n$ matrix M with integer entries satisfies* (3) *and* (4) *in Theorem* 3.1. *If $D := MU \cap \mathbb{Z}^n$, then the set $\mathbf{T} := \{\sum_{k=1}^{\infty} M^{-k} d_{i_k} \mid d_{i_k} \in D\}$ is lattice-tiles $\mathbb{R}^n$ with lattice $\mathbb{Z}^n$.*

Proof. See figure 2 or [1], [7], or [11]. □

Proof of Theorem 3.1. For $x \in \mathbb{R}^n$, by virtue of Corollary 3.2, we have $x = z + t$ for $z \in \mathbb{Z}^n$ and $t \in \mathbf{T}$. With conditions (1) and (2) in Theorem 3.1, we have

$$z = M^l a_l + M^{l-1} a_{l-1} + \cdots + M a_2 + M a_1 + a_0 + M^{-1} a_{-1} + M^{-2} a_{-2} + \cdots .$$

Like a decimal expansion in $\mathbb{R}$ we have a representation for x,

$$a_l a_{l-1} \cdots a_2 a_1 a_0 \cdot a_{-1} a_{-2} \cdots .$$

□

EXAMPLE 1. Consider a 2×2 matrix $M = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$ with $|\det M| = 5$. It is easy to see that $C \subset D$ where $D = \left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix}\right\}$ (in fact $C = D$), $\lim_{k\to\infty} M^k U = \mathbb{R}^n$, and the eigenvalues of M are

$\lambda_1 = 2+i$, $\lambda_2 = 2-i$ with $|\lambda_i| = \sqrt{5} > 1$, $i = 1,2$. If a matrix A is integrally similar to a matrix $\left(\begin{smallmatrix} A_1 & 0 \\ B & A_2 \end{smallmatrix}\right)$, then $|\det A| = |\det A_1| \cdot |\det A_2|$. Furthermore, the characteristic polynomial of A is irreducible over $\mathbb{Q}$ implies that A is integrally irreducible. Thus if $|\det A| = p$, where p is prime, then A is integrally irreducible. We have shown that the matrix $\left(\begin{smallmatrix} 2 & 1 \\ -1 & 2 \end{smallmatrix}\right)$ fulfills the conditions (1)–(4) in Theorem 3.1. A point $\left(\begin{smallmatrix} 4.6 \\ -6.1 \end{smallmatrix}\right)$ in $\mathbb{R}^2$ can be represented as following:

$$\begin{pmatrix} 4.6 \\ -6.1 \end{pmatrix} := \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 0 \end{pmatrix},$$

in the sense of $\left(\begin{smallmatrix} 2 & 1 \\ -1 & 2 \end{smallmatrix}\right)^2 \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 2 & 1 \\ -1 & 2 \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} 0 \\ -1 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 2 & 1 \\ -1 & 2 \end{smallmatrix}\right)^{-1} \left(\begin{smallmatrix} -1 \\ 0 \end{smallmatrix}\right)$.

4. Applications and concluding remarks

There is an application of the number system in $\mathbb{Z}^n$ described in this paper. The Cuntz algebra O_n is the C^*-algebra generated by N elements $s_1, \ldots, s_N$ satisfying

$$s_i^* s_j = \delta_{ij} I \quad \text{and} \quad \sum_{i=1}^{N} s_i s_i^* = I. \tag{4.1}$$

One of the recent developments of the study of O_N is the representations on Hilbert spaces $L^2(\mathbb{R}^n)$ or $L^2(\mathbb{T}^n)$. The infinite nature of O_N with Cuntz properties fits into a fractal property which gives certain type of representations from wavelet, [3] or [6]. Another type of representations of the Cuntz algebra O_N is closed related to the study of the endomorphisms of $B(\mathcal{H})$, where $B(\mathcal{H})$ is the C^*-algebra of bounded linear operators on a separable, infinite dimensional Hilbert space $\mathcal{H}$, [3], [4], and [10]. There is a correspondence between endomorphisms of $B(\mathcal{H})$ of Powers index N and representations of O_N up to unitary action. If the representation of O_N is irreducible, then the corresponding endomorphism is an ergodic of Powers index N, vice versa. A UHF algebra is a norm separable C^*-algebra which is the norm closure of an increasing sequence of type I_{n_i}–factors and so a UHF algebra can be identified with the tensor product matrix algebra, $\bigotimes_{i=1}^{\infty} M_{n_i}$, $n_i \in \{2,3,\ldots\}$. When $n_i = N$, for all i, a UHF algebra is denoted by UHF_N which we can understand as a subalgebra of O_N, where the element $e_{i_1 j_1} \otimes e_{i_2 j_2} \otimes \cdots \otimes e_{i_l j_l} \otimes I \otimes \cdots$ in UHF_N is identified with $s_{i_1} \cdots s_{i_l} s_{j_l}^* \cdots s_{j_1}^*$ in O_N. Thus a representation of UHF_N also induces an endomorphism on $B(\mathcal{H})$. If the image of

UHF_N under representation is weakly dense in $B(\mathcal{H})$, the corresponding endomorphism is shift of Powers index N [4] and [10]. However, the representation of O_N and UHF_N are famous examples whose representations are bad [2], [3], [4], and [8] among many others. The representation π of O_N, and UHF_N satisfying

$$\pi(s_i)(e_z) \in \{e_x : x \in \mathbb{Z}^n\},$$

is related with tight frame in wavelet, [6], and endomorphism on the Hilbert space $B(L^2(\mathbb{R}^n))$, was studied in [2] and [3], where $\{e_x : x \in \mathbb{Z}^n\}$ is an orthonormal basis of $L^2(\mathbb{R}^n)$ or $L^2(\mathbb{T}^n)$ where $\mathbb{T} = \mathbb{R}/2n\pi$. This kind of representations are predominant in papers, [2], [3], [4], and [8]. The hard part is to find the irreducible subrepresentations. Using our number system in $\mathbb{Z}^n$, we are able to show that the representation π of O_n defined by

$$\pi(s_i)e_z = e_{Mz+d_i}$$

with $d_i \in D := MU \cap \mathbb{Z}^n$ is irreducible under conditions in Theorem 2. Under same condition in Theorem 2, if we take a residue set D modulo $M\mathbb{Z}^n$ in $\mathbb{Z}^n$, we then have rich representations of O_N and UHF_N, but these are not irreducible, in general. We use our number system in $\mathbb{Z}^n$ to show that the representations defined by $\pi(s_i)e_z = e_{Mz+d_i}$ with a residue set D modulo $M\mathbb{Z}^n$ in $\mathbb{Z}^n$, decomposed into finite irreducible subrepresentations, see [8].

A number system in $\mathbb{R}^n$, at this moment, is rather novel. When M is an integrally irreducible matrix whose eigenvalues have modulus greater than 1, then the set $\mathbf{T} := \left\{\sum_{k=1}^{\infty} M^{-k} d_k \mid d_k \in D\right\}$ has the following tiling property: the Lebesgue measure of $(x+\mathbf{T}) \cap (y+\mathbf{T})$ is 0 or 1 for $x, y \in \mathbb{Z}^n$. See figure 2. Since the set $\mathbf{T}$ is compact, there are cases when $(x+\mathbf{T}) \cap (y+\mathbf{T}) \neq \varnothing$ even though it has Lebesgue measure 0. As a result, the radix-representation in $\mathbb{R}^n$,

$$a_l a_{l-1} \cdots a_2 a_1 a_0 \cdot a_{-1} a_{-2} \cdots ,$$

described in Theorem 3.1, is not unique. If we take a subset $\tilde{\mathbf{T}}$ of the set $\mathbf{T}$ satisfying

$$\bigcup_{x \in \mathbb{Z}^n} \left(x + \tilde{\mathbf{T}}\right) = \mathbb{R}^n$$

and

$$\left(x + \tilde{\mathbf{T}}\right) \cap \left(y + \tilde{\mathbf{T}}\right) = \varnothing \quad \text{if } x \neq y$$

for $x, y \in \mathbb{Z}^n$, the radix-representation $a_l a_{l-1} \cdots a_2 a_1 a_0 \cdot a_{-1} a_{-2} \cdots$ is unique. For example, in $\mathbb{R}$, this situation can be remedied by setting

$\tilde{\mathbf{T}} := \mathbf{T} - \mathbf{T}_9$ for

$$\mathbf{T} = \left\{ \sum_{k=1}^{\infty} 10^{-k} d_{i_k} \;\middle|\; d_{i_k} \in \{0, 1, \ldots, 9\} \text{ for all } k \right\} = [0, 1],$$

$$\mathbf{T}_9 = \{ t \in \mathbf{T} \mid \text{there exists a positive integer } n_0 \text{ such that } d_k = 9 \text{ for all } k \geq n_0 \}.$$

We do not have a general result in $\mathbb{R}^n$, but the only case is studied. With the matrix M and the digit set D in the above Example 1, we are currently investigating possible unique radix-representations in $\mathbb{R}^n$ using four $\tilde{\mathbf{T}}$'s defined as follows:

$$\tilde{\mathbf{T}} = \mathbf{T} - \mathbf{T}_i, \quad i = 1, 2, 3, 4,$$

with

$$\mathbf{T} = \left\{ \sum_{k=1}^{\infty} M^{-k} d_{i_k} \;\middle|\; d_{i_k} \in D \right\},$$

$$\mathbf{T}_1 = \mathbf{T}_{\{\binom{1}{0},\binom{0}{1}\}} \cup \mathbf{T}_{\{\binom{1}{0},\binom{0}{-1}\}},$$

$$\mathbf{T}_2 = \mathbf{T}_{\{\binom{0}{1},\binom{1}{0}\}} \cup \mathbf{T}_{\{\binom{0}{1},\binom{-1}{0}\}},$$

$$\mathbf{T}_3 = \mathbf{T}_{\{\binom{-1}{0},\binom{0}{1}\}} \cup \mathbf{T}_{\{\binom{-1}{0},\binom{0}{-1}\}},$$

and

$$\mathbf{T}_4 = \mathbf{T}_{\{\binom{0}{-1},\binom{-1}{0}\}} \cup \mathbf{T}_{\{\binom{0}{-1},\binom{1}{0}\}},$$

where

$$\mathbf{T}_{\{\binom{1}{0},\binom{0}{1}\}} := \left\{ \sum_{k=1}^{\infty} M^{-k} d_{i_k} \;\middle|\; d_{i_k} \in \left\{ \binom{1}{0}, \binom{0}{1} \right\} \right\}$$

with analogous definitions for $\mathbf{T}_{\{\binom{1}{0},\binom{0}{-1}\}}$, $\mathbf{T}_{\{\binom{0}{1},\binom{1}{0}\}}$, etc. The resulting structure suggests general results for unique radix-representation in $\mathbb{R}^n$. But, at least, we have a number system in $\mathbb{R}^n$ which is similar to the familiar one in $\mathbb{R}$.

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Department of Mathematics
Chung-Ang University
Seoul 156-756, Korea
E-mail: jeong@cau.ac.kr