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CONFORMAL CHANGE OF THE TENSOR $U^\nu{}_{\omega\mu}$ IN 7-DIMENSIONAL g-UFT

Chung Hyun Cho

Abstract. We investigate change of the tensor $U^\nu{}_{\omega\mu}$ induced by the conformal change in 7-dimensional g-unified field theory. These topics will be studied for the second class with the first category in 7-dimensional case.

1. Introduction

The conformal change in a generalized 4-dimensional Riemannian space connected by an Einstein's connection was primarily studied by Hlavatý [9]. Chung [4] also investigated the same topic in 4-dimensional $^*g$-unified field theory.

The Einstein's connection induced by the conformal change for the second class with the first category of the torsion tensor $S_{\omega\mu}{}^\nu$ in 7-dimensional case were investigated by Cho [2].

In the present paper, we investigate change of tensor $U^\nu{}_{\omega\mu}$ induced by the conformal change in 7-dimensional g-unified field theory. These topics will be studied for the second class with the first category in 7-dimensional case.

2. Preliminaries

This chapter is a brief collection of basic concepts, notations, theorems, and results needed in our further considerations. They may be referred to Chung [3], Cho [1], [2].

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2.1. n-dimensional g-unified field theory

The n-dimensional g-unified field theory (n-g-UFT hereafter) was originally suggested by Hlavatý [9] and systematically introduced by Chung [5].

Let $X_n (n \geq 2)$ be an n-dimensional generalized Riemannian manifold, referred to a real coordinate system x^ν obeying coordinate transformations $x^\nu \to x^{\nu'}$, for which

$$Det\left(\left(\frac{\partial x}{\partial x'}\right)\right) \neq 0. \tag{2.1}$$

In the usual Einstein's n-dimensional unified field theory, the manifold X_n is endowed with a general real nonsymmetric tensor $g_{\lambda\mu}$ which may be split into its symmetric part $h_{\lambda\mu}$ and skew-symmetric part $k_{\lambda\mu}$:

$$g_{\lambda\mu} = h_{\lambda\mu} + k_{\lambda\mu} \tag{2.2}$$

where

$$Det((g_{\lambda\mu})) \neq 0 \qquad Det((h_{\lambda\mu})) \neq 0. \tag{2.3}$$

Therefore we may define a unique tensor $h^{\lambda\nu} = h^{\nu\lambda}$ by

$$h_{\lambda\mu} h^{\lambda\nu} = \delta^\nu_\mu. \tag{2.4}$$

In our n-g-UFT, the tensors $h_{\lambda\mu}$ and $h^{\lambda\nu}$ will serve for raising and/or lowering indices of the tensors in X_n in the usual manner.

The manifold X_n is connected by a general real connection $\Gamma^\nu_{\omega\mu}$ with the following transformation rule :

$$\Gamma^{\nu'}_{\omega'\mu'} = \frac{\partial x^{\nu'}}{\partial x^\alpha}\left(\frac{\partial x^\beta}{\partial x^{\omega'}} \cdot \frac{\partial x^\gamma}{\partial x^{\mu'}} \Gamma^\alpha_{\beta\gamma} + \frac{\partial^2 x^\alpha}{\partial x^{\omega'} \partial x^{\mu'}}\right) \tag{2.5}$$

and satisfies the system of Einstein's equations

$$D_\omega g_{\lambda\mu} = 2 S_{\omega\mu}{}^\alpha g_{\lambda\alpha} \tag{2.6}$$

where D_ω denotes the covariant derivative with respect to $\Gamma^\nu_{\lambda\mu}$ and

$$S_{\lambda\mu}{}^\nu = \Gamma^\nu_{[\lambda\mu]} \tag{2.7}$$

is the *torsion tensor* of $\Gamma^\nu_{\lambda\mu}$. The connection $\Gamma^\nu_{\lambda\mu}$ satisfying (2.6) is called the *Einstein's connection.*

In our further considerations, the following scalars, tensors, abbreviations, and notations for $p = 0, 1, 2, \cdots$ are frequently used :

$$\begin{aligned} &\mathfrak{g} = Det((g_{\lambda\mu})) \neq 0, \quad \mathfrak{h} = Det((h_{\lambda\mu})) \neq 0, \\ &\mathfrak{k} = Det((k_{\lambda\mu})), \end{aligned} \tag{2.8a}$$

(2.8b) $$g = \frac{\mathfrak{g}}{\mathfrak{h}}, \quad k = \frac{\mathfrak{k}}{\mathfrak{h}},$$

(2.8c) $$K_p = k_{[\alpha_1}{}^{\alpha^1} \cdots k_{\alpha_p]}{}^{\alpha^p}, \quad (p = 0, 1, 2, \cdots)$$

(2.8d) $$^{(0)}k_\lambda{}^\nu = \delta_\lambda{}^\nu, \quad ^{(1)}k_\lambda{}^\nu = k_\lambda{}^\nu, \quad ^{(p)}k_\lambda{}^\alpha = {}^{(p-1)}k_\lambda{}^\alpha k_\alpha{}^\nu,$$

(2.8e) $$K_{\omega\mu\nu} = \nabla_\nu k_{\omega\mu} + \nabla_\omega k_{\nu\mu} + \nabla_\mu k_{\omega\nu},$$

(2.8f) $$\sigma = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

where ∇_ω is the symbolic vector of the convariant derivative with respect to the Christoffel symbols $\{^{\nu}_{\lambda\mu}\}$ defined by $h_{\lambda\mu}$. The scalars and vectors introduced in (2.8) satisfy

(2.9a) $$K_0 = 1; K_n = k \quad \text{if } n \text{ is even}; \quad K_p = 0 \quad \text{if } p \text{ is odd},$$

(2.9b) $$g = 1 + K_2 + \cdots + K_{n-\sigma},$$

(2.9c) $$^{(p)}k_{\lambda\mu} = (-1)^{p(p)}k_{\mu\lambda}, \quad ^{(p)}k^{\lambda\mu} = (-1)^{p(p)}k^{\nu\lambda}.$$

Furthermore, we also use the following useful abbreviations, denoting an arbitrary tensor $T_{\omega\mu\nu}$, skew-symmetric in the first two indices, by T:

(2.10a) $$\overset{pqr}{T} = \overset{pqr}{T}{}_{\omega\mu\nu} = T_{\alpha\beta\gamma}{}^{(p)}k_\omega{}^{\alpha(q)}k_\mu{}^{\beta(r)}k_\nu{}^\gamma,$$

(2.10b) $$T = T_{\omega\mu\nu} = \overset{000}{T},$$

(2.10c) $$2\overset{pqr}{T}{}_{\omega[\lambda\mu]} = \overset{pqr}{T}{}_{\omega\lambda\mu} - \overset{pqr}{T}{}_{\omega\mu\lambda},$$

(2.10d) $$2\overset{(pq)r}{T}{}_{\omega\lambda\mu} = \overset{pqr}{T}{}_{\omega\lambda\mu} + \overset{qpr}{T}{}_{\omega\lambda\mu}.$$

We then have

(2.11) $$\overset{pqr}{T}{}_{\omega\lambda\mu} = -\overset{qpr}{T}{}_{\lambda\omega\mu}.$$

If the system (2.6) admits $\Gamma^\nu_{\lambda\mu}$, using the above abbreviations it was shown that the connection is of the form

(2.12) $$\Gamma^\nu_{\omega\mu} = \{^{\nu}_{\omega\mu}\} + S_{\omega\mu}{}^\nu + U^\nu{}_{\omega\mu}$$

where

(2.13) $$U_{\nu\omega\mu} = 2\overset{001}{S}{}_{\nu(\omega\mu)}.$$

The above two relations show that our problem of determining $\Gamma^{\nu}_{\omega\mu}$ in terms of $g_{\lambda\mu}$ is reduced to that of studying the tensor $S_{\omega\mu}{}^{\nu}$. On the other hand, it has also been shown that the tensor $S_{\omega\mu}{}^{\nu}$ satisfies

$$S = B - 3 \overset{(110)}{S} \tag{2.14}$$

where

$$2B_{\omega\mu\nu} = K_{\omega\mu\nu} + 3K_{\alpha[\mu\beta}k^{\alpha}_{\omega]}k^{\beta}_{\nu}. \tag{2.15}$$

2.2. Some results for the second class in 7-g-UFT

In this section, we introduce some results of 7-g-UFT without proof, which are needed in our subsequent considerations.

They may be referred to Kim and Others [6], [7], [8].

DEFINITION 2.1. In 7-g-UFT, the tensor $g_{\lambda\mu}(k_{\lambda\mu})$ is said to be the second class, if $K_2 \neq 0$, $K_4 = 0$.

THEOREM 2.2. (*Main Recurrence Relations*). *For the second class in 7-UFT, the following recurrence relation hold*

$$^{(p+3)}k_{\lambda}{}^{\nu} = -K_2^{(p+1)}k_{\lambda}{}^{\nu}, \quad (p = 0, 1, 2, \cdots). \tag{2.16}$$

THEOREM 2.3. (*For the second class in 7-g-UFT*). *A necessary and sufficient condition for the existence and uniqueness of the solution of* (2.5) *is*

$$1 - (K_2)^2 \neq 0. \tag{2.187}$$

If the condition (2.17) *is satisfied, the unique solution of* (2.14) *is given by*

$$(1 - K_2^2)(S - B) = -2 \overset{(10)1}{B} + (K_2 - 1) \overset{110}{B} + 2 \overset{(20)2}{B} + 2 \overset{112}{B}. \tag{2.18}$$

3. Conformal change of the 7-dimensional tensor $U^{\nu}{}_{\omega\mu}$ for the second class

In this final chapter we investigate the change $U^{\nu}{}_{\omega\mu} \to \overline{U}^{\nu}{}_{\omega\mu}$ of the vector induced by the conformal change of the tensor $g_{\lambda\mu}$, using the recurrence relations and theorems introduced in the preceding chapter.

We say that X_n and $\overline{X}_n$ are conformal if and only if

$$\overline{g}_{\lambda\mu}(x) = e^{\Omega} g_{\lambda\mu}(x) \tag{3.1}$$

where $\Omega = \Omega(x)$ is an at least twice differentiable function. This conformal change enforces a change of the tensor $U^{\nu}{}_{\omega\mu}$. An explicit representation of the change of 7-dimensional tensor $U^{\nu}{}_{\omega\mu}$ for the second class will be exhibited in this chapter.

AGREEMENT 3.1. *Throughout this section, we agree that, if T is a function of $g_{\lambda\mu}$, then we denote $\overline{T}$ the same function of $\overline{g}_{\lambda\mu}$. In particular, if T is a tensor, so is $\overline{T}$. Furthermore, the indices of $T(\overline{T})$ will be raised and/or lowered by means of $h^{\lambda\nu}(\overline{h}^{\lambda\nu})$ and/or $h_{\lambda\nu}(\overline{h}_{\lambda\nu})$.*

The results in the following theorems are needed in our further considerations. They may be referred to Cho [1], [2].

THEOREM 3.2. *In n-g-UFT, the conformal change* (3.1) *induces the following changes:*

$$(3.2a)\qquad {}^{(p)}\overline{k}_{\lambda\mu} = e^{\Omega(p)}k_{\lambda\mu}, \quad {}^{(p)}\overline{k}_{\lambda} = {}^{(p)}k^{\nu}_{\lambda}, \quad {}^{(p)}\overline{k}^{\lambda\mu} = e^{-\Omega(p)}k^{\lambda\mu},$$

$$(3.2b)\qquad \overline{g} = g, \quad \overline{K}_p = K_p, \qquad (p = 1, 2, \cdots).$$

THEOREM 3.3. (*For all classes in 7-g-UFT*). *The change of the tensor $B_{\omega\mu\nu}$ induced by the conformal change* (3.1) *may be given by*

$$(3.3)\qquad \begin{aligned}\overline{B}_{\omega\mu\nu} &= e^{\Omega}(B_{\omega\mu\nu} + k_{\nu[\omega}\Omega_{\mu]} - k_{\omega\mu}\Omega_{\nu} \\ &-h_{\nu[\omega}k_{\mu]}{}^{\delta}\Omega_{\delta} + 2{}^{(2)}k_{\nu[\omega}k_{\mu]}{}^{\delta}\Omega_{\delta} + k_{\omega\mu}{}^{(2)}k_{\nu}{}^{\delta}\Omega_{\delta}).\end{aligned}$$

Now, we are ready to derive representations of the changes $U^{\nu}{}_{\omega\mu} \to \overline{U}^{\nu}{}_{\omega\mu}$ in 7-g-UFT for the second class with the first category induced by the conformal change (3.1).

THEOREM 3.4. *The conformal change* (3.1) *induces the following change:*

$$(3.4)\qquad \begin{aligned}\overline{\overset{ppq}{B}}_{\omega\mu\nu} &= e^{\Omega}[\overset{ppq}{B}_{\omega\mu\nu} + (-1)^{p}\{2^{(p+q+2)}k_{\nu[\omega}{}^{(p+1)}k_{\mu]}{}^{\delta} \\ &+{}^{(2p+1)}k_{\omega\mu}{}^{(2+q)}k_{\nu}{}^{\delta} - {}^{(2p+1)}k_{\omega\mu}{}^{(q)}k_{\nu}{}^{\delta} \\ &+{}^{(p+q+1)}k_{\nu[\omega}{}^{(p)}k_{\mu]}{}^{\delta} - {}^{(p+q)}k_{\nu[\omega}{}^{(p+1)}k_{\mu]}{}^{\delta}\}\Omega_{\delta}].\end{aligned}$$

$$\begin{pmatrix} p = 0, 1, 2, 3, 4, \cdots \\ q = 0, 1, 2, 3, 4, \cdots \end{pmatrix}$$

THEOREM 3.5. (*For the second class with the first category*) *The conformal change* (3.1) *induces the following change.*

$$(3.5)\qquad \begin{aligned}\overline{\overset{001}{B}}_{\nu(\omega\mu)} &= e^{\Omega}(\overset{001}{B}_{\nu(\omega\mu)} - K_2 k_{\nu[\omega}k_{\mu]}{}^{\delta}\Omega_{\delta} \\ &+2K_2 k_{\nu[\omega}k_{\mu]}{}^{\delta}\Omega_{\delta} + {}^{(2)}k_{\nu(\omega}\Omega_{\mu)} - {}^{(2)}k_{\omega\nu}\Omega_{\nu}).\end{aligned}$$

Proof. The relation (3.5) obtained by (2.16), (3.3), (3.4). □

THEOREM 3.6. *The change* ${S_{\omega\mu}}^{\nu} \to {\overline{S}_{\omega\mu}}^{\nu}$ *induced by conformal change* (3.1) *may be represented by*

$$\begin{aligned}
{\overline{S}_{\omega\mu}}^{\nu} &= {S_{\omega\mu}}^{\nu} + 1 - h^{\nu}{}_{[\omega}k_{\mu]}{}^{\delta}\Omega_{\delta} \\
&+(K_2-1)k_{\omega\mu}\Omega^{\nu} + (1-K_2)k_{\omega\mu}{}^{(2)}k^{\nu\delta}\Omega_{\delta} \\
&+\tfrac{1}{{K_2}^2-1}[(-1+K_2)k^{\nu}{}_{[\omega}\Omega_{\mu]} \\
&+(-1+2K_2+{K_2}^2)^{(2)}k^{\nu}{}_{[\omega}k_{\mu]}{}^{\delta}\Omega_{\delta} \\
&+(K_2+{K_2}^2-2{K_2}^3)k^{\nu}{}_{[\omega}{}^{(2)}k_{\mu]}{}^{\delta}\Omega_{\delta}].
\end{aligned} \tag{3.6}$$

THEOREM 3.7. *The change* $U^{\nu}{}_{\omega\mu} \to \overline{U}^{\nu}{}_{\omega\mu}$ *induced by the conformal change* (3.1) *may be represented by*

$$\begin{aligned}
\overline{U}^{\nu}{}_{\omega\mu} &= U^{\nu}{}_{\omega\mu} + 1 + \tfrac{1}{K_2+1}({}^{(2)}k^{\nu}{}_{(\omega}\Omega_{\mu)} - {}^{(2)}k_{\omega\mu}h^{\nu\delta}\Omega_{\delta} \\
&+(-2K_2-1)K_2^2k^{\nu}_{(\omega}{}^{(2)}k^{\delta}_{\mu)}\Omega_{\delta}) \\
&+\tfrac{1}{K_2^2-1}((2K_2^4-3K_2^3+3K_2^2 \\
&+3K_2-3)k^{\nu}{}_{(\omega}k_{\mu)}{}^{\delta}\Omega_{\delta}).
\end{aligned} \tag{3.7}$$

Proof. In virtue of (2.13) and Agreement (3.1), we have

$$\overline{U}^{\nu}{}_{\omega\mu} = 2\overset{\overline{001}}{S}{}^{\nu}{}_{(\omega\mu)} \tag{3.8}$$

The relation (3.7) follows by substituting (2.10), (2.16), (3.5), (3.6) into (3.8). □

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Department of Mathematics, Inha University, Incheon 402-751, Korea
E-mail: chcho@inha.ac.kr