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LOCALLY NILPOTENT GROUPS WITH THE MINIMAL CONDITION ON NORMAL SUBGROUPS OF INFINITE INDEX

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ABSTRACT. A group G is said to satisfy the minimal condition on normal subgroups of infinite index if there does not exist an infinite properly descending chain $G_1 > G_2 > \cdots$ of normal subgroups of infinite index in G. We characterize the structure of locally nilpotent groups satisfying this chain condition.

1. Introduction

A group G is said to satisfy the weak maximal condition on normal subgroups if there does not exist an infinite properly ascending chain of normal subgroups $G_1 < G_2 < \cdots$ such that $|G_{i+1} : G_i| = \infty$ for each i. A group G is said to satisfy the weak minimal condition on normal subgroups if it has no infinite properly descending chain of normal subgroups $G_1 > G_2 > \cdots$ such that $|G_i : G_{i+1}| = \infty$ for each i. In the late 1970s and early 1980s Kurdachenko considered groups satisfying the weak maximal or the weak minimal conditions on normal subgroups and their connection with each other and with other finiteness conditions, particularly within the class of locally nilpotent groups. Kurdachenko [1] proved that a torsion(torsion-free) locally nilpotent group satisfies the weak maximal condition on normal subgroups if and only if it is Chernikov (minimax respectively). In addition, in [2] he gave necessary and sufficient conditions for a locally nilpotent group to satisfy the weak minimal condition on normal subgroups.

A group is said to satisfy *max-∞* (the maximal condition on infinite subgroups) if there is no infinite properly ascending chain of infinite subgroups. A group satisfying the conditions *max-∞s* (the maximal

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condition on infinite subnormal subgroups) or *max-∞n* (the maximal condition on infinite normal subgroups) can be defined similarly by imposing the conditions on infinite subnormal or infinite normal subgroups respectively. A group G is said to satisfy *min-∞* (the minimal condition on subgroups of infinite index) if there does not exist an infinite properly descending chain of subgroups with infinite index in G. Again, we can define groups satisfying *min-∞s* (the minimal condition on subnormal subgroups of infinite index) or *min-∞n* (the minimal condition on normal subgroups of infinite index) analogously.

Groups satisfying *max-∞* or *min-∞* and the structure of solvable groups with *max-∞s* or *min-∞s* were investigated in [3, 4]. Also in [5] locally nilpotent groups satisfying *max-∞n* were considered. In this paper we focus on the structure of locally nilpotent groups satisfying *min-∞n*.

It is clear that if a group satisfies min-∞n then it satisfies the weak minimal condition on normal subgroups. The additive group of p-adic rationals $\mathbb{Q}_p$ satisfies both the weak maximal and weak minimal conditions, but it satisfies neither max-∞ nor min-∞. Thus the condition min-∞n is stronger than the weak minimal condition on normal subgroups. A motivation for our study is to estimate the influence of the condition min-∞n on the subgroup lattice of the infinite normal subgroups.

2. Preliminary results

First we note two easy facts.

LEMMA 2.1. *Let G be a group with min-∞n. If N is a normal subgroup of G of infinite index, then N satisfies min-G (the minimal condition for G-invariant subgroups).*

LEMMA 2.2. *The property min-∞n is quotient closed.*

The following results are useful in classifying locally nilpotent groups with min-∞.

LEMMA 2.3 ([3], Theorem 2.8). *A polycyclic-by-finite group satisfies min-∞ if and only if it is infinite cyclic-by-finite.*

LEMMA 2.4 ([3], Theorem 3.2). *An abelian group satisfies min-∞ if and only if either it satisfies min or it is infinite cyclic-by-finite.*

3. Locally nilpotent groups with min-∞n

Note that the conditions *min-n* (the minimal condition on normal subgroups) and *min* (the minimal condition on subgroups) are the same property for locally nilpotent groups. We begin by describing nilpotent groups with min-∞n.

LEMMA 3.1. *A nilpotent group G satisfies min-∞n if and only if either it has min or it is central infinite cyclic-by-finite.*

Proof. Sufficiency has already been proved by Lemma 2.3. Suppose that G satisfies min-∞n, but not min. If G_{ab} has min, then so does G, a contradiction. Thus G_{ab} does not satisfy min, so G_{ab} is infinite cyclic-by-finite by Lemma 2.4. Hence G_{ab} is finitely generated and so is G. This implies that $Z(G)$ contains an element g of infinite order. For otherwise $Z(G)$ is finitely generated abelian torsion and so is finite, from which it follows that G is finite, a contradiction. Now $\langle g\rangle \lhd G$ since $g \in Z(G)$. Note that $G/\langle g\rangle$ is finite, for otherwise $\langle g\rangle$ satisfies min, which cannot be true. Therefore $G/\langle g\rangle$ is finite and hence G has a central infinite cyclic subgroup of finite index. □

We are now in position to characterize locally nilpotent groups satisfying min-∞n, but not min.

THEOREM 3.2. *Let G be a locally nilpotent group with min-∞n, but not min. Then G has a normal subgroup M such that:*

1. *$G/M = \langle xM\rangle$ is infinite cyclic;*
2. *M is a torsion group with min-G;*
3. *$M/[M, x^k]$ is finite for all $k > 0$;*
4. *M lies in the hypercenter of G.*

Proof. Let T be a torsion subgroup of G. Then G/T is torsion-free nilpotent with min-∞n (and so weak min-n). Hence G/T is nilpotent by the proof of Theorem 2 in [1]. Now since G/T is nilpotent with min-∞n, but not min, it is central infinite cyclic-by-finite by Theorem 3.1 and so it is finite-by-infinite cyclic. Hence there is a maximal normal torsion subgroup M of G containing N such that $G/M = \langle xM\rangle$ is infinite cyclic and M/N is finite. Then M is torsion with min-G and lies in the hypercenter of G since G is hypercentral by Theorem 1 in [2].

It remains to show that $M/[M, x^k]$ is finite for all $k > 0$. We note that

$$\langle x^k, [M, x^k]\rangle \lhd \langle x, M\rangle = G$$

for all $k > 0$. If $M/[M, x^k]$ is infinite, then $\langle x^k, [M, x^k]\rangle$ has infinite index in $\langle x^k, M\rangle$. Thus $\langle x^k, [M, x^k]\rangle$ has min-G. But then $\langle x^{kl}, [M, x^{kl}]\rangle$ is a G-invariant subgroup of $\langle x^k, [M, x^k]\rangle$ for each $l > 0$, which cannot be true. Therefore $M/[M, x^k]$ is finite for all $k > 0$. □

We now consider the converse of Theorem 3.2.

THEOREM 3.3. *Assume that G is a group such that $G = \langle x\rangle \ltimes M$ where M is a torsion subgroup with min-G and $|x| = \infty$. Assume also that M lies in the hypercenter of G. If $M/[M, x^k]$ is finite for all $k > 0$, then G is a locally nilpotent group which satisfies min-∞n, but not min.*

Proof. We first show that M contains every normal subgroup L of G with infinite index. Note that G is locally nilpotent. Since G/M is infinite cyclic, LM/M is either trivial or infinite cyclic. If LM/M is infinite cyclic, then $LM/M = \langle x^k M\rangle$ and so $x^k a \in L$ for some $k > 0$, $a \in M$. Now $\langle x, a\rangle$ is finitely generated nilpotent, so its torsion subgroup T is finite and $[T, x^l] = 1$ for some $l > 0$ since $G/C_G(T)$ is isomorphic with a subgroup of $\mathrm{Aut}(T)$. Hence L contains the element $(x^k a)^l = x^{kl}\overline{a}$ where $\overline{a} \in T$. If $|\overline{a}| = n$, then $x^{kln} = (x^{kl}\overline{a})^n \in L$. It follows that $[M, x^{kln}] \leqslant L$ and $|G : L| < \infty$. By this contradiction $LM/M = 1$; hence $L \leqslant M$.

Now assume that G does not satisfy min-∞n and let $G_1 > G_2 > \cdots$ be an infinite descending chain of normal subgroups of G with infinite index. Then $G_i \leqslant M$ for all i by the previous paragraph. Thus M does not satisfy min-G, a contradiction. Hence G satisfies min-∞n; it cannot satisfy min because $G/M \simeq C_\infty$. □

We note that an example of a locally nilpotent group with min-∞n, but not min is furnished by Example 3.4 in [5].

EXAMPLE 3.4. Let $M = A_1 \times A_2 \times \cdots$ be an infinite elementary abelian p-group, where $A_i = \langle a_i\rangle$ is cyclic of order p and let $X = \langle x\rangle$ be an infinite cyclic group acting on M via

$$\begin{aligned} a_1^x &= a_1; \\ a_{i+1}^x &= a_{i+1}a_i \end{aligned}$$

for all $i = 1, 2, \ldots$. Let G be the corresponding semidirect product $X \ltimes M$. Let L be an infinite normal subgroup of G with $L \leqslant M$. Suppose $L \neq M$. Then $a_i \notin L$ for some least $i > 0$, and $\langle a_1, a_2, \ldots, a_{i-1}\rangle \leqslant L$. Since L is infinite, $L \nleqslant \langle a_1, a_2, \ldots, a_{i-1}\rangle$. Hence L has an element u such that

$$u = {a_1}^{k_1}\, {a_2}^{k_2} \cdots {a_r}^{k_r}$$

with $r \geq i$ and $k_r \not\equiv 0 \pmod{p}$. Now inductive computation shows that

$$[u,\, _{r-i}x] = {a_1}^{k_{r-i+1}}\, {a_2}^{k_{r-i+2}} \cdots {a_i}^{k_r}.$$

Let $v = {a_1}^{k_{r-i+1}}\, {a_2}^{k_{r-i+2}} \cdots {a_{i-1}}^{k_{r-1}}$. Then $v \in L$ and $[u,\, _{r-i}x] = v\, {a_i}^{k_r} \in L$ since $L \lhd G$. Hence ${a_i}^{k_r} \in L$ and $a_i \in L$ since $k_r \not\equiv 0 \pmod{p}$, a contradiction. Consequently $L = M$ and so M is a minimal infinite normal subgroup of G and so it has min-G. Now argue as in the proof of Theorem 3.2 that $[M, x^k]$ is infinite, so $M = [M, x^k]$ for all $k > 0$. Thus $G = X \ltimes M$ satisfies min-∞n, but not min by Theorem 3.3.

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