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DENSENESS OF TEST FUNCTIONS IN THE SPACE OF EXTENDED FOURIER HYPERFUNCTIONS

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ABSTRACT. We research properties of analytic functions which are exponentially decreasing or increasing. Also we show that the space of test functions is dense in the space of extended Fourier hyperfunctions, and that the Fourier transform of the space of extended Fourier hyperfunctions into itself is an isomorphism and Parseval's inequality holds.

§0. Introduction

In this paper, making use of the same method as in [3], we research properties of the space $\mathcal{G}'$ of extended Fourier hyperfunctions introduced in [1], which hold true in the space of distributions.

In §2, we give a criterion of test functions for extended Fourier hyperfunctions(Theorem 2.1) and research properties of analytic functions which are exponentially decreasing or increasing(Proposition 2.3 and 2.4). Also we show that every analytic function extended to any strip in $\mathbb{C}^n$ which is estimated with the aid of a special exponential function $\exp(\mu|x|)$ is a multiplier on the space $\mathcal{G}$ of test functions for extended Fourier hyperfunctions(Proposition 2.5).

In §3, we show that the space $\mathcal{G}$ is nondense in $F_{(h,\nu)}$, but in a weaker topology every function in $F_{(h,\nu)}$ can be approximated with functions belonging to the space $\mathcal{G}$(Theorem 3.2), and that $\mathcal{G}$ is dense in $\mathcal{G}'$(Theorem 3.7). And we show that the convolution of an extended Fourier hyperfunction and a test function is an entire function which increases exponentially(Theorem 3.5) and the convolution of an extended Fourier hyperfunction and two test functions is associative(Theorem 3.6).

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In §4, we show that the Fourier transform $\mathcal{F} : \mathcal{G} \to \mathcal{G}(\mathcal{G}' \to \mathcal{G}')$ is an isomorphism(Theorem 4.3), and that Parseval's inequality holds true. Regarded all function in $\mathcal{M}$ ($\mathcal{G}$ resp.) as an entire function estimated with a special exponential function $\exp(\mu|x|)$(with any exponential function $\exp(\mu|x|)$ resp.), we show that $\mathcal{G}$ is an ideal in $\mathcal{M}$, i.e., each element in $\mathcal{M}$ is a multiplier on $\mathcal{G}$(Proposition 4.9).

§1. Preliminaries

As a norm in $\mathbb{R}^n_x$ (in $\mathbb{C}^n_z$, resp.) we take $|x| = \sum_{j=1}^n |x_j|$ ($|z| = \sum_{j=1}^n |z_j|$, resp.), and the volume element $dx = dx_1 \cdots dx_n$ is fixed. Put $D = (D_1, \ldots, D_n)$; $D_k = i^{-1}\partial_k$, $\partial_k = \partial/\partial x_k$, $k = 1, \ldots, n$, $i = \sqrt{-1}$.

We denote by $\mathbb{R}^n_\xi$ the dual space of $\mathbb{R}^n_x$. Let $\xi = (\xi_1, \ldots, \xi_n)$ be coordinates in $\mathbb{R}^n_\xi$ such that the duality is expressed by the bilinear form $< x, \xi >= x_1\xi_1 + \cdots + x_n\xi_n$. If $\beta = (\beta_1, \ldots, \beta_n)$ and $\gamma = (\gamma_1, \cdots, \gamma_n)$ are multi-indices of nonnegative integers, then $|\beta| = \beta_1 + \cdots + \beta_n$, $\beta + \gamma = (\beta_1 + \gamma_1, \cdots, \beta_n + \gamma_n)$, $\beta! = \beta_1! \cdots \beta_n!$, $\xi^\beta = {\xi_1}^{\beta_1} \cdots {\xi_n}^{\beta_n}$, $D^\beta = D_1^{\beta_1} \cdots D_n^{\beta_n}$, and $\partial^\beta = \partial_1^{\beta_1} \cdots \partial_n^{\beta_n}$.

Let E_1 and E_2 be topological vector spaces embedded in a topological space E. Denote by $E_1 \bigcap E_2$ and $E_1 + E_2$ the subspace of elements of E_1 being contained in E_2 and the space of sums $\varphi_1 + \varphi_2$, $\varphi_1 \in E_1$, $\varphi_2 \in E_2$, respectively. The topologies of E_1 and E_2 induce topologies in $E_1 \bigcap E_2$ and $E_1 + E_2$. In case E_1 and E_2 are Banach spaces, the Banach norm

$$|\varphi, E_1 \bigcap E_2| = |\varphi, E_1| + |\varphi, E_2| \tag{1.1}$$

and the norm

$$|\varphi, E_1 + E_2| = \inf_{\varphi_1 + \varphi_2 = \varphi} (|\varphi_1, E_1| + |\varphi_2, E_2|) \tag{1.2}$$

are defined on $E_1 \bigcap E_2$ and $E_1 + E_2$, respectively.

Let I denote an open unit cube in $\mathbb{R}^n$:

$$I = \{\omega = (\omega_1, \ldots, \omega_n) \in \mathbb{R}^n \mid |\omega_j| < 1, j = 1, \ldots, n\}$$

and let $I^{(\kappa)}$, $\kappa = 1, \ldots, 2^n$, be the vertices of the cube, i.e., the various vector whose coordinates assume the values ± 1.

§2. The space $F_{(h,\nu)}$ of analytic functions decreasing or increasing exponentially

Let $F_{(h,\nu)}$ be the space of continuously differentiable functions $\varphi(x)$ for which the norm

$$|\varphi|_{(h,\nu)} = \sup_{x\in\mathbb{R}^n,\alpha} \frac{|\partial^\alpha\varphi(x)|\exp(\nu|x|)}{h^{-|\alpha|}\alpha!}, \; h>0,\; \nu\in\mathbb{R} \tag{2.1}$$

is finite.

Then the (continuous) embeddings

$$F_{(h,\nu)} \subset F_{(h',\nu')}, \; h \geq h' > 0, \; \nu \geq \nu' \tag{2.2}$$

take place.

THEOREM 2.1. *$f(x) \in F_{(h,\nu)}$ if and only if $f(x)$ can be continued holomorphically to the tube domain $D_h = \{x+yi \in \mathbb{C}^n |\; |y_j| < h, j = 1,2,\cdots,n\}$ such that*

$$|f(x+yi)| \; \leq C\exp(-\nu|x|) \tag{2.3}$$

Proof. Let $f(x) \in F_{(h,\nu)}$ and $x_0 \in \mathbb{R}^n$. When $|x_i - x_{0i}| < h, i = 1,2,...,n$, Taylor's formula can be written

$$\begin{aligned} f(x) = &\sum_{|\alpha|<k} \frac{\partial^\alpha f(x_0)}{\alpha!}(x-x_0)^\alpha \\ &+ k\int_0^1 (1-t)^{k-1} \sum_{|\alpha|=k} \frac{\partial^\alpha f(x_0+t(x-x_0))}{\alpha!}(x-x_0)^\alpha dt. \end{aligned}$$

The absolute value of the remainder is less than or equal to

$$|f|_{(h,\nu)}\exp(-\nu|x_0|)\exp(nh|\nu|)(k+1)^n(|x_{i_0} - x_{0i_0}|/h)^k,$$

where $|x_{i_0} - x_{0i_0}| = \max_{1\leq i\leq n}\{|x_i - x_{0i}|\}$. Since $\lim_{k\to\infty} k^\alpha(1+p)^{-k} = 0$ for $p > 0$ and $\alpha \in \mathbb{R}$, this tends to zero for $|x_i - x_{0i}| < h$. This implies that for $|y_j| < h$

$$\begin{aligned} |f(x+yi)| &\leq \sum_\alpha \frac{|\partial^\alpha f(x)|}{\alpha!}|y^\alpha| \\ &\leq |f|_{(h,\nu)}\exp(-\nu|x|)\sum_\alpha h^{-|\alpha|}|y^\alpha| \\ &\leq K|f|_{(h,\nu)}\exp(-\nu|x|), \end{aligned}$$

whence we obtain (2.3).

Conversely, if $f(x)$ satisfies (2.3), then from Cauchy's integral formula:

$$(2.4)\qquad \partial^\alpha f(x) = \frac{\alpha!}{(2\pi i)^n} \int_{|z_j - x_j| = h-\epsilon} \frac{f(z)}{(z-x)^{\alpha+1}} dz,\ 1 = (1, ..., 1)$$

we obtain

$$(2.5)\qquad |\partial^\alpha f(x)| \le C \frac{\alpha!}{(h-\epsilon)^{|\alpha|}} \exp(-\nu|x|) \exp(n|\nu|(h-\epsilon)).$$

If $\epsilon \to 0_+$, then we see that $f \in F_{(h,\nu)}$. □

From Theorem 2.1 we can see that $F_{(h,\nu)}$ consists of analytic functions extended to the tube domain D_h which increase, for $|x| \to \infty$, not stronger than the exponential function $\exp(-\nu|x|)$.

PROPOSITION 2.2. *$f_n \to 0$ in $F_{(h,\nu)}$ if and only if*

$$\sup_{z \in D_h} |f_n(z)| \exp(\nu|Rez|) \to 0.$$

Proof. It follows from the proof of Theorem 2.1. □

We can easily see from Theorem 2.1 and Cauchy's integral formula (2.4) that the following holds:

PROPOSITION 2.3.
(1) *If $\varphi_i \in F_{(h_i,\nu_i)}$, $i = 1, 2$, then $\varphi_1\varphi_2 \in F_{(h,\nu_1+\nu_2)}$, $h = \min\{h_1, h_2\}$,*
(2) *If $\varphi \in F_{(h,\nu)}$, then $x^\alpha\varphi \in F_{(h,\nu-\epsilon)}$, $\partial^\alpha\varphi \in F_{(h/2,\nu)}$, $\epsilon > 0$,*
(3) *If $f \in F_{(h,\nu)}$, then $f(x)\exp(-\sum_{i=1}^n x_i^2) \in F_{(h,\mu)}$, $\mu > \nu$.*

For $\Gamma \in \mathbb{R}^n$ let $F_{[h,\Gamma]}$ be the space of infinitely differentiable functions φ for which the norm

$$|\varphi|_{[h,\Gamma]} = \sup_{x \in \mathbb{R}^n, \alpha} \frac{|\partial^\alpha \varphi(x)| \exp(< x, \Gamma >)}{h^{-|\alpha|}\alpha!}.$$

is finite.

PROPOSITION 2.4.

(i) *For $h, \nu > 0$ we have*

$$F_{(h,\nu)} = \bigcap_{\kappa=1}^{2^n} F_{[h,\nu I^{(\kappa)}]} \tag{2.6}$$

and (2.1) *is equivalent to the natural norm of the right-hand space of* (2.6)*:*

$$\sum_{\kappa=1}^{2^n} |\varphi|_{[h,\nu I^{(\kappa)}]}. \tag{2.1'}$$

(i′) *For $h, \nu > 0$ the space $F_{(h,\nu)}$ consists of those and only those elements of the intersection $\bigcap_{\Gamma\in\nu I} F_{[h,\Gamma]}$ for which the norm*

$$'|\varphi|_{(h,\nu)} = \sup_{\Gamma\in\nu I} |\varphi|_{[h,\Gamma]} \tag{2.1''}$$

is finite.

(ii) *For $h > 0, \nu < 0$ we have*

$$F_{(h,\nu)} = \sum_{\kappa=1}^{2^n} F_{[h,\nu I^{(\kappa)}]}, \tag{2.7}$$

and (2.1) *is equivalent to the natural norm of the right-hand space of* (2.7).

Proof. (i) It is obvious that

$$\exp(\nu|x_i|) \le \exp(\nu x_i) + \exp(-\nu x_i) \le 2\exp(\nu|x_i|).$$

Multiplying these inequalities for $i = 1, ..., n$ we find

$$\exp(\nu|x|) \le \sum_{\kappa=1}^{2^n} \exp(< \nu I^{(\kappa)}, x >) \le 2^n \exp(\nu|x|), \tag{2.8}$$

which implies that (2.1) and (2.1′) are equivalent for $h, \nu > 0$.

(i′) Let $\Gamma = (\omega_1, ..., \omega_n)$ and let $|\omega_j| < \nu, j = 1, ..., n$. Multiplying the inequalities

$$\exp(\omega_i x_i) \le \exp(\nu x_i) + \exp(-\nu x_i) \tag{2.9}$$

for $i = 1, ..., n$, we obtain

$$(2.8') \qquad \exp(<\Gamma, x>) \leq \sum_{\kappa=1}^{2^n} \exp(<\nu I^{(\kappa)}, x>),$$

whence

$$'|\varphi|_{(h,\nu)} \leq \sum_{\kappa=1}^{2^n} |\varphi|_{[h,\nu I^{(\kappa)}]} \leq 2^n \; |\varphi|_{(h,\nu)}.$$

Conversely, let $'|\varphi|_{(h,\nu)} < \infty$. Given $x \in \mathbb{R}^n$, we take $\Gamma = (\epsilon_1\rho, ..., \epsilon\rho)$, $\rho < \nu$, in (2.8′), where $\epsilon_i = \pm 1$ and the sign of ϵ_i coincides with that of x_i. Then we derive

$$\frac{\exp(\rho|x|)|\partial^\alpha \varphi(x)|}{h^{-|\alpha|}\alpha!} \leq {}'|\varphi|_{(h,\nu)}$$

for all $x \in \mathbb{R}^n$ and $\rho < \nu$. By continuity, this inequality is retained for $\rho = \nu$ as well. Taking the supremum over $x \in \mathbb{R}^n$ and α we obtain (i′).

(ii) By virtue of the obvious inequality

$$\exp(\nu|x|) \leq \exp(<\Gamma, x>), \; \forall \nu < 0, \; \Gamma = (\omega_1, ..., \omega_n), \; |\omega_j| \leq |\nu|,$$

the spaces $F_{[h,\nu I^{(\kappa)}]}$, $\kappa = 1, 2, ..., 2^n$, and, consequently, their linear hull as well are embedded in $F_{(h,\nu)}$. And if $\psi = \sum_{\kappa=1}^{2^n} \psi^{(\kappa)}$, $\psi^{(\kappa)} \in F_{[h,\nu I^{(\kappa)}]}$, then, by triangle inequality,

$$|\psi|_{(h,\nu)} \leq \sum_{\kappa=1}^{2^n} |\psi^{(\kappa)}|_{[h,\nu I^{(\kappa)}]}.$$

Taking the infimum in the right-hand side over all the representation $\psi = \sum_{\kappa=1}^{2^n} \psi^{(\kappa)}$ we prove that the right-hand side space of (2.7) is embedded into the left-hand side space.

To prove the opposite embedding we construct a system of functions $\chi^{(\kappa)}$, $\kappa = 1, ..., 2^n$, $\chi^{(\kappa)} \geq 0$, possessing the following properties:

(a) $\sum_{\kappa=1}^{2^n} \chi^{(\kappa)}(x) = 1$

(b) If $\psi \in F_{(h,\nu)}$, then $\chi^{(\kappa)}\psi \in F_{[h,\nu I^{(\kappa)}]}$, and $|\chi^{(\kappa)}(x)\psi|_{[h,\nu I^{(\kappa)}]} \leq const|\psi|_{(h,\nu)}$.

The embedding of the left-hand space of (2.7) into the right-hand side space is a trivial consequence of (a) and (b).

If $x_{i_1}, ..., x_{i_k} \geq 0$ and $x_{i_{k+1}}, ..., x_{i_n} < 0$, let $(\epsilon_1, ..., \epsilon_n)$ be the coordinates of a vertex $I^{(\kappa)}$, where $\epsilon_{i_1} = \cdots = \epsilon_{i_k} = 1$ and $\epsilon_{i_{k+1}} = \cdots = \epsilon_{i_n} = -1$. Then we put

$$\chi^{(\kappa)}(x) = \prod_{l=1}^{k} \exp(-x_{i_l}^2) \prod_{l=k+1}^{n} (1 - \exp(-x_{i_l}^2)). \tag{2.10}$$

It is obvious that (a) is fulfilled.

Since $< I^{(\kappa)}, x >= \sum_{l=1}^{k} x_{i_l} - \sum_{l=k+1}^{n} x_{i_l} = |x|$, it follows from the proof of Theorem 2.1 that (b) holds, whence the proposition is proved.□

By virtue of (2.2), we can define the spaces $\mathcal{G}$ and $\mathcal{M}$ with the aid of the operations of projective and inductive limits:

$$\begin{aligned} \mathcal{G} &= \bigcap_{h,\nu} F_{(h,\nu)}, \\ \mathcal{M} &= \bigcap_{h>0} F_{(h,-\infty)}, \ F_{(h,-\infty)} = \bigcup_{\nu} F_{(h,\nu)}. \end{aligned} \tag{2.11}$$

Using the system of norms (2.1) we introduce the structure of a countably normed space in $\mathcal{G}$, i.e., the system of neighborhoods in $\mathcal{G}$ is determined by

$$|\varphi|_{(h,\nu)} < \epsilon, \ h, \nu \in \mathbb{Q}^+, \ \epsilon > 0. \tag{2.12}$$

The topology generated by the neighborhoods (2.12) can be interpreted as the topology of projective limit of the spaces $F_{(h,\nu)}$.

The system of norms (2.1) makes it possible to introduce a distance function in $\mathcal{G}$, i.e., to turn $\mathcal{G}$ into a Fréchet space.

The spaces $\mathcal{G}$ and $\mathcal{M}$ consist, respectively, of analytic functions extended to $\mathbb{C}^n$ which increase, for $|Rez| \to \infty$, not stronger than any exponential function $\exp(\nu|Rez|)$ and of analytic functions extended to any strip in $\mathbb{C}^n$ which are estimated with the aid of a special exponential function $\exp(\nu|Rez|)$.

PROPOSITION 2.5.

(i) *$\mathcal{M}$ is a commutative algebra relative to multiplication.*

(ii) *$\mathcal{G}$ is an ideal in $\mathcal{M}$, i.e., the operation of multiplication is defined:*

$$\mathcal{M} \times \mathcal{G} \to \mathcal{G} \ ((a(x), \psi(x)) \to a(x)\psi(x))$$

and this operator is continuous.

Proof. It follows from the proof of Theorem 2.1. □

§3. Denseness of the space $\mathcal{G}$

$F_{(h,\mu)}$ is not dense in $F_{(h,\nu)}$ when $\mu > \nu$. Indeed, if $F_{(h,\mu)}$ is dense in $F_{(h,\nu)}$, then for $\varphi \in F_{(h,\nu)}$ with $|\varphi| \exp(\nu|x|) > 1$ there is a function $\psi \in F_{(h,\mu)}$ such that $|\varphi - \psi|_{(h,\nu)} < 1$.

On the other hand, we have

$$
\begin{aligned}
1 >&|\varphi - \psi|_{(h,\nu)} \\
\geq&|\varphi| \exp(\nu|x|) - |\psi| \exp(\nu|x|) \\
\geq&|\varphi| \exp(\nu|x|) - |\psi|_{(h,\mu)} \exp(-(\mu - \nu)|x|).
\end{aligned} \tag{3.1}
$$

Passing to the limit for $|x| \to \infty$, it is a contradiction.

PROPOSITION 3.1. *If $F_{(h,\mu)}$ is dense in $F_{(h,\nu)}$ relative to the topology of $F_{(h,\nu-\eta)}$, $\mu > \nu$, $\eta > 0$*

Proof. Let $f \in F_{(h,\nu)}$ and $\varphi(x) = \exp(-\sum_{i=1}^{n} x_i^2)$. Then it follows from Theorem 2.1 that $f(x)\varphi(\epsilon x) \in F_{(h,\mu)}$ and

$$\sup_{z \in D_h} |f(z)\varphi(\epsilon z) - f(z)| \exp((\nu - \eta)|Rez|) \to 0 \text{ as } \epsilon \to 0_+,$$

whence it follows from Proposition 2.2. □

The space $\mathcal{G}$ is nondense in $F_{(h,\nu)}$. In fact, if $\overline{\mathcal{G}} = F_{(h,\nu)}$, then for $\varphi \in F_{(h,\nu)}$ with $|\varphi| \exp(\nu|x|) > 1$, then there is a function $\psi \in \mathcal{G}$ such that $|\varphi - \psi|_{(h,\nu)} < 1$. We can see from (3.1) that it is a contradiction.

However, in a weaker topology all the element of $F_{(h,\nu)}$ can be approximated with functions belonging to $\mathcal{G}$.

THEOREM 3.2. *$\mathcal{G}$ is dense in $F_{(h,\nu)}$ relative to the topology of*

$$F_{(h/e,\nu-\tau)}, \tau > 0.$$

Proof. Let

$$f \in F_{(h,\nu)},\ \varphi_j(x) = (j+1)^n \nu^n \pi^{-n/2} \exp(-\nu^2 (j+1)^2 \sum_{k=1}^{n} x_k^2),$$

and let

$$f_j(x) = f * \varphi_j(x).$$

Then it follows from Theorem 2.1 that $f_j \in F_{(\infty,\nu)}$.

First of all, we show that $F_{(\infty,\nu)}$ is dense in $F_{(h,\nu)}$ relative to the topology of $F_{(h/e,\nu)}$, i.e., $|f_j - f|_{(h/e,\nu)} \to 0$ as $j \to \infty$.

From the direct estimates we obtain that

$$\begin{aligned}
&|\partial^\alpha f_j(x) - \partial^\alpha f(x)| \\
&\leq (j+1)^n \nu^n \pi^{-n/2} \int |\nabla \partial^\alpha f(x-\theta t)\cdot t| \exp(-\nu^2 (j+1)^2 \sum_{k=1}^n t_k^2) dt \\
&\leq \pi^{-1/2} \frac{n 2^{n-1}}{\nu(j+1)} h^{-1} (h/e)^{-|\alpha|} \alpha! |f|_{(h,\nu)} \\
&\times \exp(-\nu|x|) \exp(\frac{(n-1)\theta^2}{4(j+1)^2}) [\exp(-\frac{\theta^2}{4(j+1)^2}) + \frac{\pi^{1/2}\theta}{j+1}]
\end{aligned}$$

for some $0 < \theta < 1$. This implies that $|f_j - f|_{(h/e,\nu)} \to 0$ as $j \to \infty$.

Next, we show that $\mathcal{G}$ is dense in $F_{(\infty,\nu)}$ relative to the topology of $F_{(h/e,\nu-\tau)}$.

Let $f \in F_{(\infty,\nu)}$. Then we see from Theorem 2.1 that

$$f(x)\exp(-\sum_{k=1}^n (\epsilon x_k)^2) \in \mathcal{G}.$$

From Proposition 2.2 it suffices to prove that

$$\sup_{z \in D_{h/e}} |f(z)\exp(-\sum_{k=1}^n (\epsilon z_k)^2) - f(z)| \exp((\nu-\eta)|x|) \to 0 \text{ as } \epsilon \to 0_+.$$

By Theorem 2.1 we obtain the following estimates: For $z \in D_{h/e}$

$$\begin{aligned}
&|f(z)\exp(-\sum_{k=1}^n (\epsilon z_k)^2) - f(z)| \exp(\nu|x|) \\
&\leq C\epsilon^2 \exp(n(\epsilon h/e)^2)\{(1+2(\epsilon h/e)^2)|x|^2 + (2h/e)|x| + n(\epsilon h/e)^2\},
\end{aligned}$$

whence it follows, and hence the theorem is proved. □

Let the space $\mathcal{G}'$ be a space of continuous linear functionals on $\mathcal{G}$. We denote by $(f,\varphi) = f(\varphi)$ the value of the functional $f \in \mathcal{G}'$ on the element $\varphi \in \mathcal{G}$.

Note that if $f \in \mathcal{G}'$, then there are positive numbers h, μ and a constant K such that

$$(3.2) \qquad |(f, \varphi)| \leq K|\varphi|_{(h,\mu)}, \ \forall \varphi \in \mathcal{G}.$$

This implies that the functional f is continuous in the norm $|\cdot|_{(h,\mu)}$, i.e., belongs to the Banach conjugate space $(F_{(h,\mu)})'$ of $F_{(h,\mu)}$.

The embeddings (2.2) induce the adjoint embeddings

$$(3.3) \qquad (F_{(h',\nu')})' \subset (F_{(h,\nu)})', \ h \geq h' > 0, \ \nu \geq \nu',$$

and we can consider the union $\bigcup_{h,\nu}(F_{(h,\nu)})'$. We have already shown that $\mathcal{G}' \subset \bigcup_{h,\nu}(F_{(h,\nu)})'$. Since the opposite inclusion is obvious, we have thus proved.

THEOREM 3.3. *$\mathcal{G}'$ regarded as a vector space coincides with the union of $(F_{(h,\nu)})'$:*

$$(3.4) \qquad \mathcal{G}' = \bigcup_{h,\nu)} (F_{(h,\nu)})'.$$

The right-hand space of (3.4) can be equipped with the topology of inductive limit, and in the left-hand space of (3.4) we can introduce the topology of the strong conjugate space of $\mathcal{G}$.

Note that the space $\mathcal{G}'$ is a reflexive and is a regular inductive limit, which implies the coincidence of two above-mentioned topologies in $\mathcal{G}'$(see [3]).

The space $\mathcal{G}'$ are called the space of extended Fourier hyperfunctions.

If $f \in F_{(h,\nu)}$, $g \in F_{(h,-\nu+\epsilon)}$, $h, \epsilon > 0$, then the bilinear form

$$(3.5) \qquad (f, g) = \int f(x)g(x)dx$$

is defined and depends continuously on f and g (in the corresponding topologies). Hence,

$$(3.6) \qquad \mathcal{G} \subset F_{(h,\nu)} \subset (F_{(h,-\nu+\epsilon)})' \subset \mathcal{G}'.$$

If $u \in \mathcal{G}'$ and $\varphi \in \mathcal{G}$, we denote by $u * \varphi$ as follows:

$$(3.7) \qquad (u * \varphi)(x) = u_y(\varphi(x - y)).$$

We see that integration by parts implies

$$\int (\partial^\alpha u)(x)\varphi(x)dx = (-1)^{|\alpha|}\int u(x)(\partial^\alpha \varphi)(x)dx \tag{3.8}$$

for $u, \varphi \in \mathcal{G}$. The mapping $\varphi \to \int (\partial^\alpha u)(x)\varphi(x)dx$ and $\psi \to \int u(x)\psi(x)dx$ are continuous linear functionals on $\mathcal{G}$. Denoting them by $\partial^\alpha u$ and u, (3.8) can be written in the form

$$(\partial^\alpha u)(\varphi) = (-1)^{|\alpha|}u(\partial^\alpha \varphi). \tag{3.9}$$

But the right-hand side of (3.9) is well-defined whenever $u \in \mathcal{G}'$ and $\varphi \in \mathcal{G}$. Furthermore, $\varphi \to u(\partial^\alpha \varphi)$ is continuous on $\mathcal{G}$ (being the composition of two continuous functions). Therefore, we can define the partial derivative $\partial^\alpha u$ is an element of $\mathcal{G}'$.

In a similar way, the translation operator τ_h, $h \in \mathbb{R}^n$ and the reflection operator I on $\mathcal{G}'$are defined as follows:

$$(\tau_h u)(\varphi) = u(\tau_{-h}\varphi),\ Iu(\varphi) = u(I\varphi)\ \forall \varphi \in \mathcal{G}$$

THEOREM 3.4. *If $u \in \mathcal{G}'$ and $\varphi \in \mathcal{G}$, we have $u * \varphi \in C^\infty$. The derivatives of the convolution are given by*

$$\partial^\alpha (u * \varphi) = (\partial^\alpha u) * \varphi = u * (\partial^\alpha \varphi). \tag{3.10}$$

Proof. If $x_j \to x$, it is clear that $\varphi(x_j - y) \to \varphi(x - y)$ in $\mathcal{G}$ as a function of y. Hence $u * \varphi$ is continuous. To complete the proof we only have to prove (3.10) when $|\alpha| = 1$: it then follows inductively for all α and shows that $u * \varphi \in C^\infty$. Thus let e_k be the unit vector along the x_k-axis and consider the difference quotient

$$h^{-1}((u * \varphi)(x + he_k) - (u * \varphi)(x)) = u_y((\varphi(x + he_k - y) - \varphi(x - y))/h).$$

When $h \to 0$, the difference quotient $\varphi(x + he_k - y) - \varphi(x - y))/h$ converges to $\partial_k \varphi(x - y) \in \mathcal{G}$ as a function of y for fixed x. Hence we obtain

$$\partial_k(u * \varphi) = u * (\partial_k \varphi).$$

Since $\partial_k \varphi(x - y) = -\partial_{y_k}\varphi(x - y)$, it follows from the definition of the derivative of a Fourier hyperfunction that

$$(\partial_k u) * \varphi = u * (\partial_k \varphi),$$

and this completes the proof. □

THEOREM 3.5. *If $u \in \mathcal{G}'$ and $\varphi \in \mathcal{G}$, then the convolution $f(x) = (u * \varphi)(x)$ belongs to $F_{(\infty,-\mu)}$ for some $\mu > 0$.*

Proof. From (3.2) we obtain the following estimates: For some C, h, $\mu > 0$

$$
\begin{aligned}
&|(u(\partial^\alpha \varphi(x-y))| \\
&\le C|\partial^\alpha \varphi(x-\cdot)|_{(h,\mu)} \\
&= C \sup_{y,\beta} \frac{|\partial^{\alpha+\beta}\varphi(x-y)| \exp(\mu|y|)}{h^{-|\beta|}\beta!} \\
(3.11)\qquad &\le C|\varphi|_{(h',\mu)} \sup_{y,\beta} \frac{h'^{-|\alpha+\beta|}(\alpha+\beta)! \exp(-\mu|x-y|)\exp(\mu|y|)}{h^{-|\beta|}\beta!} \\
&\le C|\varphi|_{(h',\mu)} \sup_{y,\beta} (h'/2)^{-|\alpha|}(h'/2h)^{-|\beta|}\alpha! \exp(\mu|x|).
\end{aligned}
$$

Then if $h'/2 \ge h$, (3.11) implies that

$$|\partial^\alpha (u * \varphi)(x)| \le C|\varphi|_{(h',\mu)}(h'/2)^{-|\alpha|}\alpha! \exp(\mu|x|).$$

If $h'/2 \le h$, then we obtain

$$|(\partial^\alpha \varphi(x-\cdot))|_{(h,\mu)} \le C|\varphi|_{(2h,\mu)}(h'/2)^{-|\alpha|}\alpha! \exp(\mu|x|),$$

whence the theorem is proved. □

A direct application of Fubini's theorem shows that if u, φ, and ψ are all in $\mathcal{G}$, then

$$\int (u * \varphi)(x)\psi(x)dx = \int u(x)(I\varphi * \psi)(x)dx.$$

The mappings $\psi \to \int (u * \varphi)(x)\psi(x)dx$ and $\theta \to \int u(x)\theta(x)dx$ are continuous linear functionals on $\mathcal{G}$. If we denote these functionals by $u * \varphi$ and u, the equality can be written in the form:

$$(3.12)\qquad (u * \varphi)(\psi) = u(I\varphi * \psi).$$

If $u \in \mathcal{G}'$ and φ, ψ belong to $\mathcal{G}$, the right-hand side of (3.12) is well-defined since $I\varphi * \psi \in \mathcal{G}$. Furthermore, the mapping $\psi \to u(I\varphi * \psi)$, being the composition of two continuous functions, is continuous. Thus, we can define the convolution of an extended Fourier hyperfunction u with a test function φ, $u * \varphi$, by means of the equality (3.12).

Note that a sequence $\{\varphi_j\}$ converges to 0 in $\mathcal{G}$ if and only if for any $h, \mu > 0$

$$\frac{\partial^\alpha \varphi_j(x)\exp(\mu|x|)}{h^{-|\alpha|}\alpha!} \to 0$$

uniformly with respect to x, α when $j \to \infty$.

THEOREM 3.6. *If $u \in \mathcal{G}'$ and $\varphi, \psi \in \mathcal{G}$, then*

$$(u * \varphi) * \psi = u * (\varphi * \psi). \tag{3.13}$$

Proof. For $\epsilon > 0$ we form the Riemann sum

$$f_\epsilon(x) = \epsilon^n \sum_g \varphi(x - g\epsilon)\psi(g\epsilon), \tag{3.14}$$

where g run through all points with integer coordinates. Let

$$f_{\epsilon,m}(x) = \epsilon^n \sum_{|g_j| \le m} \varphi(x - g\epsilon)\psi(g\epsilon).$$

Then we can see from Theorem 2.1 and 3.4 that $f_{\epsilon,m}(x) \to f_\epsilon(x)$ in $\mathcal{G}$, and that $f_\epsilon(x) \to (\varphi * \psi)(x)$ in $\mathcal{G}$. Hence

$$\begin{aligned} &(u * (\varphi * \psi))(x) \\ = &\lim_{\epsilon \to 0_+} \lim_{m \to \infty} (u * f_{\epsilon,m})(x) \\ = &\lim_{\epsilon \to 0_+} \lim_{m \to \infty} \epsilon^n \sum_{|g_j| \le m} (u * \varphi)(x - g\epsilon)\psi(g\epsilon), \end{aligned}$$

and the theorem is proved. □

THEOREM 3.7. *The canonical embedding $\mathcal{G} \subset \mathcal{G}'$ is continuous and image of $\mathcal{G}$ is dense in $\mathcal{G}'$.*

Proof. First of all, we show that if $u \in \mathcal{G}'$, there is a sequence $u_j \in F_{(\infty,-\mu)}$ such that $u_j \to u$ in the weak topology in $\mathcal{G}'$.

Let $\varphi_j(x) = (j+1)^n \nu^n \pi^{-n/2} \exp(-\nu^2 (j+1)^2 \sum_{k=1}^n x_k{}^2)$ and $u_j(x) = u * \varphi_j(x)$. Then we obtain from Theorem 3.5 that $u_j \in F_{(\infty,-\mu)}$ for some $\mu > 0$. It follows from Theorem 3.6 that for every $\psi \in G$

$$u_j(\psi) = ((u * \varphi_j) * I\psi)(0) = (u * (\varphi_j * I\psi))(0).$$

Since $\varphi_j * I\psi \to I\psi$ in $\mathcal{G}$, it follows.

Next, we show that if $\varphi \in F_{(\infty,-\mu)}$, there is a sequence $\{\varphi_j\}$ in $\mathcal{G}$ such that $\varphi_j \to \varphi$ in the weak topology in $\mathcal{G}'$.

Let $\varphi_j(x) = \varphi(x) \exp(-j^{-1} \sum_{k=1}^n x_k^2)$. Then it follows that $\varphi_j \in \mathcal{G}$ and

$$\varphi_j(\psi) = \int \varphi_j(x)\psi(x)dx \to \varphi(\psi) = \int \varphi(x)\psi(x)dx,$$

whence it follows, and hence the theorem is proved. □

§4. The Fourier(-Laplace) operator in $\mathcal{G}$

Denote by $(\mathcal{F}\varphi)(\xi)$ the Fourier transform of a function $\varphi(x) \in \mathcal{G}$:

$$(\mathcal{F}\varphi)(\xi) = \hat{\varphi}(\xi) = (2\pi)^{-n/2} \int \exp(-i < x, \xi >)\varphi(x)dx.$$

PROPOSITION 4.1. *The Fourier transform $\mathcal{F} : \mathcal{G} \to \mathcal{G}$ is continuous and linear. The inverse Fourier transform $\mathcal{F}^{-1} : \mathcal{G} \to \mathcal{G}$ is also continuous and linear.*

Proof. Let $\varphi \in F_{(h,\nu)}$, $\forall h, \nu > 0$. Then we have

$$\begin{aligned} |\xi^\beta \partial^\alpha \hat{\varphi}(\xi)| &\leq (2\pi)^{-n/2} \int |\partial^\beta (x^\alpha \varphi(x))| dx \\ &\leq (2\pi)^{-n/2} \sum_{\gamma}^{\beta} \frac{\beta!}{\gamma!(\beta-\gamma)!} \frac{\alpha!}{(\alpha-\gamma)!} \int |x^{\alpha-\gamma} \partial^{\beta-\gamma} \varphi(x))| dx \\ &\leq C |\varphi|_{(h,\nu)} \alpha! \beta! (\nu/2)^{-|\alpha|} (h/2)^{-|\beta|} 2^{-|\beta|}, \end{aligned}$$

whence we obtain

$$\frac{(h/2)^{|\beta|} |\xi^\beta|}{\beta!} \frac{|\partial^\alpha \hat{\varphi}(\xi)|}{(\nu/2)^{-|\alpha|} \alpha!} \leq C |\varphi|_{(h,\nu)} 2^{-|\beta|}.$$

Summing the above inequality with respect to β and taking the supremum over all x and α, we find

$$|\hat{\varphi}|_{(\nu/2, h/2)} \leq K |\varphi|_{(h,\nu)}. \tag{4.1}$$

This proves that the mapping $\varphi \to \hat{\varphi}$ is continuous in the topology of $\mathcal{G}$. □

THEOREM 4.2. *Fourier's inverse theorem holds:*

$$\mathcal{F}^{-1}\mathcal{F}\varphi(x) = (2\pi)^{-n/2} \int \exp(i < x, \xi >) \mathcal{F}\varphi(\xi) d\xi = \varphi(x), \tag{4.2}$$

i.e., we have $\mathcal{F}^{-1}\mathcal{F}\varphi = \varphi$, and similarly $\mathcal{F}\mathcal{F}^{-1}\varphi = \varphi$.

Proof. We have

$$\int \psi(\xi)\hat{\varphi}(\xi)\exp(i < x,\xi >)d\xi = \int \hat{\psi}(y)\varphi(x+y)dy, \ (\varphi, \ \psi \in \mathcal{G}).$$

If we take $\psi(\epsilon\xi)$ for $\psi(\xi)$, $\epsilon > 0$, then

$$(2\pi)^{-n/2}\int \exp(-i < y,\xi >)\psi(\epsilon\xi)d\xi = \epsilon^{-n}\hat{\psi}(y/\epsilon).$$

Hence

$$\int \psi(\epsilon\xi)\hat{\varphi}(\xi)\exp(i < x,\xi >)d\xi = \int \hat{\psi}(y)\varphi(x+\epsilon y)dy.$$

We shall take, $\psi(x) = \exp(-\sum_{k=1}^{n} x_k^2/2)$ and let $\epsilon \to 0_+$. Then

$$\psi(0)\int \hat{\varphi}(\xi)\exp(i < x,\xi >)d\xi = \varphi(x)\int \hat{\psi}(y)dy.$$

This proves (4.2). □

Based on the conjugacy, the Fourier operator on $\mathcal{G}'$ is defined: For $f \in \mathcal{G}'$ we put

$$(\mathcal{F}f, \mathcal{F}^{-1}\varphi) = (f, \varphi). \tag{4.3}$$

As has already been said, we have $\mathcal{F}^{-1}\mathcal{G} = \mathcal{G}$, and therefore the function $\mathcal{F}f$ is defined throughout $\mathcal{G}$.

According to classical Parserval's theorem, (4.3) holds for any $f, \varphi \in \mathcal{G}$. It follows that the Fourier operator on $\mathcal{G}$ commutes with the canonical embedding $\mathcal{G} \to \mathcal{G}'$, i.e., for $f \in \mathcal{G}$ we can interpret as the Fourier transform in the sense of $\mathcal{G}'$. If we put $\varphi(x) = \hat{\psi}(x)$, where $\psi(\xi) \in \mathcal{G}$, this results in

$$(\hat{f}, \psi) = (f, \hat{\psi}),$$

i.e.,

$$\mathcal{F} : \mathcal{G}' \to \mathcal{G}'$$

is the adjoint operator of $\mathcal{F} : \mathcal{G} \to \mathcal{G}$ and hence is an isomorphism. We have thus proved

THEOREM 4.3. *The spaces $\mathcal{G}$ and $\mathcal{G}'$ are Fourier self-dual:*

$$\mathfrak{F}\mathcal{G} = \mathcal{G}, \ \mathfrak{F}\mathcal{G}' = \mathcal{G}'. \tag{4.4}$$

REMARK. Since $\mathcal{G}$ is a dense subset in $\mathcal{G}'$, the Fourier operator on $\mathcal{G}'$ can be also be defined as the extension by continuity of (4.1).

Let $\nu > 0$. Let $F^{(\nu,s)}$ denote the Banach space of functions $\psi(\zeta)$ holomorphic in the tube domain D_ν and having a finite norm

$$|\psi|^{(\nu,s)} = \sup_{\zeta \in D_\nu} \exp(s|\zeta|)|\psi(\zeta)|. \tag{4.5}$$

PROPOSITION 4.4. *The map $F_{(h,\nu)} \to F^{(h,\nu)} \ : \ f(x) \to f(x+yi)$ is a topological isomorphism and there are constants $C_1, \ C_2 > 0$ such that*

$$C_1|f|^{(h,\nu)} \le |f|_{(h,\nu)} \le C_2|f|^{(h,\nu)}.$$

Proof. It follows from the proof of Theorem 2.1. □

REMARK. From Proposition 4.4 we can see that

$$\mathcal{M} = \bigcap_{h>0} F^{(h,-\infty)}. \tag{2.11'}$$

PROPOSITION 4.5. *If $\{f_n(\zeta)\}$ is a sequence of functions in $F^{(h,\nu)}$ such that $f_n(\zeta) \to f(\zeta)$ in $F^{(h,\nu)}$, then $f \in F^{(h,\nu)}$.*

Proof. It is obvious that $f(\zeta)$ is holomorphic in the tube domain D_ν. Since there is a natural number N such that $|f_n(\zeta) - f(\zeta)|^{(h,\nu)} < 1$ for $n > N$, we obtain

$$|f(\zeta)| \exp(\nu|\zeta|) \le 1 + |f_n(\zeta)| \exp(\nu|\zeta|).$$

Taking the supremum in the above inequality over all $\zeta \in D_\nu$, it is proved. □

COROLLARY 4.6. *If $\{f_n(x)\}$ is a sequence of functions in $F_{(h,\nu)}$ such that $f_n(x) \to f(x)$ in $F_{(h,\nu)}$, then $f \in F_{(h,\nu)}$.*

Proof. It follows from Proposition 2.2 and 4.5. □

If $\varphi \in F_{(h,\nu)}$, $h, \nu > 0$, then for any $\zeta \in D_\nu$, the absolutely convergent Fourier-Laplace integral is defined:

$$(4.6) \qquad \mathfrak{F} : \varphi(x) \to \hat{\varphi}(\zeta) = (2\pi)^{-n/2} \int \exp(-i < x, \zeta >)\varphi(x)dx,$$

and Parseval's inequality holds:

$$(4.7) \qquad |\hat{\varphi}|^{(\nu,h)} \leq K|\varphi|_{(h,\nu)}.$$

Indeed, since

$$\hat{\varphi}(\xi) = (2\pi)^{-n/2} \int \exp(-i < x + yi, \xi > \varphi(x + yi)dx,$$

$$y_j = -(h - \epsilon)\frac{|\xi_j|}{\xi_j}, \ \epsilon > 0,$$

we obtain that for $|Im\zeta_j| < \nu$

$$|\hat{\varphi}(\zeta)| \leq (2\pi)^{-n/2}|\varphi|^{(h,\nu)} \int \exp(< x, Im\zeta >) \times \exp(-(h - \epsilon)|\xi|) \exp(-\nu|x + yi|)dx.$$

If $\epsilon \to 0_+$, then (4.7) follows from Proposition 4.4.

On the other hand, since $F_{(\nu,h)}$ is a subspace of the Schwartz space $\mathcal{S}$ of rapidly decreasing functions, for a function $\psi(\xi) \in F_{(\nu,h)}$ there exists a function $\varphi \in \mathcal{S}$ such that $\hat{\varphi}(\xi) = \psi(\xi)$. Therefore for $\hat{\varphi}(\zeta) \in F^{(\nu,h)}$ the classical inverse Fourier-Laplace transform is defined:

$$(4.6') \qquad \varphi(x) = (2\pi)^{-n/2} \int \exp(i < x, \xi + i\omega >)\hat{\varphi}(\xi + i\omega)d\xi,$$

where $\omega_j = (\nu - \epsilon)\frac{|x_j|}{x_j}$, $0 < \epsilon < \nu$. Then for $|y_j| < h$ we have

$$\begin{aligned} &|\varphi(x + yi)| \\ &\leq (2\pi)^{-n/2} \int \exp(- < x, \omega >) \exp(- < y, \xi >)|\hat{\varphi}(\xi + i\omega)|d\xi \\ &\leq C|\hat{\varphi}|^{(\nu,h)} \exp(-(\nu - \epsilon)|x|) \int \exp(- < y, \xi >) \exp(-h|\xi|)d\xi. \end{aligned}$$

If $\epsilon \to 0_+$, then it follows from Proposition 4.4 that Parserval's inequality holds:

$$(4.7') \qquad |\varphi|_{(h,\nu)} \leq K|\hat{\varphi}|^{(\nu,h)}.$$

As a consequence of what has been said, we obtain the following:

THEOREM 4.7. *The Fourier-Laplace transform* $\mathfrak{F} : F_{(h,\nu)} \to F^{(\nu,h)}$: $\varphi(x) \to \hat{\varphi}(\zeta)$ *is a topological isomorphism.*

THEOREM 4.8. *The Fourier-Laplace transform operator determines an isomorphism*

$$\mathfrak{F}\mathcal{G} = \bigcap_{h,\nu>0} F^{(\nu,h)}. \tag{4.8}$$

The right-hand space (4.8) consists of entire functions $\psi(\zeta)$ such that for any $h, \nu > 0$ there exists a constant $K_{h\nu}$ such that $|\psi(\zeta)| \leq K_{h\nu} \exp(-h|\zeta|)$, $\forall \zeta \in \overline{T}_\nu$.

PROPOSITION 4.9. (i) $\mathcal{M}$ *is a commutative algebra relative to multiplication;*

(ii) $\mathfrak{F}\mathcal{G}$ *is an ideal in* $\mathcal{M}$. *i.e., the operation of multiplication*

$$\mathcal{M} \times \mathcal{G} \to \mathcal{G} \ ((a(\zeta), \psi(\zeta)) \to a(\zeta)\psi(\zeta))$$

is defined.

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