ON GRAM'S DETERMINANT
IN n-INNER PRODUCT SPACES

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ABSTRACT. An analogue of the Gram's inequality for n-inner product spaces is given. Further, a number of inequalities involving Gram's determinant are stated and proved in terms of n-inner products.

1. Introduction

A concepts of n-inner products and n-inner product spaces, especially in the case $n = 2$, have been intensively studied by many authors in the last three decades. A systematic presentation of the recent results related to the theory of n-inner product spaces as well as an extensive list of the related references can be found in the book [1]. Here we give the basic definitions and the elementary properties of n-inner products.

Let X be a linear space of dimension greater than 1 over the field $K = \mathbf{R}$ of real numbers or the field $K = \mathbf{C}$ of complex numbers. Suppose that $(\cdot, \cdot | \cdot, \cdots, \cdot)$ is a K-valued function defined on $X^{n+1} = \underbrace{X \times X \times \cdots \times X}_{n+1 \text{ times}}$, $n \geq 2$, satisfying the following conditions

(nI_1) $(x, x | z_2, \cdots, z_n) \geq 0$ and $(x, x | z_2, \cdots, z_n) = 0$ if and only if the vectors $x, z_2, \cdots, z_n$ are linearly dependent,

(nI_2) $(x, x | z_2, \cdots, z_n) = (z_2, z_2 | x, \cdots, z_n)$,

(nI_3) $(x, y | z_{i_2}, \cdots, z_{i_n}) = (x, y | z_2, \cdots, z_n)$ for any permutation $(i_2, \cdots, i_n)$ of $(2, \cdots, n)$,

(nI_4) $(y, x | z_2, \cdots, z_n) = \overline{(x, y | z_2, \cdots, z_n)}$,

(nI_5) $(\alpha x, y | z_2, \cdots, z_n) = \alpha \, (x, y | z_2, \cdots, z_n)$, for any scalar $\alpha \in K$,

(nI_6) $(x + x', y | z_2, \cdots, z_n) = (x, y | z_2, \cdots, z_n) + (x', y | z_2, \cdots, z_n)$.

Here $(x, y | z_2, \cdots, z_n)$ denotes the value which the function $(\cdot, \cdot | \cdot, \cdots, \cdot)$ assigns to the $(n+1)$-tuple $(x, y, z_2, \cdots, z_n)$ of vectors from X. Also, $\overline{(x, y | z_2, \cdots, z_n)}$ denotes the complex conjugate of the scalar $(x, y | z_2, \cdots, z_n)$.

A function $(\cdot, \cdot | \cdot, \cdots, \cdot)$ is called an *n-inner product* on X and $(X, (\cdot, \cdot | \cdot, \cdots, \cdot))$ is called an *n-inner product space* (or an *n-pre-Hilbert space*). Some basic properties of the n-inner product $(\cdot, \cdot | \cdot, \cdots, \cdot)$ can be immediately obtained as follows:

(1) If $K = \mathbf{R}$, then (nI_4) reduces to

$$(y, x | z_2, \cdots, z_n) = (x, y | z_2, \cdots, z_n).$$

(2) From (nI_2) and (nI_3), it follows that

$$(x, x | z_2, \cdots, z_n) = (z_i, z_i | z_2, \cdots, z_{i-1}, x, z_{i+1}, \cdots, z_n)$$

holds for any $i \in \{2, \cdots, n\}$.

(3) From (nI_4) and (nI_5), we get

$$(0, y | z_2, \cdots, z_n) = 0, \quad (x, 0 | z_2, \cdots, z_n) = 0$$

and also

(1.1) $$(x, \alpha y | z_2, \cdots, z_n) = \overline{\alpha} \, (x, y | z_2, \cdots, z_n).$$

(4) Using (nI_2)–(nI_6), we get

$$
\begin{aligned}
& (z_2, z_2 | x \pm y, \cdots, z_n) \\
& = (x \pm y, x \pm y | z_2, \cdots, z_n) \\
& = (x, x | z_2, \cdots, z_n) + (y, y | z_2, \cdots, z_n) \pm 2\mathrm{Re}\,(x, y | z_2, \cdots, z_n)
\end{aligned}
$$

and

(1.2)
$$
\begin{aligned}
& \mathrm{Re}\,(x, y | z_2, \cdots, z_n) \\
& = \frac{1}{4} \left[(z_2, z_2 | x + y, \cdots, z_n) - (z_2, z_2 | x - y, \cdots, z_n) \right].
\end{aligned}
$$

In the real case $K = \mathbf{R}$, (1.2) reduces to

(1.3)
$$
\begin{aligned}
& (x, y | z_2, \cdots, z_n) \\
& = \frac{1}{4} \left[(z_2, z_2 | x + y, \cdots, z_n) - (z_2, z_2 | x - y, \cdots, z_n) \right]
\end{aligned}
$$

and, using this formula, it is easy to see that, for any $\alpha \in \mathbf{R}$, we have

$$(1.4) \qquad (x, y | \alpha z_2, \cdots, z_n) = \alpha^2 (x, y | z_2, \cdots, z_n).$$

In the complex case $K = \mathbf{C}$, using (1.1) and (1.2), we get

$$
\begin{aligned}
& \operatorname{Im} (x, y | z_2, \cdots, z_n) \\
& = \operatorname{Re} [-i (x, y | z_2, \cdots, z_n)] \\
& = \frac{1}{4} [(z_2, z_2 | x + iy, \cdots, z_n) - (z_2, z_2 | x - iy, \cdots, z_n)],
\end{aligned}
$$

which in combination with (1.2) yields

$$
\begin{aligned}
& (x, y | z_2, \cdots, z_n) \\
& = \frac{1}{4} [(z_2, z_2 | x + y, \cdots, z_n) - (z_2, z_2 | x - y, \cdots, z_n)] \\
& \quad + \frac{i}{4} [(z_2, z_2 | x + iy, \cdots, z_n) - (z_2, z_2 | x - iy, \cdots, z_n)].
\end{aligned}
$$

Using the above formula and (1.1), we get, for any $\alpha \in \mathbf{C}$,

$$(1.5) \qquad (x, y | \alpha z_2, \cdots, z_n) = |\alpha|^2 (x, y | z_2, \cdots, z_n).$$

However, for $\alpha \in \mathbf{R}$, (1.5) reduces to (1.4). Also, from (1.5) and (nI_3), we get

$$
\begin{aligned}
& (x, y | z_2, \cdots, z_{i-1}, \alpha z_i, z_{i+1}, \cdots, z_n) \\
& = |\alpha|^2 (x, y | z_2, \cdots, z_{i-1}, z_i, z_{i+1}, \cdots, z_n)
\end{aligned}
$$

and

$$(x, y | z_2, \cdots, z_{i-1}, 0, z_{i+1}, \cdots, z_n) = 0$$

for any $i \in \{2, \cdots, n\}$.

(5) Suppose that $x, y, z_2, \cdots, z_n \in X$ are given vectors. If $x, z_2, \cdots, z_n$ are linearly independent, then we can define the vector

$$u = y - \frac{(y, x | z_2, \cdots, z_n)}{(x, x | z_2, \cdots, z_n)} x.$$

We have $(u, u | z_2, \cdots, z_n) \geq 0$, which, by the properties (nI_4)-(nI_6), reduces to the inequality

$$(y, y | z_2, \cdots, z_n) - \frac{|(x, y | z_2, \cdots, z_n)|^2}{(x, x | z_2, \cdots, z_n)} \geq 0,$$

or, equivalently,

$$(1.6) \qquad |(x, y | z_2, \cdots, z_n)|^2 \leq (x, x | z_2, \cdots, z_n)(y, y | z_2, \cdots, z_n).$$

Moreover, the equality in (1.6) holds if and only if $u, z_2, \cdots, z_n$ are linearly dependent, that is,

$$y = \frac{(y, x | z_2, \cdots, z_n)}{(x, x | z_2, \cdots, z_n)} x + a, \quad a \in L(z_2, \cdots, z_n).$$

Similarly, if $y, z_2, \cdots, z_n$ are linearly independent, then we consider the vector

$$v = x - \frac{(x, y | z_2, \cdots, z_n)}{(y, y | z_2, \cdots, z_n)} y$$

and the inequality $(v, v | z_2, \cdots, z_n) \geq 0$, which is also equivalent to the inequality (1.6). In this case, the equality holds in (1.6) if and only if

$$x = \frac{(x, y | z_2, \cdots, z_n)}{(y, y | z_2, \cdots, z_n)} y + a, \quad a \in L(z_2, \cdots, z_n).$$

Finally, if $x, z_2, \cdots, z_n$ are linearly dependent and $y, z_2, \cdots, z_n$ are linearly dependent too, then we have

$$(x, x | z_2, \cdots, z_n) = 0 \quad \text{and} \quad (y, y | z_2, \cdots, z_n) = 0.$$

Therefore, we get

$$0 \leq (x + y, x + y | z_2, \cdots, z_n) = 2\text{Re}(x, y | z_2, \cdots, z_n),$$
$$0 \leq (x - y, x - y | z_2, \cdots, z_n) = -2\text{Re}(x, y | z_2, \cdots, z_n),$$

which implies that

$$(1.7) \qquad \qquad \text{Re}(x, y | z_2, \cdots, z_n) = 0.$$

In the real case $K = \mathbf{R}$, (1.7) reduces to $(x, y | z_2, \cdots, z_n) = 0$, which means that both sides of (1.6) are equal to zero. In the complex case $K = \mathbf{C}$, we have additionally

$$0 \leq (x + iy, x + iy | z_2, \cdots, z_n) = 2\text{Im}(x, y | z_2, \cdots, z_n),$$
$$0 \leq (x - iy, x - iy | z_2, \cdots, z_n) = -2\text{Im}(x, y | z_2, \cdots, z_n),$$

which implies that
$$\mathrm{Im}\,(x, y | z_2, \cdots, z_n) = 0.$$

This in combination with (1.7) yields $(x, y | z_2, \cdots, z_n) = 0$ and (1.6) is again true since the sides of (1.6) are equal to zero. We conclude that the inequality (1.6) is valid for any choice of vectors $x, y, z_2, \cdots, z_n \in X$. Moreover, for any vector $a \in L(z_2, \cdots, z_n)$, we have $(a, a | z_2, \cdots, z_n) = 0$ since $a, z_2, \cdots, z_n$ are linearly dependent and so, from (1.6), it follows that
$$(x, a | z_2, \cdots, z_n) = 0, \quad (a, y | z_2, \cdots, z_n) = 0$$

for any vectors $x, y \in X$ and $a \in L(z_2, \cdots, z_n)$. Also, if $z_2, \cdots, z_n$ are linearly dependent, then, from (1.6), we get

$$(x, y | z_2, \cdots, z_n) = 0$$

for any vectors $x, y \in X$. Using these facts and the above discussion on the equality cases, we easily see that the equality in (1.6) holds if and only if the vectors $x, y, z_2, \cdots, z_n$ are linearly dependent.

(6) In any given n-inner product space $(X, (\cdot, \cdot | \cdot, \cdots, \cdot))$, we can define a function $\| \cdot, \cdots, \cdot \|$ on $X^n = \underbrace{X \times \cdots \times X}_{n \text{ times}}$ as

(1.8) $$\|x_1, x_2, \cdots, x_n\| = \sqrt{(x_1, x_1 | x_2, \cdots, x_n)}$$

for any $x_1, x_2, \cdots, x_n \in X$. It is easy to see that this function satisfies the following conditions:

(nN_1) $\|x_1, x_2, \cdots, x_n\| \geq 0$ and $\|x_1, x_2, \cdots, x_n\| = 0$ if and only if the vectors $x_1, x_2, \cdots, x_n$ are linearly dependent,

(nN_2) $\|x_{i_1}, x_{i_2}, \cdots, x_{i_n}\| = \|x_1, x_2, \cdots, x_n\|$ for any permutation $(i_1, i_2, \cdots, i_n)$ of $(1, 2, \cdots, n)$,

(nN_3) $\|\alpha x_1, x_2, \cdots, x_n\| = |\alpha| \|x_1, x_2, \cdots, x_n\|$ for any scalar $\alpha \in K$,

(nN_4) $\|x'_1 + x''_1, x_2, \cdots, x_n\| \leq \|x'_1, x_2, \cdots, x_n\| + \|x''_1, x_2, \cdots, x_n\|$.

A function $\| \cdot, \cdots, \cdot \|$ defined on X^n and satisfying the conditions $(nN_1) \sim (nN_4)$ is called an *n-norm* on X and $(X, \| \cdot, \cdots, \cdot \|)$ is called an *n-normed space*. Whenever an n-inner product space $(X, (\cdot, \cdot | \cdot, \cdots, \cdot))$ is given, we consider it as an n-normed space $(X, \| \cdot, \cdots, \cdot \|)$ with the n-norm defined by (1.8).

A natural extension of the Cauchy-Schwarz-Bunjakowsky inequality

(1.9) $$|(x, y)|^2 \leq (x, x)(y, y)$$

in an inner product space $(X, (\cdot, \cdot))$ is the Gram's inequality

(1.10) $\Gamma(x_1, x_2, \cdots, x_k) \geq 0,$

which holds for any choice of vectors $x_1, x_2, \cdots, x_k \in X$ and is strict unless $x_1, x_2, \cdots, x_k$ are linearly dependent. Also, there are a number of inequalities of various types related to the Gram's determinant

$$\Gamma(x_1, x_2, \cdots, x_k) = \begin{vmatrix} (x_1, x_1) & (x_1, x_2) & \cdots & (x_1, x_k) \\ (x_2, x_1) & (x_2, x_2) & \cdots & (x_2, x_k) \\ \vdots & \vdots & \ddots & \vdots \\ (x_k, x_1) & (x_k, x_2) & \cdots & (x_k, x_k) \end{vmatrix}$$

(see, for instance, [2, pp. 381–385] or [3, Ch. XX]). The inequality (1.6) is an analogue of the Cauchy–Schwarz-Bunjakowsky inequality (1.9) for n-inner product spaces.

The aim of this paper is to give an analogue of the Gram's inequality (1.10) for n-inner product spaces as well as the analogues for n-inner product spaces of some classical inequalities involving Gram's determinant.

In Section 2, we give a definition of Gram's determinant in n-inner product spaces and then prove a version of Gram's inequality (1.10) for n-inner product spaces. Also we give a versions of Parseval's identity and of Bessel's inequality in n-inner product spaces.

In Section 3, we prove some further inequalities involving n-inner product analogue of Gram's determinant.

In Section 4, we give a version for n-inner product spaces of the well known inequality which can be regarded as a generalization via Gram's determinant of the Cauchy-Schwarz inequality for sequences (see, for instance, [3, p. 599]).

In Section 5, we give a n-inner product analogue of one well known result which can be regarded as a generalization via Gram's determinant of Bessel's inequality (see [3, pp. 396–397]). Also, we give two interesting consequences of this result (Corollary 6 and Theorem 8) which are in turn n-inner product analogues of the known classical results (see [3, pp. 603–604]).

2. Gram's inequality

Let $(X, (\cdot, \cdot | \cdot, \cdots, \cdot))$ be an n-inner product space over the field of real numbers $K = \mathbf{R}$ or the field of complex numbers $K = \mathbf{C}$. For any

given vectors $x_1, \cdots, x_k \in X$ and $z_2, \cdots, z_n \in X$, define the matrix $G(x_1, \cdots, x_k | z_2, \cdots, z_n)$ by

$$G(x_1, \cdots, x_k | z_2, \cdots, z_n)$$
$$= \begin{bmatrix} (x_1, x_1 | z_2, \cdots, z_n) & (x_1, x_2 | z_2, \cdots, z_n) & \cdots & (x_1, x_k | z_2, \cdots, z_n) \\ (x_2, x_1 | z_2, \cdots, z_n) & (x_2, x_2 | z_2, \cdots, z_n) & \cdots & (x_2, x_k | z_2, \cdots, z_n) \\ \vdots & \vdots & \ddots & \vdots \\ (x_k, x_1 | z_2, \cdots, z_n) & (x_k, x_2 | z_2, \cdots, z_n) & \cdots & (x_k, x_k | z_2, \cdots, z_n) \end{bmatrix}$$

and Gram's determinant $\Gamma(x_1, \cdots, x_k | z_2, \cdots, z_n)$ of the vectors $x_1, \cdots, x_k$ with respect to the vectors $z_2, \cdots, z_n$ as

(2.1) $\Gamma(x_1, \cdots, x_k | z_2, \cdots, z_n) = \det G(x_1, \cdots, x_k | z_2, \cdots, z_n).$

THEOREM 1. *Let $x_1, \cdots, x_k \in X$ and $z_2, \cdots, z_n \in X$ be given vectors in an n-inner product space X. Then we have*

(2.2) $\Gamma(x_1, \cdots, x_k | z_2, \cdots, z_n) \geq 0.$

Further, the equality holds in (2.2) if and only if the vectors $x_1, \cdots, x_k$, $z_2, \cdots, z_n$ are linearly dependent.

Proof. First, we consider the case of equality in (2.2). Suppose that the vectors $x_1, \cdots, x_k, z_2, \cdots, z_n$ are linearly dependent. Then we have

(2.3) $\alpha_1 x_1 + \cdots + \alpha_k x_k + \beta_2 z_2 + \cdots + \beta_n z_n = 0$

for some scalars $\alpha_1, \cdots, \alpha_k, \beta_2, \cdots, \beta_n \in K$ and at least one of them is different from zero. From (2.3), we get

$$(\alpha_1 x_1 + \cdots + \alpha_k x_k + \beta_2 z_2 + \cdots + \beta_n z_n, x_j | z_2, \cdots, z_n)$$
$$= 0, \quad j = 1, \cdots, k,$$

that is, since $(\beta_2 z_2 + \cdots + \beta_n z_n, x_j | z_2, \cdots, z_n) = 0$,

(2.4) $\alpha_1 (x_1, x_j | z_2, \cdots, z_n) + \cdots + \alpha_k (x_k x_j | z_2, \cdots, z_n)$
$$= 0, \quad j = 1, \cdots, k.$$

If $\alpha_1 = \cdots = \alpha_k = 0$, then (2.3) reduces to

$$\beta_2 z_2 + \cdots + \beta_n z_n = 0$$

with $\beta_i \neq 0$ for at least one $i \in \{2, \cdots, n\}$. This means that $z_2, \cdots, z_n$ are linearly dependent and obviously $\Gamma(x_1, x_2, \cdots, x_k | z_2, \cdots, z_n) = 0$ since, in this case, all the elements of determinant are equal to zero. Further, if $\alpha_i \neq 0$ for at least one $i \in \{1, \cdots, k\}$, then the system (2.4) has nontrivial solution $(\alpha_1, \cdots, \alpha_k)$ which means that the matrix of the system, which is equal to the transpose of the matrix $G(x_1, \cdots, x_k | z_2, \cdots, z_n)$, must be singular and hence $\Gamma(x_1, \cdots, x_k | z_2, \cdots, z_n) = 0$. Therefore, if the vectors $x_1, \cdots, x_k, z_2, \cdots, z_n$ are linearly dependent, then $\Gamma(x_1, \cdots, x_k | z_2, \cdots, z_n) = 0$. Conversely, suppose that $\Gamma(x_1, \cdots, x_k | z_2, \cdots, z_n) = 0$. Then the system (2.4) has nontrivial solution $(\alpha_1, \cdots, \alpha_k)$. But (2.4) can be rewritten as

$$(2.5) \qquad (\alpha_1 x_1 + \cdots + \alpha_k x_k, x_j | z_2, \cdots, z_n) = 0, \quad j = 1, \cdots, k.$$

Multiplying the jth equation in (2.5) by $\overline{\alpha_j}$ and then summing over $j = 1, \cdots, k$, we get

$$(\alpha_1 x_1 + \cdots + \alpha_k x_k, \alpha_1 x_1 + \cdots + \alpha_k x_k | z_2, \cdots, z_n) = 0.$$

This means that the vectors $\alpha_1 x_1 + \cdots + \alpha_k x_k, z_2, \cdots, z_n$ are linearly dependent and so there exist the scalars $\alpha, \gamma_2, \cdots, \gamma_n \in K$ such that $\alpha \neq 0$ or $\gamma_i \neq 0$ for at least one $i \in \{2, \cdots, n\}$ and

$$\alpha(\alpha_1 x_1 + \cdots + \alpha_k x_k) + \gamma_2 z_2 + \cdots + \gamma_n z_n = 0.$$

Since $\alpha_j \neq 0$ for at least one $j \in \{1, \cdots, k\}$, we conclude that the vectors $x_1, x_2, \cdots, x_k, z_2, \cdots, z_n$ are linearly dependent.

Suppose that the vectors $x_1, \cdots, x_k, z_2, \cdots, z_n$ are linearly independent. Then, for $r \in \{1, \cdots, k\}$, the vectors $x_1, \cdots, x_r, z_2, \cdots, z_n$ are linearly independent and

$$\Gamma(x_1, \cdots, x_r | z_2, \cdots, z_n) \neq 0, \quad r = 1, \cdots, k.$$

Define the vectors $y_1, \cdots, y_k$ by

$$y_1 = x_1$$

and
(2.6)

$$y_r = \begin{vmatrix} & & & x_1 \\ & G(x_1, \cdots, x_{r-1} | z_2, \cdots, z_n) & & \vdots \\ & & & x_{r-1} \\ (x_r, x_1 | z_2, \cdots, z_n) & \cdots & (x_r, x_{r-1} | z_2, \cdots, z_n) & x_r \end{vmatrix}$$

for $r = 2, \cdots, k$. Expanding the determinant in (2.6) over the last column, we get

$$(2.7) \quad y_r = \Delta_1 x_1 + \cdots + \Delta_{r-1} x_{r-1} + \Gamma\left(x_1, \cdots, x_{r-1} | z_2, \cdots, z_n\right) x_r$$

and
(2.8)
$$(y_r, x_s | z_2, \cdots, z_n)$$
$$= \Delta_1(x_1, x_s | z_2, \cdots, z_n) + \cdots + \Delta_{r-1}(x_{r-1}, x_s | z_2, \cdots, z_n)$$
$$+ \Gamma(x_1, \cdots, x_{r-1} | z_2, \cdots, z_n)(x_r, x_s | z_2, \cdots, z_n)$$

$$= \left| \begin{array}{c|c} & (x_1, x_s | z_2, \cdots, z_n) \\ G(x_1, \cdots, x_{r-1} | z_2, \cdots, z_n) & \vdots \\ & (x_{r-1}, x_s | z_2, \cdots, z_n) \\ \hline (x_r, x_1 | z_2, \cdots, z_n) \cdots (x_r, x_{r-1} | z_2, \cdots, z_n) & (x_r, x_s | z_2, \cdots, z_n) \end{array} \right|$$

for $r = 2, \cdots, k$ and $1 \le s \le r$. If $1 \le s < r$, then the determinant in (2.8) has two equal columns and hence

$$(y_r, x_s | z_2, \cdots, z_n) = 0, \quad 1 \le s < r.$$

For $s = r$, it follows from (2.8) that

$$(y_r, x_r | z_2, \cdots, z_n) = \Gamma\left(x_1, \cdots, x_r | z_2, \cdots, z_n\right).$$

Now, using the expansion (2.7) and the above equalities, we get

$$(y_r, y_r | z_2, \cdots, z_n)$$
$$= \Gamma\left(x_1, \cdots, x_{r-1} | z_2, \cdots, z_n\right) (y_r, y_r | z_2, \cdots, z_n)$$
$$= \Gamma\left(x_1, \cdots, x_{r-1} | z_2, \cdots, z_n\right) \overline{\Gamma\left(x_1, \cdots, x_r | z_2, \cdots, z_n\right)}$$
$$\neq 0$$

and hence
$$(y_r, y_r | z_2, \cdots, z_n) > 0, \quad r = 2, \cdots, k.$$

In fact, we have

$$(2.9) \quad \begin{aligned} \Gamma\left(x_1, \cdots, x_r | z_2, \cdots, z_n\right) &= \frac{\overline{(y_r, y_r | z_2, \cdots, z_n)}}{\Gamma\left(x_1, \cdots, x_{r-1} | z_2, \cdots, z_n\right)} \\ &= \frac{(y_r, y_r | z_2, \cdots, z_n)}{\Gamma\left(x_1, \cdots, x_{r-1} | z_2, \cdots, z_n\right)} \end{aligned}$$

for $r = 2, \cdots, k$. Now, it follows that

$$\Gamma(x_1|z_2, \cdots, z_n) = (x_1, x_1|z_2, \cdots, z_n) > 0$$

by the assumed independence of $x_1, z_2, \cdots, z_n$. Using this and (2.9) with $r = 2$, we get further

$$\Gamma(x_1, x_2|z_2, \cdots, z_n) = \frac{(y_2, y_2|z_2, \cdots, z_n)}{\Gamma(x_1|z_2, \cdots, z_n)} = \frac{(y_2, y_2|z_2, \cdots, z_n)}{\Gamma(x_1|z_2, \cdots, z_n)} > 0.$$

Continuing in this way, we conclude that

$$\Gamma(x_1, \cdots, x_r|z_2, \cdots, z_n) > 0, \quad r \in \{1, \cdots, k\}.$$

This completes the proof.   □

REMARK 1. The inequality (2.2) is an analogue of the Gram's inequality for n-inner product spaces. In the case when $k = 2$, (2.2) reduces to

$$(x_1, x_1|z_2, \cdots, z_n)(x_2, x_2|z_2, \cdots, z_n) - |(x_1, x_2|z_2, \cdots, z_n)|^2 \geq 0$$

with equality if and only if the vectors $x_1, x_2, z_2, \cdots, z_n$ are linearly dependent. This is just the inequality (1.6) and so Gram's inequality can be regarded as a generalization of the Cauchy-Schwarz-Bunjakowsky inequality.

Note that, in the case when the vectors $x_1, x_2, \cdots, x_k, z_2, \cdots, z_n$ are linearly independent, we can define the vectors $y_1, y_2, \cdots, y_k$ as in the proof above and, from (2.7), it follows that

$$L(y_1, y_2, \cdots, y_r) = L(x_1, x_2, \cdots, x_r), \quad r = 1, \cdots, k.$$

Moreover, from the proof above, we see that

$$(y_r, y_s|z_2, \cdots, z_n) = 0, \quad 1 \leq s < r \leq k.$$

Also, it follows that

$$\|y_1, z_2, \cdots, z_n\|^2 = (y_1, y_1|z_2, \cdots, z_n) = \Gamma(x_1|z_2, \cdots, z_n)$$

and

$$\begin{aligned} \|y_r, z_2, \cdots, z_n\|^2 &= (y_r, y_r|z_2, \cdots, z_n) \\ &= \Gamma(x_1, \cdots, x_{r-1}|z_2, \cdots, z_n)\Gamma(x_1, \cdots, x_r|z_2, \cdots, z_n) \end{aligned}$$

for $r = 2, \cdots, k$.

Now, the following result is evident:

COROLLARY 1. *Let $x_1, x_2, \cdots, x_k, z_2, \cdots, z_n \in X$ be given linearly independent vectors in n-inner product space X. Let the vectors $y_1, y_2, \cdots, y_k$ be defined as in the proof of Theorem 1. Define the vectors $e_1, e_2, \cdots, e_k$ by*

$$e_1 = \frac{y_1}{\|y_1|z_2, \cdots, z_n\|} = \frac{x_1}{\Gamma(x_1|z_2, \cdots, z_n)^{1/2}}$$

and

$$e_r = \frac{y_r}{\|y_r|z_2, \cdots, z_n\|}$$
$$= \frac{y_r}{[\Gamma(x_1, \cdots, x_{r-1}|z_2, \cdots, z_n)\,\Gamma(x_1, \cdots, x_r|z_2, \cdots, z_n)]^{1/2}}$$

for $r = 2, \cdots, k$. Then we have the following:
  (1) *For $r, s \in \{1, 2, \cdots, k\}$,*

$$(e_r, e_s|z_2, \cdots, z_n) = \delta_{rs} = \begin{cases} 0 & \text{for } r \neq s, \\ 1 & \text{for } r = s. \end{cases}$$

  (2) *For $r \in \{1, 2, \cdots, k\}$,*

$$L(e_1, e_2, \cdots, e_r) = L(x_1, x_2, \cdots, x_r).$$

REMARK 2. Note that the requirement that the vectors $x_1, x_2, \cdots, x_k, z_2, \cdots, z_n$ are linearly independent is equivalent to the requirement that these vectors satisfy the following three conditions:
  (i) $x_1, x_2, \cdots, x_k$ are linearly independent,
  (ii) $z_2, \cdots, z_n$ are linearly independent,
  (iii) $L(x_1, x_2, \cdots, x_k) \cap L(z_2, \cdots, z_n) = \{0\}$.

Now, suppose that $x_1, x_2, \cdots$ is an infinite sequence of linearly independent vectors from the space X and there exist linearly independent vectors $z_2, \cdots, z_n \in X$ such that

$$L(x_1, x_2, \cdots) \cap L(z_2, \cdots, z_n) = \{0\}.$$

Then we can construct an infinite sequence of vectors $e_1, e_2, \cdots$ such that the conclusions of Corollary 1 are valid for all $r, s \in \mathbf{N}$.

Suppose now that Y is a finite-dimensional linear subspace of a n-inner product space $(X, (\cdot, \cdot | \cdot, \cdots, \cdot))$ and $z_2, \cdots, z_n \in X$ are linearly independent vectors such that

$$Y \cap L(z_2, \cdots, z_n) = \{0\}.$$

If $\dim Y = k$, then, by the Corollary 1, we can construct the base $\{e_1, \cdots, e_k\}$ for Y such that

$$(2.10) \qquad (e_i, e_j | z_2, \cdots, z_n) = \delta_{ij}, \quad i, j \in \{1, \cdots, k\}.$$

Any vector $x \in Y$ has unique representation of the form $x = \sum_{i=1}^{k} \alpha_i e_i$. Using (2.10), we get $(x, e_j | z_2, \cdots, z_n) = \alpha_j$ for all $j = 1, \cdots, k$ and so

$$x = \sum_{i=1}^{k} (x, e_i | z_2, \cdots, z_n) e_i$$

for all $x \in Y$. Therefore, if $x, y \in Y$ are two given vectors from the subspace Y, then, using (2.10), we get

$$
\begin{aligned}
& (x, y | z_2, \cdots, z_n) \\
& = \left(\sum_{i=1}^{k} (x, e_i | z_2, \cdots, z_n) e_i, \sum_{j=1}^{k} (y, e_j | z_2, \cdots, z_n) e_j \Big| z_2, \cdots, z_n \right) \\
& = \sum_{i=1}^{k} \sum_{j=1}^{k} (x, e_i | z_2, \cdots, z_n) \overline{(y, e_j | z_2, \cdots, z_n)} \delta_{ij} \\
& = \sum_{i=1}^{k} (x, e_i | z_2, \cdots, z_n) (e_i, y | z_2, \cdots, z_n),
\end{aligned}
$$

(2.11)

which is an analogue of Parseval's identity for n-inner product spaces. Especially, for any $x \in Y$, (2.11) with $y = x$ becomes

$$\|x, z_2, \cdots, z_n\|^2 = (x, x | z_2, \cdots, z_n) = \sum_{i=1}^{k} |(x, e_i | z_2, \cdots, z_n)|^2.$$

Further, for any $x \in X$, define the vectors $u \in Y$ and $v \in X$ by

$$u = \sum_{i=1}^{k} (x, e_i | z_2, \cdots, z_n) e_i, \quad v = x - u.$$

For $j = 1, \cdots, k$, we have

$$\left(v, e_j | z_2, \cdots, z_n\right) = \left(x - \sum_{i=1}^{k} (x, e_i | z_2, \cdots, z_n) \, e_i, e_j | z_2, \cdots, z_n\right)$$

$$= (x, e_j | z_2, \cdots, z_n) - \sum_{i=1}^{k} (x, e_i | z_2, \cdots, z_n) \, \delta_{ij}$$

$$= 0,$$

which implies that $(v, y | z_2, \cdots, z_n) = 0$ for every $y \in Y$.

THEOREM 2. *Let Y be a finite-dimensional linear subspace of n-inner product space X and $z_2, \cdots, z_n \in X$ be linearly independent vectors such that*

$$(2.12) \qquad Y \cap L(z_2, \cdots, z_n) = \{0\}.$$

Then every $x \in X$ can be uniquely represented as

$$x = u + v,$$

where $u \in Y$ and $v \in X$ with

$$(v, y | z_2, \cdots, z_n) = 0, \quad y \in Y.$$

Proof. The existence of the proposed representation for $x \in X$ is already proved. It remains to prove the uniqueness. Now, suppose that

$$x = u + v = u' + v',$$

where $u, u' \in Y$ and $(v, y | z_2, \cdots, z_n) = (v', y | z_2, \cdots, z_n) = 0$ for all $y \in Y$. Then we have

$$v - v' = u' - u \in Y$$

and

$$(u' - u, u' - u | z_2, \cdots, z_n)$$
$$= (v - v', u' - u | z_2, \cdots, z_n)$$
$$= (v, u' - u | z_2, \cdots, z_n) - (v', u' - u | z_2, \cdots, z_n)$$
$$= 0.$$

This implies that $u' - u, z_2, \cdots, z_n$ are linearly dependent, which means that $u' - u \in L(z_2, \cdots, z_n)$ since $z_2, \cdots, z_n$ are linearly independent. Because of (2.12), it is possible only when $u' - u = 0$. Thus we must have $v - v' = u' - u = 0$, that is $v = v'$ and $u = u'$. This completes the proof. □

COROLLARY 2. *Let Y be a finite-dimensional linear subspace of n-inner product space X and $z_2, \cdots, z_n \in X$ be linearly independent vectors such that (2.12) holds. If $\{e_1, \cdots, e_k\}$ is the base for Y such that (2.10) holds, then, for any $x \in X$,*

$$(2.13) \qquad \sum_{i=1}^{k} |(x, e_i | z_2, \cdots, z_n)|^2 \leq \|x, z_2, \cdots, z_n\|^2.$$

The equality in (2.13) holds if and only if $x = u + v$ for some $u \in Y$ and some $v \in L(z_2, \cdots, z_n)$.

Proof. By Theorem 2, any $x \in X$ can be represented as $x = u + v$, where $u \in Y$ and $(v, y | z_2, \cdots, z_n) = 0$ for all $y \in Y$. Moreover, we have

$$x = u + v, \quad u = \sum_{i=1}^{k} (x, e_i | z_2, \cdots, z_n) e_i, \quad (v, u | z_2, \cdots, z_n) = 0,$$

which yields that

$$\begin{aligned}
\|x, z_2, \cdots, z_n\|^2 &= (u + v, u + v | z_2, \cdots, z_n) \\
&= (u, u | z_2, \cdots, z_n) + (u, v | z_2, \cdots, z_n) \\
&\quad + (v, u | z_2, \cdots, z_n) + (v, v | z_2, \cdots, z_n) \\
&= \|u, z_2, \cdots, z_n\|^2 + \|v, z_2, \cdots, z_n\|^2 \\
&\geq \|u, z_2, \cdots, z_n\|^2 \\
&= \sum_{i=1}^{k} |(x, e_i | z_2, \cdots, z_n)|^2.
\end{aligned}$$

Therefore, (2.13) is valid. Further, it is evident that we have the equality if and only if $\|v, z_2, \cdots, z_n\|^2 = 0$, which is equivalent to the requirement that $v, z_2, \cdots, z_n$ are linearly dependent, that is, $v \in L(z_2, \cdots, z_n)$ since $z_2, \cdots, z_n$ are assumed to be linearly independent. This completes the proof. $\square$

REMARK 3. The inequality (2.13) is an analogue of Bessel's inequality for n-inner product spaces. It is easy to see that it is also valid for an infinite sequence of vectors. Namely, if $e_1, e_2, \cdots$ is an infinite sequence of vectors from X and $z_2, \cdots, z_n \in X$ are linearly independent vectors such that

$$L(e_1, e_2, \cdots) \cap L(z_2, \cdots, z_n) = \{0\}$$

and

$$(e_i, e_j | z_2, \cdots, z_n) = \delta_{ij}, \quad i, j = 1, 2, \cdots,$$

then we can apply Corollary 2 to the subspace $Y = L(e_1, \cdots, e_k)$ to obtain the inequality (2.13) for any fixed $k \in \mathbf{N}$. When $k \to \infty$, we get that, for any $x \in X$,

$$\sum_{i=1}^{\infty} |(x, e_i | z_2, \cdots, z_n)|^2 \leq \|x, z_2, \cdots, z_n\|^2.$$

3. Some inequalities involving Gram's determinant

Throughout this section, we assume the notation from the previous two sections. We prove some inequalities involving the Gram's determinant in n-inner product space defined by (2.1). First, we need one technical result.

LEMMA 1. *Let Y be any linear subspace of an n-inner product space X and $z_2, \cdots, z_n \in X$ be linearly independent vectors from X. Suppose that $x \in X$ can be represented as*

$$x = u + v,$$

where $u \in Y$ and $(v, y | z_2, \cdots, z_n) = 0$ for all $y \in Y$. Then, for arbitrarily chosen vectors $x_1, \cdots, x_m \in Y$, we have the following:

$$
\begin{aligned}
(3.1) \quad & \Gamma(x, x_1, \cdots, x_m | z_2, \cdots, z_n) \\
& = \Gamma(u, x_1, \cdots, x_m | z_2, \cdots, z_n) \\
& \quad + \|v, z_2, \cdots, z_n\|^2 \Gamma(x_1, \cdots, x_m | z_2, \cdots, z_n).
\end{aligned}
$$

Especially, if $u \in L(x_1, \cdots, x_m)$, then

$$
\begin{aligned}
(3.2) \quad & \Gamma(x, x_1, \cdots, x_m | z_2, \cdots, z_n) \\
& = \|v, z_2, \cdots, z_n\|^2 \Gamma(x_1, \cdots, x_m | z_2, \cdots, z_n).
\end{aligned}
$$

Proof. Under given assumptions, we have, for all $j = 1, \cdots, m$,

$$(x_j, x | z_2, \cdots, z_n) = (x_j, u + v | z_2, \cdots, z_n) = (x_j, u | z_2, \cdots, z_n),$$
$$(x, x_j | z_2, \cdots, z_n) = (u + v, x_j | z_2, \cdots, z_n) = (u, x_j | z_2, \cdots, z_n).$$

Also, it follows that

$$
\begin{aligned}
(x, x | z_2, \cdots, z_n) &= (u + v, u + v | z_2, \cdots, z_n) \\
&= (u, u | z_2, \cdots, z_n) + (v, v | z_2, \cdots, z_n) \\
&= (u, u | z_2, \cdots, z_n) + \| v, z_2, \cdots, z_n \|^2 .
\end{aligned}
$$

Using this and the elementary properties of determinant, we get

$$
\Gamma(x, x_1, \cdots, x_m | z_2, \cdots, z_n)
$$

$$
= \begin{vmatrix}
\begin{array}{l}(u, u | z_2, \cdots, z_n) \\ + \| v, z_2, \cdots, z_n \|^2\end{array} & (u, x_1 | z_2, \cdots, z_n) & \cdots & (u, x_m | z_2, \cdots, z_n) \\
(x_1, u | z_2, \cdots, z_n) & (x_1, x_1 | z_2, \cdots, z_n) & \cdots & (x_1, x_m | z_2, \cdots, z_n) \\
\vdots & \vdots & \ddots & \vdots \\
(x_m, u | z_2, \cdots, z_n) & (x_m, x_1 | z_2, \cdots, z_n) & \cdots & (x_m, x_m | z_2, \cdots, z_n)
\end{vmatrix}
$$

$$
= \begin{vmatrix}
(u, u | z_2, \cdots, z_n) & (u, x_1 | z_2, \cdots, z_n) & \cdots & (u, x_m | z_2, \cdots, z_n) \\
(x_1, u | z_2, \cdots, z_n) & (x_1, x_1 | z_2, \cdots, z_n) & \cdots & (x_1, x_m | z_2, \cdots, z_n) \\
\vdots & \vdots & \ddots & \vdots \\
(x_m, u | z_2, \cdots, z_n) & (x_m, x_1 | z_2, \cdots, z_n) & \cdots & (x_m, x_m | z_2, \cdots, z_n)
\end{vmatrix}
$$

$$
+ \begin{vmatrix}
\| v, z_2, \cdots, z_n \|^2 & (u, x_1 | z_2, \cdots, z_n) & \cdots & (u, x_m | z_2, \cdots, z_n) \\
0 & (x_1, x_1 | z_2, \cdots, z_n) & \cdots & (x_1, x_m | z_2, \cdots, z_n) \\
\vdots & \vdots & \ddots & \vdots \\
0 & (x_m, x_1 | z_2, \cdots, z_n) & \cdots & (x_m, x_m | z_2, \cdots, z_n)
\end{vmatrix}
$$

$$
= \Gamma(u, x_1, \cdots, x_m | z_2, \cdots, z_n)
$$

$$
+ \| v, z_2, \cdots, z_n \|^2 \Gamma(x_1, \cdots, x_m | z_2, \cdots, z_n),
$$

which is just the identity (3.1). The identity (3.2) follows directly from (3.1) since $u \in L(x_1, \cdots, x_m)$ implies that $u, x_1, \cdots, x_m, z_2, \cdots, z_n$ are linearly dependent and hence $\Gamma(u, x_1, \cdots, x_m | z_2, \cdots, z_n) = 0$. This completes the proof. □

Now, we can prove some inequalities involving the Gram's determinant.

THEOREM 3. *Let* $x_1, \cdots, x_m \in X$ *be given vectors in an* n-*inner product space* X *and* $z_2, \cdots, z_n \in X$ *be linearly independent vectors from* X *such that*

$$
(3.3) \qquad L(x_1, \cdots, x_m) \cap L(z_2, \cdots, z_n) = \{0\}.
$$

Then we have the following:

$$(3.4) \quad \begin{aligned} &\Gamma\left(x_1, \cdots, x_m | z_2, \cdots, z_n\right) \\ &\leq \|x_1, z_2, \cdots, z_n\|^2 \cdots \|x_m, z_2, \cdots, z_n\|^2. \end{aligned}$$

For $m \geq 2$, the equality in (3.4) holds if and only one of the following two conditions is satisfied:

(i) *there exists at least one $j \in \{1, \cdots, m\}$ such that $x_j = 0$,*
(ii) *the vectors $x_1, \cdots, x_m$ are linearly independent and*

$$\left(x_i, x_j | z_2, \cdots, z_n\right) = 0, \quad 1 \leq i < j \leq m.$$

Proof. For $m = 1$, we have $\Gamma\left(x_1 | z_2, \cdots, z_n\right) = \|x_1, z_2, \cdots, z_n\|^2$ and (3.4) is trivially satisfied. So, take $m \geq 2$ and, first, suppose that $x_1, \cdots, x_m$ are linearly dependent. Then the left-hand side in (3.4) is equal to zero, while the right-hand side is nonnegative and so (3.4) is valid. Further, the equality occurs in (3.4) in this case if and only if $\|x_j, z_2, \cdots, z_n\|^2 = 0$ for at least one $j \in \{1, \cdots, m\}$. On the other hand, $\|x_j, z_2, \cdots, z_n\|^2 = 0$ holds if and only if $x_j, z_2, \cdots, z_n$ are linearly dependent that is if and only if $x_j \in L\left(z_2, \cdots, z_n\right)$ since $z_2, \cdots, z_n$ are assumed to be linearly independent. By (3.3), $x_j \in L\left(z_2, \cdots, z_n\right)$ is equivalent with $x_j = 0$. Next, suppose that $x_1, \cdots, x_m$ are linearly independent. Then $x_1, \cdots, x_m, z_2, \cdots, z_n$ are also linearly independent since $z_2, \cdots, z_n$ are assumed to be linearly independent and (3.3) holds. Hence $\Gamma\left(x_1, \cdots, x_m | z_2, \cdots, z_n\right) > 0$. Define $Y = L\left(x_2, \cdots, x_m\right)$. By Theorem 2, the vector x_1 can be represented as

$$x_1 = u + v, \quad u \in Y, \quad (v, y | z_2, \cdots, z_n) = 0, \quad y \in Y.$$

Applying Lemma 1, we get

$$(3.5) \quad \begin{aligned} &\Gamma\left(x_1, \cdots, x_m | z_2, \cdots, z_n\right) \\ &= \|v, z_2, \cdots, z_n\|^2 \Gamma\left(x_2, \cdots, x_m | z_2, \cdots, z_n\right). \end{aligned}$$

On the other hand, $(v, u | z_2, \cdots, z_n) = 0$ implies

$$(3.6) \quad \begin{aligned} \|x_1, z_2, \cdots, z_n\|^2 &= \|u, z_2, \cdots, z_n\|^2 + \|v, z_2, \cdots, z_n\|^2 \\ &\geq \|v, z_2, \cdots, z_n\|^2. \end{aligned}$$

Since $\Gamma(x_2, \cdots, x_m | z_2, \cdots, z_n) > 0$, from (3.5) and (3.6), it follows that

$$
\begin{aligned}
(3.7) \quad & \Gamma(x_1, \cdots, x_m | z_2, \cdots, z_n) \\
& \leq \|x_1, z_2, \cdots, z_n\|^2 \Gamma(x_2, \cdots, x_m | z_2, \cdots, z_n).
\end{aligned}
$$

Moreover, the equality in (3.7) holds if and only if $\|u, z_2, \cdots, z_n\|^2 = 0$, which is possible if and only if $u, z_2, \cdots, z_n$ are linearly dependent that is if and only if $u = 0$ since $z_2, \cdots, z_n$ are assumed to be linearly independent and (3.3) implies $Y \cap L(z_2, \cdots, z_n) = \{0\}$. Now, $u = 0$ is equivalent with $x_1 = v$, that is, $(x_1, y | z_2, \cdots, z_n) = 0$ for all $y \in Y$ or, equivalently,

$$
(x_1, x_i | z_2, \cdots, z_n) = 0, \quad i = 2, \cdots, m.
$$

Applying same observations to

$$
\Gamma(x_2, \cdots, x_m | z_2, \cdots, z_n), \cdots, \Gamma(x_{m-1}, x_m | z_2, \cdots, z_n),
$$

then we easily get the proposed conclusions. This completes the proof. $\square$

THEOREM 4. *Let* $x_1, \cdots, x_m \in X$ *be linearly independent vectors in an n-inner product space X and*

$$
Y := L(x_1, \cdots, x_m).
$$

Let $z_2, \cdots, z_n \in X$ *be linearly independent vectors such that*

$$
Y \cap L(z_2, \cdots, z_n) = \{0\}.
$$

Then, for any $x \in X$, we have the following:

$$
\begin{aligned}
\inf_{y \in Y} \|x - y, z_2, \cdots, z_n\| &= \min_{y \in Y} \|x - y, z_2, \cdots, z_n\| \\
&= \left[\frac{\Gamma(x, x_1, \cdots, x_m | z_2, \cdots, z_n)}{\Gamma(x_1, \cdots, x_m | z_2, \cdots, z_n)} \right]^{1/2}.
\end{aligned}
$$

Proof. Let $x \in X$ be given. By Theorem 2, x can be uniquely represented as

$$
x = u + v, \quad u \in Y, \quad (v, y | z_2, \cdots, z_n) = 0, \quad y \in Y.
$$

Now, if $y \in Y$ is arbitrarily chosen, then

$$x - y = u - y + v, \quad u - y \in Y, \quad (v, u - y | z_2, \cdots, z_n) = 0.$$

Therefore, it follows that

$$\|x - y, z_2, \cdots, z_n\|^2 = \|u - y, z_2, \cdots, z_n\|^2 + \|v, z_2, \cdots, z_n\|^2$$
$$\geq \|v, z_2, \cdots, z_n\|^2$$

and the equality occurs when $y = u$. We conclude that

(3.8)
$$\inf_{y \in Y} \|x - y, z_2, \cdots, z_n\|$$
$$= \min_{y \in Y} \|x - y, z_2, \cdots, z_n\| = \|v, z_2, \cdots, z_n\|.$$

On the other hand, by Lemma 1, we have

$$\Gamma(x, x_1, \cdots, x_m | z_2, \cdots, z_n)$$
$$= \|v, z_2, \cdots, z_n\|^2 \, \Gamma(x_1, \cdots, x_m | z_2, \cdots, z_n).$$

Also, $\Gamma(x_1, \cdots, x_m | z_2, \cdots, z_n) > 0$ since $x_1, \cdots, x_m, z_2, \cdots, z_n$ are linearly independent under given assumptions and so

$$\|v, z_2, \cdots, z_n\|^2 = \frac{\Gamma(x, x_1, \cdots, x_m | z_2, \cdots, z_n)}{\Gamma(x_1, \cdots, x_m | z_2, \cdots, z_n)},$$

which in combination with (3.8) proves our assertion. This completes the proof. $\qquad \square$

COROLLARY 3. *Let* $x_1, \cdots, x_m, z_2, \cdots, z_n \in X$ *be linearly independent vectors in an* n-*inner product space* X *for* $m \geq 2$. *Then we have the following:*

(3.9)
$$\frac{\Gamma(x_1, \cdots, x_m | z_2, \cdots, z_n)}{\Gamma(x_1, \cdots, x_k | z_2, \cdots, z_n)} \leq \frac{\Gamma(x_2, \cdots, x_m | z_2, \cdots, z_n)}{\Gamma(x_2, \cdots, x_k | z_2, \cdots, z_n)}$$
$$\leq \cdots$$
$$\leq \frac{\Gamma(x_k, \cdots, x_m | z_2, \cdots, z_n)}{\Gamma(x_k | z_2, \cdots, z_n)}$$
$$\leq \Gamma(x_{k+1}, \cdots, x_m | z_2, \cdots, z_n)$$

for $1 \leq k < m$. Moreover, the equality

$$\frac{\Gamma\left(x_{r-1}, \cdots, x_m | z_2, \cdots, z_n\right)}{\Gamma\left(x_{r-1}, \cdots, x_k | z_2, \cdots, z_n\right)} = \frac{\Gamma\left(x_r, \cdots, x_m | z_2, \cdots, z_n\right)}{\Gamma\left(x_r, \cdots, x_k | z_2, \cdots, z_n\right)}$$

occurs for some $r \in \{2, \cdots, k\}$ if and only if

$$x_{r-1} = u_r + v_r, \quad u_r \in L\left(x_r, \cdots, x_k\right), \quad (v_r, x_i | z_2, \cdots, z_n) = 0$$

for all $i = r, \cdots, m$. The equality

$$\frac{\Gamma\left(x_k, \cdots, x_m | z_2, \cdots, z_n\right)}{\Gamma\left(x_k | z_2, \cdots, z_n\right)} = \Gamma\left(x_{k+1}, \cdots, x_m | z_2, \cdots, z_n\right)$$

occurs if and only if $(x_k, x_i | z_2, \cdots, z_n) = 0$ for all $i = k+1, \cdots, m$.

Proof. First, take $k = 1$. Then (3.9) reduces to

$$(3.10) \qquad \frac{\Gamma\left(x_1, \cdots, x_m | z_2, \cdots, z_n\right)}{\Gamma\left(x_1 | z_2, \cdots, z_n\right)} \leq \Gamma\left(x_2, \cdots, x_m | z_2, \cdots, z_n\right),$$

which is, in fact, the inequality (3.7). Also, the equality in this inequality occurs if and only if $(x_1, x_i | z_2, \cdots, z_n) = 0$ for all $i = 2, \cdots, m$ as we proved for the inequality (3.7). Further, suppose that $1 < k < m$. Replacing $x_1, \cdots, x_m$ in (3.10) by $x_k, \cdots, x_m$, we obtain the last inequality in (3.10) and obviously the assertion on the equality case is true.

Next, for $r \in \{2, \cdots, k\}$, define the subspaces Y_r and Y_r' by

$$Y_r = L\left(x_r, \cdots, x_m\right), \quad Y_r' = L\left(x_r, \cdots, x_k\right).$$

By Theorem 2, the vector x_{r-1} can be uniquely represented in the following two forms

$$x_{r-1} = u_r + v_r, \quad u_r \in Y_r, \quad (v_r, x_i | z_2, \cdots, z_n) = 0$$

for all $i = r, \cdots, m$ and

$$x_{r-1} = u_r' + v_r', \quad u_r' \in Y_r', \quad (v_r', x_i | z_2, \cdots, z_n) = 0$$

for all $i = r, \cdots, k$. Then, applying Theorem 4, we get

$$\inf_{y \in Y_r} \|x_{r-1} - y, z_2, \cdots, z_n\|^2 = \|v_r, z_2, \cdots, z_n\|^2$$

$$= \frac{\Gamma\left(x_{r-1}, x_r, \cdots, x_m | z_2, \cdots, z_n\right)}{\Gamma\left(x_r, \cdots, x_m | z_2, \cdots, z_n\right)}$$

and

$$\inf_{y \in Y_r'} \|x_{r-1} - y, z_2, \cdots, z_n\|^2 = \|v_r', z_2, \cdots, z_n\|^2$$
$$= \frac{\Gamma(x_{r-1}, x_r, \cdots, x_k | z_2, \cdots, z_n)}{\Gamma(x_r, \cdots, x_k | z_2, \cdots, z_n)}.$$

But we have $Y_r' \subseteq Y_r$, which implies that

$$\inf_{y \in Y_r} \|x_{r-1} - y, z_2, \cdots, z_n\|^2 \leq \inf_{y \in Y_r'} \|x_{r-1} - y, z_2, \cdots, z_n\|^2,$$

that is,

$$\frac{\Gamma(x_{r-1}, x_r, \cdots, x_m | z_2, \cdots, z_n)}{\Gamma(x_r, \cdots, x_m | z_2, \cdots, z_n)} \leq \frac{\Gamma(x_{r-1}, x_r, \cdots, x_k | z_2, \cdots, z_n)}{\Gamma(x_r, \cdots, x_k | z_2, \cdots, z_n)}$$

or, equivalently,

$$(3.11) \quad \frac{\Gamma(x_{r-1}, x_r, \cdots, x_m | z_2, \cdots, z_n)}{\Gamma(x_{r-1}, x_r, \cdots, x_k | z_2, \cdots, z_n)} \leq \frac{\Gamma(x_r, \cdots, x_m | z_2, \cdots, z_n)}{\Gamma(x_r, \cdots, x_k | z_2, \cdots, z_n)}.$$

Moreover, the equality in (3.11) occurs if and only if

$$(3.12) \qquad \|v_r, z_2, \cdots, z_n\|^2 = \|v_r', z_2, \cdots, z_n\|^2.$$

Now, from $x_{r-1} = u_r + v_r = u_r' + v_r'$, it follows that

$$v_r' = u_r - u_r' + v_r, \quad u_r - u_r' \in Y_r \quad \text{and} \quad (v_r, u_r - u_r' | z_2, \cdots, z_n) = 0,$$

which implies

$$\|v_r', z_2, \cdots, z_n\|^2 = \|u_r - u_r', z_2, \cdots, z_n\|^2 + \|v_r, z_2, \cdots, z_n\|^2.$$

From the above inequality and (3.12), we get $\|u_r - u_r', z_2, \cdots, z_n\|^2 = 0$, which is possible only with $u_r - u_r' = 0$. This means that $u_r = u_r'$ and $v_r = v_r'$. In fact, (3.12) is equivalent to the requirement

$$x_{r-1} = u_r + v_r, \quad u_r = u_r' \in L(x_r, \cdots, x_k), \quad (v_r, x_i | z_2, \cdots, z_n) = 0$$

for $i = r, \cdots, m$. This completes the proof. $\qquad \square$

COROLLARY 4. *Let $x_1, \cdots, x_m \in X$ be arbitrarily chosen vectors in an n-inner product space X for $m \geq 2$ and $z_2, \cdots, z_n \in X$ be linearly independent vectors such that*

$$L(x_1, \cdots, x_m) \cap L(z_2, \cdots, z_n) = \{0\}.$$

Then we have the following:

$$
\begin{aligned}
(3.13) \quad & \Gamma(x_1, \cdots, x_k, x_{k+1}, \cdots, x_m | z_2, \cdots, z_n) \\
& \leq \Gamma(x_1, \cdots, x_k | z_2, \cdots, z_n) \Gamma(x_{k+1}, \cdots, x_m | z_2, \cdots, z_n)
\end{aligned}
$$

for $1 \leq k < m$. Moreover, the equality in (3.13) occurs if and only if one of the following three conditions is satisfied:

(i) *the vectors $x_1, \cdots, x_k$ are linearly dependent,*
(ii) *the vectors $x_{k+1}, \cdots, x_m$ are linearly dependent,*
(iii) *the vectors $x_1, \cdots, x_m$ are linearly independent and*

$$(x_i, x_j | z_2, \cdots, z_n) = 0, \quad i \in \{1, \cdots, k\}, \ j \in \{k+1, \cdots, m\}.$$

Proof. If $x_1, \cdots, x_m$ are linearly dependent, then (3.13) trivially holds since the left hand side is zero and the right hand side is nonnegative. Also, the equality in this case occurs in (3.13) if and only if the right hand side is zero which is equivalent with the requirement that either the vectors $x_1, \cdots, x_k$ are linearly dependent or the vectors $x_{k+1}, \cdots, x_m$ are linearly dependent. Further, if the vectors $x_1, \cdots, x_m$ are linearly independent, then $x_1, \cdots, x_m, z_2, \cdots, z_n$ are also linearly independent and we can apply the first and the last inequality from (3.9) to obtain the inequality

$$\frac{\Gamma(x_1, \cdots, x_m | z_2, \cdots, z_n)}{\Gamma(x_1, \cdots, x_k | z_2, \cdots, z_n)} \leq \Gamma(x_{k+1}, \cdots, x_m | z_2, \cdots, z_n),$$

which is equivalent to (3.13). Also, the equality occurs in (3.13) if and only if we have the equalities throughout in (3.9), that is,

$$
(3.14) \quad \frac{\Gamma(x_k, \cdots, x_m | z_2, \cdots, z_n)}{\Gamma(x_k | z_2, \cdots, z_n)} = \Gamma(x_{k+1}, \cdots, x_m | z_2, \cdots, z_n)
$$

and

$$
(3.15) \quad \frac{\Gamma(x_{r-1}, \cdots, x_m | z_2, \cdots, z_n)}{\Gamma(x_{r-1}, \cdots, x_k | z_2, \cdots, z_n)} = \frac{\Gamma(x_r, \cdots, x_m | z_2, \cdots, z_n)}{\Gamma(x_r, \cdots, x_k | z_2, \cdots, z_n)}
$$

for all $r \in \{2, \cdots, k\}$. Now, (3.14) is equivalent with

$$(3.16) \qquad (x_k, x_i | z_2, \cdots, z_n) = 0, \quad i = k+1, \cdots, m.$$

Next, (3.15) with $r = k$ is equivalent with

$$(3.17) \qquad \begin{aligned} & x_{k-1} = u_k + v_k, \quad u_k \in L(x_k), \quad (v_k, x_i | z_2, \cdots, z_n) = 0, \\ & i = k, \cdots, m. \end{aligned}$$

It is easy to see that (3.16) and (3.17) together are equivalent with

$$(x_i, x_j | z_2, \cdots, z_n) = 0, \quad i \in \{k-1, k\}, \ j \in \{k+1, \cdots, m\}.$$

Continuing the argument in this way for $r = k-1, \cdots, 2$, it follows that we have the equalities throughout in (3.9) if and only if

$$(x_i, x_j | z_2, \cdots, z_n) = 0, \quad i \in \{1, \cdots, k\}, \ j \in \{k+1, \cdots, m\}.$$

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

4. A generalization of Cauchy-Schwarz inequality

Let $(X, (\cdot, \cdot | \cdot, \cdots, \cdot))$ be an n-inner product space over the field of real numbers $K = \mathbf{R}$ or the field of complex numbers $K = \mathbf{C}$. For given $m \in \mathbf{N}$, consider two sequences of vectors $x_1, \cdots, x_m \in X$ and $y_1, \cdots, y_m \in X$. Then, for any given vectors $z_2, \cdots, z_n \in X$, we can define the square matrix A of order m by

$$(4.1) \qquad A = \begin{bmatrix} (x_1, y_1 | z_2, \cdots, z_n) & \cdots & (x_1, y_m | z_2, \cdots, z_n) \\ \vdots & \ddots & \vdots \\ (x_m, y_1 | z_2, \cdots, z_n) & \cdots & (x_m, y_m | z_2, \cdots, z_n) \end{bmatrix}.$$

If we define

$$Y = L(x_1, \cdots, x_m, y_1, \cdots, y_m),$$

then Y is a finite-dimensional linear subspace of X of dimension $\dim Y = k$. If $z_2, \cdots, z_n$ are linearly independent and

$$Y \cap L(z_2, \cdots, z_n) = \{0\},$$

then we can choose the base $\{e_1, \cdots, e_k\}$ for Y such that

$$(e_i, e_j | z_2, \cdots, z_n) = \delta_{ij}, \quad i, j \in \{1, \cdots, k\}.$$

Using the Parseval's identity (2.11), it is easy to see that A can be represented as

(4.2)
$$A = \begin{bmatrix} (x_1, e_1 | z_2, \cdots, z_n) & \cdots & (x_1, e_k | z_2, \cdots, z_n) \\ \vdots & \ddots & \vdots \\ (x_m, e_1 | z_2, \cdots, z_n) & \cdots & (x_m, e_k | z_2, \cdots, z_n) \end{bmatrix}$$
$$\times \begin{bmatrix} (e_1, y_1 | z_2, \cdots, z_n) & \cdots & (e_1, y_m | z_2, \cdots, z_n) \\ \vdots & \ddots & \vdots \\ (e_k, y_1 | z_2, \cdots, z_n) & \cdots & (e_k, y_m | z_2, \cdots, z_n) \end{bmatrix}.$$

If $m > k$, then obviously the vectors $x_1, \cdots, x_m$ must be linearly dependent (the same is true for the vectors $y_1, \cdots, y_m$), which implies that the rows (columns) of the matrix A are linearly dependent and hence

$$\det A = 0.$$

LEMMA 2. *If $m \le k$, then we have the identity*

(4.3) $$\det A = \sum_{1 \le j_1 < j_2 < \cdots < j_m \le k} \xi(j_1, j_2, \cdots, j_m) \, \eta(j_1, j_2, \cdots, j_m),$$

where

(4.4)
$$\xi(j_1, j_2, \cdots, j_m)$$
$$= \begin{vmatrix} (x_1, e_{j_1} | z_2, \cdots, z_n) & \cdots & (x_1, e_{j_m} | z_2, \cdots, z_n) \\ \vdots & \ddots & \vdots \\ (x_m, e_{j_1} | z_2, \cdots, z_n) & \cdots & (x_m, e_{j_m} | z_2, \cdots, z_n) \end{vmatrix}$$

and

(4.5)
$$\eta(j_1, j_2, \cdots, j_m)$$
$$= \begin{vmatrix} (e_{j_1}, y_1 | z_2, \cdots, z_n) & \cdots & (e_{j_1}, y_m | z_2, \cdots, z_n) \\ \vdots & \ddots & \vdots \\ (e_{j_m}, y_1 | z_2, \cdots, z_n) & \cdots & (e_{j_m}, y_m | z_2, \cdots, z_n) \end{vmatrix}.$$

Proof. Applying Binet-Cauchy's theorem (see, for example, [2. p. 179]), we get (4.3) directly from (4.2). □

THEOREM 5. *Let $x_1, \cdots, x_m$ be given vectors in an n-inner product space X. Set $Y = L(x_1, \cdots, x_m)$ and take any linearly independent vectors $z_2, \cdots, z_n \in X$ such that $Y \cap L(z_2, \cdots, z_n) = \{0\}$. If $k = \dim Y \geq m$ and $\{e_1, \cdots, e_k\}$ is the base for Y such that*

$$(e_i, e_j | z_2, \cdots, z_n) = \delta_{ij}, \quad i, j \in \{1, \cdots, k\},$$

then

$$(4.6) \quad \Gamma(x_1, \cdots, x_m | z_2, \cdots, z_n) = \sum_{1 \leq j_1 < j_2 < \cdots < j_m \leq k} |\xi(j_1, j_2, \cdots, j_m)|^2,$$

where $\xi(j_1, j_2, \cdots, j_m)$ is defined by (4.4).

Proof. Set $y_j = x_j$ for $j = 1, \cdots, m$. Then, for the matrix A defined by (4.1), we have

$$A = G(x_1, \cdots, x_m | z_2, \cdots, z_n), \quad \det A = \Gamma(x_1, \cdots, x_m | z_2, \cdots, z_n).$$

Also, for $\xi(j_1, j_2, \cdots, j_m)$ and $\eta(j_1, j_2, \cdots, j_m)$, respectively, given by (4.4) and (4.5), we have

$$\eta(j_1, j_2, \cdots, j_m) = \overline{\xi(j_1, j_2, \cdots, j_m)}.$$

Therefore, (4.3) reduces to (4.6) in this case. This completes the proof. □

THEOREM 6. *Let $\{x_1, \cdots, x_m\}$ and $\{y_1, \cdots, y_m\}$ be two sets of linearly independent vectors in an n-inner product space X. Set $Y = L(x_1, \cdots, x_m, y_1, \cdots, y_m)$ and take any linearly independent vectors $z_2, \cdots, z_n \in X$ such that*

$$Y \cap L(z_2, \cdots, z_n) = \{0\}.$$

If A is defined by (4.1), then

$$(4.7) \quad |\det A|^2 \leq \Gamma(x_1, \cdots, x_m | z_2, \cdots, z_n) \Gamma(y_1, \cdots, y_m | z_2, \cdots, z_n).$$

The equality occurs in (4.7) if and only if $\{x_1, \cdots, x_m\}$ spans the same subspace as $\{y_1, \cdots, y_m\}$ does, that is, if and only if $L(x_1, \cdots, x_m) = L(y_1, \cdots, y_m) = Y$.

Proof. Set $k = \dim Y$. Obviously $k \geq m$ under given assumptions. Take any base $\{e_1, \cdots, e_k\}$ for Y such that

$$(e_i, e_j | z_2, \cdots, z_n) = \delta_{ij}, \quad i, j \in \{1, \cdots, k\}.$$

Then the identity (4.3) is valid and we can apply Cauchy's inequality for sequences to obtain the inequality

$$|\det A|^2 \leq \sum_{1 \leq j_1 < j_2 < \cdots < j_m \leq k} |\xi(j_1, j_2, \cdots, j_m)|^2$$
$$\times \sum_{1 \leq j_1 < j_2 < \cdots < j_m \leq k} |\eta(j_1, j_2, \cdots, j_m)|^2.$$

By (4.6), the first sum on the right hand side of the above inequality is equal to

$$\Gamma(x_1, \cdots, x_m | z_2, \cdots, z_n),$$

while the second sum is equal to

$$\Gamma(y_1, \cdots, y_m | z_2, \cdots, z_n)$$

since, for the transpose M^τ of a square matrix M, we have $\det M^\tau = \det M$. Thus the above inequality is equivalent to (4.7). It remains the question on the equality case in (4.7).

The orthonormal base $\{e_1, \cdots, e_k\}$ for Y can always be chosen so that the first m vectors are obtained by applying the procedure of getting orthonormal vectors described in Corollary 1 to the vectors $x_1, \cdots, x_m$. It is easy to see that, in this case, we can express the vectors $x_1, \cdots, x_m$ in the form

$$x_1 = \gamma_1^{1/2} e_1,$$

$$x_r = \left(\frac{\gamma_r}{\gamma_{r-1}}\right)^{1/2} [\alpha_{r,1} e_1 + \cdots + \alpha_{r,r-1} e_{r-1} + e_r], \quad r = 2, \cdots, m,$$

where

$$\gamma_r = \Gamma(x_1, \cdots, x_r | z_2, \cdots, z_n), \quad r = 1, 2, \cdots, m.$$

Therefore, for $j = 1, \cdots, m$, we get

$$(x_1, y_j | z_2, \cdots, z_n) = \gamma_1^{1/2} (e_1, y_j | z_2, \cdots, z_n)$$

and

$$(x_r, y_j | z_2, \cdots, z_n)$$

$$= \left(\frac{\gamma_r}{\gamma_{r-1}}\right)^{1/2} \times [\alpha_{r,1} (e_1, y_j | z_2, \cdots, z_n) + \cdots$$

$$+ \alpha_{r,r-1} (e_{r-1}, y_j | z_2, \cdots, z_n) + (e_r, y_j | z_2, \cdots, z_n)]$$

for $r = 2, \cdots, m$. Using this and the elementary properties of determinant, we get

$$\det A = \Gamma(x_1, \cdots, x_m | z_2, \cdots, z_n)^{1/2} \det B,$$

where

$$B = \begin{bmatrix} (e_1, y_1 | z_2, \cdots, z_n) & (e_1, y_2 | z_2, \cdots, z_n) & \cdots & (e_1, y_m | z_2, \cdots, z_n) \\ (e_2, y_1 | z_2, \cdots, z_n) & (e_2, y_2 | z_2, \cdots, z_n) & \cdots & (e_2, y_m | z_2, \cdots, z_n) \\ \vdots & \vdots & \ddots & \vdots \\ (e_m, y_1 | z_2, \cdots, z_n) & (e_m, y_2 | z_2, \cdots, z_n) & \cdots & (e_m, y_m | z_2, \cdots, z_n) \end{bmatrix}.$$

This means that

$$|\det A|^2 = \Gamma(x_1, \cdots, x_m | z_2, \cdots, z_n) |\det B|^2.$$

Note that actually we have

$$\det B = \eta(1, 2, \cdots, m),$$

where $\eta(j_1, j_2, \cdots, j_m)$ for $1 \leq j_1 < j_2 < \cdots < j_m \leq k$ is given by (4.5). Now, the equality in (4.7) is equivalent with the requirement that

$$(4.8) \qquad \Gamma(y_1, \cdots, y_m | z_2, \cdots, z_n) = |\det B|^2 = |\eta(1, 2, \cdots, m)|^2.$$

On the other hand, by Theorem 5, we have

$$(4.9) \ \Gamma(y_1, \cdots, y_m | z_2, \cdots, z_n) = \sum_{1 \leq j_1 < j_2 < \cdots < j_m \leq k} |\eta(j_1, j_2, \cdots, j_m)|^2.$$

From the equalities (4.8) and (4.9), it follows that the equality in (4.7) holds if and only if

$$(4.10) \qquad \eta(j_1, j_2, \cdots, j_m) = 0, \quad (j_1, j_2, \cdots, j_m) \neq (1, 2, \cdots, m).$$

Further, for $i = m + 1, \cdots, k$, consider the vectors

$$
v_i = \begin{vmatrix}
(e_1, y_1 | z_2, \cdots, z_n) & \cdots & (e_1, y_m | z_2, \cdots, z_n) & e_1 \\
\vdots & \ddots & \vdots & \vdots \\
(e_m, y_1 | z_2, \cdots, z_n) & \cdots & (e_m, y_m | z_2, \cdots, z_n) & e_m \\
\hline
(e_i, y_1 | z_2, \cdots, z_n) & \cdots & (e_i, y_m | z_2, \cdots, z_n) & e_i
\end{vmatrix}.
$$

Expanding the above determinant over the last column and using (4.10), we get

$$
v_i = \eta(1, 2, \cdots, m)\, e_i, \quad i = m + 1, \cdots, k.
$$

On the other hand, we have, for all $j = 1, 2, \cdots, m$ and all $i = m + 1, \cdots, k$,

$$
(v_i, y_j | z_2, \cdots, z_n)
$$

$$
= \begin{vmatrix}
(e_1, y_1 | z_2, \cdots, z_n) & \cdots & (e_1, y_m | z_2, \cdots, z_n) & (e_1, y_j | z_2, \cdots, z_n) \\
\vdots & \ddots & \vdots & \vdots \\
(e_m, y_1 | z_2, \cdots, z_n) & \cdots & (e_m, y_m | z_2, \cdots, z_n) & (e_m, y_j | z_2, \cdots, z_n) \\
\hline
(e_i, y_1 | z_2, \cdots, z_n) & \cdots & (e_i, y_m | z_2, \cdots, z_n) & (e_i, y_j | z_2, \cdots, z_n)
\end{vmatrix}
$$

$$
= 0
$$

since two columns in this determinant are identical. This implies that

$$
(e_i, y_j | z_2, \cdots, z_n) = \frac{(v_i, y_j | z_2, \cdots, z_n)}{\eta(1, 2, \cdots, m)} = 0
$$

for all $i = m + 1, \cdots, k$ and $j = 1, 2, \cdots, m$. Using this and the fact that any $y \in Y$ is uniquely represented as $y = \sum_{i=1}^{k} (y, e_i | z_2, \cdots, z_n)\, e_i$, we see that, for all $j = 1, 2, \cdots, m$,

$$
y_j = \sum_{i=1}^{m} (y_j, e_i | z_2, \cdots, z_n)\, e_i \in L(e_1, \cdots, e_m) = L(x_1, \cdots, x_m).
$$

This means that $y_1, \cdots, y_m$ span the same subspace as the one spanned by $x_1, \cdots, x_m$ since $y_1, \cdots, y_m$ are linearly independent. Therefore, (4.10) is equivalent to the requirement that $L(x_1, \cdots, x_m) = L(y_1, \cdots, y_m)$. This completes the proof. $\qquad\square$

COROLLARY 5. *Let* $x_1, \cdots, x_m$ *and* y_1 *be given vectors in an* n-*inner product space* X. *Suppose that* $x_1, \cdots, x_m$ *are linearly independent and take any linearly independent vectors* $z_2, \cdots, z_n \in X$ *such that*

$$L(x_1, \cdots, x_m, y_1) \cap L(z_2, \cdots, z_n) = \{0\}.$$

Then we have the following:

(4.11)
$$\Gamma(x_1 + y_1, x_2, \cdots, x_m | z_2, \cdots, z_n)^{1/2}$$
$$\leq \Gamma(x_1, x_2, \cdots, x_m | z_2, \cdots, z_n)^{1/2}$$
$$+ \Gamma(y_1, x_2, \cdots, x_m | z_2, \cdots, z_n)^{1/2}.$$

The equality occurs in (4.11) *if and only if*

(4.12) $$y_1 = \lambda x_1 + u, \quad \lambda \geq 0, \quad u \in L(x_2, \cdots, x_m).$$

Proof. Using the elementary properties of determinant, we easily get the following identity

$$\Gamma(x_1 + y_1, x_2, \cdots, x_m | z_2, \cdots, z_n)$$
$$= \Gamma(x_1, x_2, \cdots, x_m | z_2, \cdots, z_n) + \det A$$
$$+ \overline{\det A} + \Gamma(y_1, x_2, \cdots, x_m | z_2, \cdots, z_n),$$

where

$$A = \begin{bmatrix} (x_1, y_1 | z_2, \cdots, z_n) & (x_1, x_2 | z_2, \cdots, z_n) & \cdots & (x_1, x_m | z_2, \cdots, z_n) \\ (x_2, y_1 | z_2, \cdots, z_n) & (x_2, x_2 | z_2, \cdots, z_n) & \cdots & (x_2, x_m | z_2, \cdots, z_n) \\ \vdots & \vdots & \ddots & \vdots \\ (x_m, y_1 | z_2, \cdots, z_n) & (x_m, x_2 | z_2, \cdots, z_n) & \cdots & (x_m, x_m | z_2, \cdots, z_n) \end{bmatrix}.$$

Applying Theorem 6 to the sets of vectors $\{x_1, x_2, \cdots, x_m\}$ and $\{y_1, x_2, \cdots, x_m\}$, we get

$$|\det A|$$
$$\leq \Gamma(x_1, x_2, \cdots, x_m | z_2, \cdots, z_n)^{1/2} \Gamma(y_1, x_2, \cdots, x_m | z_2, \cdots, z_n)^{1/2}.$$

Therefore, we have

$$\Gamma\left(x_1+y_1,x_2,\cdots,x_m|z_2,\cdots,z_n\right)$$

$$= \Gamma\left(x_1,x_2,\cdots,x_m|z_2,\cdots,z_n\right) + \det A + \overline{\det A}$$

$$+ \Gamma\left(y_1,x_2,\cdots,x_m|z_2,\cdots,z_n\right)$$

$$= \Gamma\left(x_1,x_2,\cdots,x_m|z_2,\cdots,z_n\right) + 2\operatorname{Re}\left[\det A\right]$$

$$+ \Gamma\left(y_1,x_2,\cdots,x_m|z_2,\cdots,z_n\right)$$

$$\leq \Gamma\left(x_1,x_2,\cdots,x_m|z_2,\cdots,z_n\right) + 2\left|\det A\right|$$

$$+ \Gamma\left(y_1,x_2,\cdots,x_m|z_2,\cdots,z_n\right)$$

$$\leq \Gamma\left(x_1,x_2,\cdots,x_m|z_2,\cdots,z_n\right)$$

$$+ \Gamma\left(x_1,x_2,\cdots,x_m|z_2,\cdots,z_n\right)^{1/2}\Gamma\left(y_1,x_2,\cdots,x_m|z_2,\cdots,z_n\right)^{1/2}$$

$$+ \Gamma\left(y_1,x_2,\cdots,x_m|z_2,\cdots,z_n\right)$$

$$= \left[\Gamma\left(x_1,x_2,\cdots,x_m|z_2,\cdots,z_n\right)^{1/2}\right.$$

$$\left.+ \Gamma\left(y_1,x_2,\cdots,x_m|z_2,\cdots,z_n\right)^{1/2}\right]^2,$$

which yields (4.11). Obviously, we have equality in (4.11) if and only if

$$(4.13) \qquad \begin{aligned} \operatorname{Re}\left[\det A\right] &= \left|\det A\right| \\ &= \Gamma\left(x_1,x_2,\cdots,x_m|z_2,\cdots,z_n\right)^{1/2} \\ &\times \Gamma\left(y_1,x_2,\cdots,x_m|z_2,\cdots,z_n\right)^{1/2}. \end{aligned}$$

The first equality in (4.13) is equivalent with $\det A \geq 0$, while the second one holds if and only if $y_1, x_2, \cdots, x_m$ are linearly dependent or $L\left(y_1,x_2,\cdots,x_m\right) = L\left(x_1,x_2,\cdots,x_m\right)$. In the case when $y_1, x_2, \cdots, x_m$ are linearly dependent, we have $y_1 = u \in L\left(x_2,\cdots,x_m\right)$ and $\det A = 0$, while in the case when

$$L\left(y_1,x_2,\cdots,x_m\right) = L\left(x_1,x_2,\cdots,x_m\right),$$

we have $y_1 = \lambda x_1 + u$ for some $\lambda \neq 0$ and some $u \in L\left(x_2,\cdots,x_m\right)$. In this case, we get $\det A = \overline{\lambda}\Gamma\left(x_1,x_2,\cdots,x_m|z_2,\cdots,z_n\right)$ and so the condition $\det A \geq 0$ is equivalent with the condition $\lambda \geq 0$. This proves that the equality occurs in (4.11) if and only if (4.12) holds. This completes the proof.   □

5. A generalization of Bessel's inequality

Let $(X, (\cdot, \cdot | \cdot, \cdots, \cdot))$ be an n-inner product space over the field of real numbers $K = \mathbf{R}$ or the field of complex numbers $K = \mathbf{C}$. In this section, we give a generalization of Bessel's inequality

$$(5.1) \qquad \sum_{i=1}^{k} |(x, e_i | z_2, \cdots, z_n)|^2 \leq \|x, z_2, \cdots, z_n\|^2 \qquad .$$

which holds for any $x \in X$ whenever $e_1, \cdots, e_k, z_2, \cdots, z_n \in X$ are linearly independent vectors such that

$$(e_i, e_j | z_2, \cdots, z_n) = \delta_{ij}, \quad i, j \in \{1, \cdots, k\}.$$

Also, we know that the equality occurs in (5.1) if and only if $x = u + v$ for some $u \in L(e_1, \cdots, e_k)$ and $v \in L(z_2, \cdots, z_n)$.

THEOREM 7. *Let X be an n-inner product space and $x_1, \cdots, x_m, z_2,$ $\cdots, z_n \in X$ be linearly independent vectors from X. For any given vector $x \in X$, define*

$$\lambda_i = (x, x_i | z_2, \cdots, z_n), \quad i = 1, \cdots, m.$$

If $\Delta = \Gamma(x_1, \cdots, x_m | z_2, \cdots, z_n)$ and Δ_i is equal to the determinant obtained from Δ by replacing the ith row of $\Gamma(x_1, \cdots, x_m | z_2, \cdots, z_n)$ with $(\lambda_1, \cdots, \lambda_m)$ for $i = 1, \cdots, m$, then we have the following:

$$(5.2) \qquad \left\| \sum_{i=1}^{m} \Delta_i x_i, z_2, \cdots, z_n \right\| \leq \Delta \|x, z_2, \cdots, z_n\|.$$

The equality in (5.2) occurs if and only if there exists a vector $v \in L(z_2, \cdots, z_n)$ such that

$$x = \frac{1}{\Delta} \sum_{i=1}^{m} \Delta_i x_i + v.$$

Proof. Note that $\Delta > 0$ and consider the vector $y \in X$ defined by

$$y = \frac{1}{\Delta} \sum_{i=1}^{m} \delta_i x_i, \quad \delta_i \in K, \ i = 1, \cdots, m.$$

Then we have

$$(y, x | z_2, \cdots, z_n) = \frac{1}{\Delta} \sum_{i=1}^{m} \delta_i \overline{(x, x_i | z_2, \cdots, z_n)} = \frac{1}{\Delta} \sum_{i=1}^{m} \delta_i \overline{\lambda_i}$$

and

$$(y, y | z_2, \cdots, z_n) = \frac{1}{\Delta^2} \sum_{i=1}^{m} \sum_{j=1}^{m} \delta_i \overline{\delta_j} \, (x_i, x_i | z_2, \cdots, z_n) \, .$$

The requirement that $(y, x | z_2, \cdots, z_n) = (y, y | z_2, \cdots, z_n)$ is therefore equivalent with

$$(5.3) \qquad \sum_{i=1}^{m} \delta_i \overline{\lambda_i} = \frac{1}{\Delta} \sum_{i=1}^{m} \sum_{j=1}^{m} \delta_i \overline{\delta_j} \, (x_i, x_i | z_2, \cdots, z_n) \, .$$

Obviously, (5.3) will be satisfied if $\delta_1, \cdots, \delta_m$ are chosen so that

$$\overline{\lambda_i} = \frac{1}{\Delta} \sum_{j=1}^{m} \overline{\delta_j} \, (x_i, x_i | z_2, \cdots, z_n) \, , \quad i = 1, \cdots, m,$$

that is,

$$(5.4) \qquad \sum_{j=1}^{m} (x_j, x_i | z_2, \cdots, z_n) \, \delta_j = \Delta \lambda_i, \quad i = 1, \cdots, m.$$

The matrix of the above system of linear equations has determinant equal to $\Gamma(x_1, \cdots, x_m | z_2, \cdots, z_n) = \Delta > 0$. Therefore, the system (5.4) has unique solution given as

$$\delta_j = \frac{1}{\Delta} \begin{vmatrix} (x_1, x_1 | z_2, \cdots, z_n) & \cdots & \Delta \lambda_1 & \cdots & (x_m, x_1 | z_2, \cdots, z_n) \\ \vdots & & \vdots & & \vdots \\ (x_1, x_m | z_2, \cdots, z_n) & \cdots & \underbrace{\Delta \lambda_m}_{j^{th} \text{column}} & \cdots & (x_m, x_m | z_2, \cdots, z_n) \end{vmatrix}$$

$$= \frac{1}{\Delta} \Delta \Delta_j$$
$$= \Delta_j, \quad j = 1, \cdots, m.$$

We conclude that, for the vector $y \in X$ defined by

$$y = \frac{1}{\Delta} \sum_{i=1}^{m} \Delta_i x_i,$$

$$(y, x | z_2, \cdots, z_n) = (y, y | z_2, \cdots, z_n)$$

and

$$(x, y | z_2, \cdots, z_n) = \overline{(y, x | z_2, \cdots, z_n)} = (y, y | z_2, \cdots, z_n).$$

Thus, using this, we get

$$
\begin{aligned}
0 \leq \; & \| x - y, z_2, \cdots, z_n \|^2 \\
= \; & (x - y, x - y | z_2, \cdots, z_n) \\
= \; & \| x, z_2, \cdots, z_n \|^2 - 2 (y, y | z_2, \cdots, z_n) + \| y, z_2, \cdots, z_n \|^2 \\
= \; & \| x, z_2, \cdots, z_n \|^2 - \| y, z_2, \cdots, z_n \|^2
\end{aligned}
$$

or, equivalently,

$$\| y, z_2, \cdots, z_n \| = \frac{1}{\Delta} \left\| \sum_{i=1}^m \Delta_i x_i, z_2, \cdots, z_n \right\| \leq \| x, z_2, \cdots, z_n \|,$$

which is equivalent to (5.2). Moreover, the equality occurs in the above inequality and hence in (5.2) if and only if $x - y, z_2, \cdots, z_n$ are linearly dependent, that is,

$$x - y = v \in L(z_2, \cdots, z_n)$$

since $z_2, \cdots, z_n$ are linearly independent. This completes the proof. $\square$

REMARK 4. (1) The inequality (5.2) can be regarded as a generalization of the Cauchy-Schwarz-Bunjakowsky inequality. Namely, for $m = 1$, we have

$$\Delta = (x_1, x_1 | z_2, \cdots, z_n), \; \Delta_1 = \lambda_1 = (x, x_1 | z_2, \cdots, z_n)$$

and (5.2) reduces to

$$
\begin{aligned}
& |(x, x_1 | z_2, \cdots, z_n)| \sqrt{(x_1, x_1 | z_2, \cdots, z_n)} \\
& \leq (x_1, x_1 | z_2, \cdots, z_n) \sqrt{(x, x | z_2, \cdots, z_n)}
\end{aligned}
$$

or, equivalently,

$$|(x, x_1 | z_2, \cdots, z_n)|^2 \leq (x_1, x_1 | z_2, \cdots, z_n)(x, x | z_2, \cdots, z_n).$$

The equality occurs if and only if

$$x = \frac{(x, x_1 | z_2, \cdots, z_n)}{(x_1, x_1 | z_2, \cdots, z_n)} x_1 + v$$

for some $v \in L(z_2, \cdots, z_n)$. This is just the Cauchy-Schwarz-Bunja-kowsky inequality stated for the vectors $x, x_1, z_2, \cdots, z_n \in X$ such that $x_1, z_2, \cdots, z_n$ are linearly independent.

(2) Suppose that $x_1, \cdots, x_m$ satisfy additional condition

$$(x_i, x_j | z_2, \cdots, z_n) = \delta_{ij}, \quad i, j \in \{1, \cdots, m\}.$$

In this case, we have

$$\Delta = 1 \quad \text{and} \quad \Delta_i = \lambda_i = (x, x_i | z_2, \cdots, z_n), \quad i = 1, \cdots, m,$$

and so (5.2) becomes

$$\left\| \sum_{i=1}^{m} (x, x_i | z_2, \cdots, z_n) x_i, z_2, \cdots, z_n \right\| \leq \| x, z_2, \cdots, z_n \|$$

or, equivalently,

$$\sum_{i=1}^{m} |(x, x_i | z_2, \cdots, z_n)|^2 \leq \| x, z_2, \cdots, z_n \|^2,$$

which is just the Bessel's inequality (5.1) where $k = m$ and $e_1, \cdots, e_k$ are replaced with $x_1, \cdots, x_m$. The equality holds in this case if and only if

$$x = \sum_{i=1}^{m} (x, x_i | z_2, \cdots, z_n) x_i + v$$

for some $v \in L(z_2, \cdots, z_n)$.

COROLLARY 6. *Let X be an n-inner product space and $a, b, z_2, \cdots, z_n \in X$ be the vectors such that $a, b, z_2, \cdots, z_n$ are linearly independent. For any given vector $x \in X$, define*

$$\mu = (x, a | z_2, \cdots, z_n), \quad \nu = (x, b | z_2, \cdots, z_n).$$

Then we have the following:

(5.5) $\Gamma(a, b | z_2, \cdots, z_n) \| x, z_2, \cdots, z_n \|^2 \geq \| \overline{\nu} a - \overline{\mu} b, z_2, \cdots, z_n \|^2.$

The equality in (5.5) occurs if and only if

$$x = \frac{(a, \overline{\nu} a - \overline{\mu} b | z_2, \cdots, z_n) b - (b, \overline{\nu} a - \overline{\mu} b | z_2, \cdots, z_n) a}{\Gamma(a, b | z_2, \cdots, z_n)} + v$$

for some vector $v \in L(z_2, \cdots, z_n)$.

Proof. We apply Theorem 7 with $m = 2$, $x_1 = a$ and $x_2 = b$. Then we have

$$\lambda_1 = \mu, \quad \lambda_2 = \nu, \quad \Delta = \Gamma\left(a, b | z_2, \cdots, z_n\right)$$

and

$$
\begin{aligned}
\Delta_1 &= \left| \begin{array}{cc} \mu & \nu \\ (b, a | z_2, \cdots, z_n) & (b, b | z_2, \cdots, z_n) \end{array} \right| \\
&= \mu\left(b, b | z_2, \cdots, z_n\right) - \nu\left(b, a | z_2, \cdots, z_n\right) \\
&= -\left(b, \overline{\nu} a - \overline{\mu} b | z_2, \cdots, z_n\right), \\
\Delta_2 &= \left| \begin{array}{cc} (a, a | z_2, \cdots, z_n) & (a, b | z_2, \cdots, z_n) \\ \mu & \nu \end{array} \right| \\
&= \nu\left(a, a | z_2, \cdots, z_n\right) - \mu\left(a, b | z_2, \cdots, z_n\right) \\
&= \left(a, \overline{\nu} a - \overline{\mu} b | z_2, \cdots, z_n\right).
\end{aligned}
$$

Consider the vector

$$\tilde{y} = \Delta_1 a + \Delta_2 b = \left(a, \overline{\nu} a - \overline{\mu} b | z_2, \cdots, z_n\right) b - \left(b, \overline{\nu} a - \overline{\mu} b | z_2, \cdots, z_n\right) a.$$

We know from (5.2) that

$$\|\tilde{y}, z_2, \cdots, z_n\| \le \Delta \|x, z_2, \cdots, z_n\| = \Gamma\left(a, b | z_2, \cdots, z_n\right) \|x, z_2, \cdots, z_n\|$$

or, equivalently,

$$\left(\tilde{y}, \tilde{y} | z_2, \cdots, z_n\right) \le \Gamma\left(a, b | z_2, \cdots, z_n\right)^2 \left(x, x | z_2, \cdots, z_n\right).$$

This is a consequence of the equality $(y, y | z_2, \cdots, z_n) = (y, x | z_2, \cdots, z_n)$ which holds for the vector

$$y = \frac{\tilde{y}}{\Delta} = \frac{\tilde{y}}{\Gamma\left(a, b | z_2, \cdots, z_n\right)}.$$

Note that the equality $(y, y | z_2, \cdots, z_n) = (y, x | z_2, \cdots, z_n)$ is equivalent with

$$\frac{\left(\tilde{y}, \tilde{y} | z_2, \cdots, z_n\right)}{\Gamma\left(a, b | z_2, \cdots, z_n\right)} = \left(\tilde{y}, x | z_2, \cdots, z_n\right).$$

Therefore, we have

$$
\Gamma\left(a, b | z_2, \cdots, z_n\right) \left\| x, z_2, \cdots, z_n \right\|^2
$$

$$
\geq \frac{\left(\tilde{y}, \tilde{y} | z_2, \cdots, z_n\right)}{\Gamma\left(a, b | z_2, \cdots, z_n\right)}
$$

$$
= \left(\tilde{y}, x | z_2, \cdots, z_n\right)
$$

$$
= \left(\left(a, \overline{\nu}a - \overline{\mu}b | z_2, \cdots, z_n\right) b - \left(b, \overline{\nu}a - \overline{\mu}b | z_2, \cdots, z_n\right) a, x | z_2, \cdots, z_n\right)
$$

$$
= \left(a, \overline{\nu}a - \overline{\mu}b | z_2, \cdots, z_n\right) \overline{\nu} - \left(b, \overline{\nu}a - \overline{\mu}b | z_2, \cdots, z_n\right) \overline{\mu}
$$

$$
= \left(\overline{\nu}a - \overline{\mu}b, \overline{\nu}a - \overline{\mu}b | z_2, \cdots, z_n\right)
$$

$$
= \left\| \overline{\nu}a - \overline{\mu}b, z_2, \cdots, z_n \right\|^2,
$$

which is just the inequality (5.5). Also, we know from Theorem 7 that the equality occurs if and only if

$$
x = y + v = \frac{\tilde{y}}{\Gamma\left(a, b | z_2, \cdots, z_n\right)} + v, \quad v \in L\left(z_2, \cdots, z_n\right).
$$

This completes the proof. $\square$

THEOREM 8. *Let X be an n-inner product space over the field $K = \mathbf{R}$ of real numbers. Let $a, b, z_2, \cdots, z_n \in X$ and $e_1, \cdots, e_m \in X$ be the vectors from X such that $a, b, z_2, \cdots, z_n$ are linearly independent and, for all $i, j \in \{1, \cdots, m\}$,*

$$
\left(e_i, e_j | z_2, \cdots, z_n\right) = \delta_{ij}
$$

and

$$
(5.6) \qquad \begin{aligned} &\left(a, e_j | z_2, \cdots, z_n\right)\left(b, e_i | z_2, \cdots, z_n\right) \\ &\neq \left(a, e_i | z_2, \cdots, z_n\right)\left(b, e_j | z_2, \cdots, z_n\right) \end{aligned}
$$

for $i \neq j$. If $p_{ij} \in \mathbf{R}$ for $i, j \in \{1, \cdots, m\}$ with $i \neq j$ are given real numbers which satisfy the conditions

$$
(5.7) \qquad p_{ij} = p_{ji}, \quad i \neq j, \ i, j \in \{1, \cdots, m\},
$$

and

$$
P = \sum_{1 \leq i < j \leq m} p_{ij} \neq 0,
$$

then, for any two scalars $\mu, \nu \in \mathbf{R}$,

$$P^2 \frac{\|\nu a - \mu b, z_2, \cdots, z_n\|^2}{\Gamma(a, b | z_2, \cdots, z_n)} \leq \sum_{i=1}^{m} \left| \sum_{i \neq j=1}^{m} \frac{p_{ij}(\nu a - \mu b, e_j | z_2, \cdots, z_n)}{\gamma_{ij}} \right|^2,$$

where γ_{ij} are defined by

$$\begin{aligned}
\gamma_{ij} = \; & (a, e_j | z_2, \cdots, z_n)(b, e_i | z_2, \cdots, z_n) \\
& - (a, e_i | z_2, \cdots, z_n)(b, e_j | z_2, \cdots, z_n).
\end{aligned}$$

Proof. For any two scalars $\mu, \nu \in \mathbf{R}$, consider the vector

$$\tilde{x} = \sum_{i=1}^{m} \sum_{i \neq j=1}^{m} \frac{p_{ij}(\nu a - \mu b, e_j | z_2, \cdots, z_n)}{\gamma_{ij}} e_i.$$

Using the properties of n-inner product $(\cdot, \cdot | \cdot, \cdots, \cdot)$, in the real case, we get

$$\begin{aligned}
(\tilde{x}, a | z_2, \cdots, z_n) = \; & \nu \sum_{i=1}^{m} \sum_{i \neq j=1}^{m} \frac{p_{ij}(a, e_j | z_2, \cdots, z_n)(a, e_i | z_2, \cdots, z_n)}{\gamma_{ij}} \\
& - \mu \sum_{i=1}^{m} \sum_{i \neq j=1}^{m} \frac{p_{ij}(b, e_j | z_2, \cdots, z_n)(a, e_i | z_2, \cdots, z_n)}{\gamma_{ij}}.
\end{aligned}$$

On the other hand, using the condition (5.7) and obvious fact that

$$\gamma_{ij} = -\gamma_{ji}, \quad i \neq j, \; i, j \in \{1, \cdots, m\},$$

we easily see that

$$\sum_{i=1}^{m} \sum_{i \neq j=1}^{m} \frac{p_{ij}(a, e_j | z_2, \cdots, z_n)(a, e_i | z_2, \cdots, z_n)}{\gamma_{ij}} = 0$$

and

$$\sum_{i=1}^{m} \sum_{i \neq j=1}^{m} \frac{p_{ij}(b, e_j | z_2, \cdots, z_n)(a, e_i | z_2, \cdots, z_n)}{\gamma_{ij}} = - \sum_{1 \leq i < j \leq m} p_{ij} = -P.$$

Therefore, we have

$$(\tilde{x}, a | z_2, \cdots, z_n) = \mu P.$$

Similarly, we get

$$(\tilde{x}, b | z_2, \cdots, z_n) = \nu P.$$

Therefore, we can apply Corollary 6 to the vector $x = \tilde{x}/P$ and so, by (5.5), we have

$$\frac{\Gamma(a, b | z_2, \cdots, z_n) \| \tilde{x}, z_2, \cdots, z_n \|^2}{P^2}$$

$$= \Gamma(a, b | z_2, \cdots, z_n) \left\| \frac{\tilde{x}}{P}, z_2, \cdots, z_n \right\|^2$$

$$\geq \| \overline{\nu} a - \overline{\mu} b, z_2, \cdots, z_n \|^2$$

or, equivalently,

$$P^2 \frac{\| \nu a - \mu b, z_2, \cdots, z_n \|^2}{\Gamma(a, b | z_2, \cdots, z_n)} \leq \| \tilde{x}, z_2, \cdots, z_n \|^2.$$

Now, applying Parseval's identity to the term $\| \tilde{x}, z_2, \cdots, z_n \|^2$, we have the conclusion. This completes the proof.   □

COROLLARY 7. *Under assumptions of Theorem 8, we have the following:*

$$\binom{m}{2}^2 \frac{\| \nu a - \mu b | z_2, \cdots, z_n \|^2}{\Gamma(a, b | z_2, \cdots, z_n)} \leq \sum_{i=1}^{m} \left| \sum_{i \neq j=1}^{m} \frac{(\nu a - \mu b, e_j | z_2, \cdots, z_n)}{\gamma_{ij}} \right|^2$$

for any two scalars $\mu, \nu \in \mathbf{R}$.

Proof. Set $p_{ij} = 1$ for all $i \neq j$, $i, j \in \{1, \cdots, m\}$ and note that

$$P = \sum_{1 \leq i < j \leq m} p_{ij} = \binom{m}{2}.$$

Then, applying Theorem 8, we have the conclusion.   □

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