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SOME RESULTS ON $\mathfrak{D}$–ADMISSIBLE $(\in, \in \vee q)$–FUZZY SUBGROUPS

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ABSTRACT. The definition of a $\mathfrak{D}$-admissible fuzzy subset for an operator domain $\mathfrak{D}$ on a group G is modified to obtain new kinds of $(\in, \in \vee q)$-fuzzy subgroups such as an $(\in, \in \vee q)$-fuzzy normal subgroup, an $(\in, \in \vee q)$-fuzzy characteristic subgroup, an $(\in, \in \vee q)$-fuzzy fully invariant subgroup which are invariant under $\mathfrak{D}$. As results, some of the fundamental properties of such $(\in, \in \vee q)$–fuzzy subgroups are obtained.

1. Introduction

The notion of a fuzzy set was introduced by Zadeh, utilizing which Rosenfeld [12] defined a fuzzy subgroup. Since then this has been further studied in details by many other researchers. Among others, Das [6] characterized fuzzy subgroups by their level subgroups and Liu defined the fuzzy normality of a fuzzy subgroup. A coherent study of fuzzy normal subgroups was initiated by Mukherjee and Bhattacharya [11]. The fuzzy subgroups of a group which are invariant under various operator domains were studied by Gupta and Sarma [8], from which various fuzzy subgroups such as a fuzzy normal subgroup, a fuzzy characteristic subgroup, a fuzzy fully invariant subgroup were introduced. Sidky and Mishref [13] continued to investigate a fuzzy characteristic subgroup and to prove some of their properties.

After Rosenfeld firstly introduced the notion of a fuzzy subgroup, in 1992, Bhakat and Das ([1], [2], [3], [4]) used the idea of a fuzzy point and its belongingness to and quasi-coincidence with a fuzzy set to define a

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$(\in, \in \vee q)$–fuzzy subgroup which is a generalization of a fuzzy subgroup defined by Rosenfeld.

The object of this paper is to study the $(\in, \in \vee q)$–fuzzy subgroups which are $\mathfrak{D}$–admissible for any operator domain $\mathfrak{D}$ on G, and to obtain some of the fundamental properties of $(\in, \in \vee q)$–fuzzy subgroups which are invariant under $\mathfrak{D}$.

2. $(\in, \in \vee q)$–fuzzy subgroups

DEFINITION 2.1. (Das [6]) Let $\lambda : X \to [0, 1]$ be a fuzzy subset of X and let $t \in (0, 1]$. The set

$$\lambda_t = \{x \in X | \lambda(x) \geq t\}$$

is called a *level subset* determined by λ and t. The subset $\{x \in X | \lambda(x) > 0\}$ is called a *support* of λ and is denoted by $Supp(\lambda)$.

DEFINITION 2.2. (Ming and Ming [10]) A fuzzy subset of X defined by

$$\lambda(y) = \begin{cases} t & \text{if } y = x \\ 0 & \text{if } y \neq x \end{cases}$$

is said to be a *fuzzy point* with support x and value t and is denoted by x_t. A fuzzy point x_t is said to *belong to* (resp. *be quasi-coincidence with*) a fuzzy subset λ, written as $x_t \in \lambda$ (resp. $x_t \; q \; \lambda$) if $\lambda(x) \geq t$ (resp. $\lambda(x) + t > 1$). If $x_t \in \lambda$ and (resp. or) $x_t \; q \; \lambda$, then we write $x_t \in \wedge q \lambda$ (resp. $x_t \in \vee q \lambda$). "$x_t \overline{\in} \lambda$, $\quad x_t \overline{\in \vee q} \; \lambda$" will respectively mean $x_t \in \lambda$ and $x_t \in \vee q \lambda$ do not hold.

For all $t, s \in [0, 1]$, $min\{t, s\}$ will be denoted by $M(t, s)$.

DEFINITION 2.3. (Bhakat and Das [3]) A fuzzy subset λ of a group G is said to be an $(\in, \in \vee q)$*–fuzzy subgroup* of G if for any $x, y \in G$ and $t, s \in [0, 1]$,

(1) $x_t, \; y_s \in \lambda$ implies $(xy)_{M(t,s)} \in \vee q \; \lambda$
(2) $x_t \in \lambda$ implies $(x^{-1})_t \in \vee q \lambda$.

By [3], the conditions (1) and (2) of Definition 2.3 are, respectively, equivalent to

(1) $\lambda(xy) \geq M(\lambda(x), \lambda(y), 0.5)$ for all $x, y \in G$
(2) $\lambda(x^{-1}) \geq M(\lambda(x), 0.5)$ for all $x \in G$.

In the study of a fuzzy subgroup, level subsets of a fuzzy subset initiated by Das have played an important role. Bhakat redefined level subsets to be suitable for an ($\in$, $\in \vee q$)–fuzzy subgroup as follows.

DEFINITION 2.4. (Bhakat [2]) Let λ be a fuzzy subset of G, and $t \in (0,1]$. Then the subset

$$\lambda_t = \{x \in G | \lambda(x) \geq t \text{ or } \lambda(x) + t > 1\} = \{x \in G | x_t \in \vee q\ \lambda\}$$

is said to be the ($\in \vee q$)–*level subset* of G determined by λ and t.

3. $\mathfrak{D}$–admissible ($\in$, $\in \vee q$)–fuzzy subgroups

For a group G, an operator domain on G is any nonempty class $\mathfrak{D}$ of endomorphisms of G. In ordinary group theory we call a subset A of G admissible under $\mathfrak{D}$, or simply $\mathfrak{D}$–admissible if $f(A) \subseteq A$ for all $f \in \mathfrak{D}$. A subgroup H of G is said to be normal (resp. characteristic, fully invariant) if H is admissible under the class of all inner automorphisms (resp. all automorphisms, all endomorphisms) of G.

Bhakat introduced a $\mathfrak{D}$–admissible fuzzy subset in the fuzzy setting as shown in the following.

DEFINITION 3.1. Let $\mathfrak{D}$ be an operator domain on G. A fuzzy subset λ of G is said to be *admissible* under $\mathfrak{D}$ (or simply $\mathfrak{D}$–*admissible*) if

$$x_t \in \lambda \quad \text{implies} \quad x_t \in \vee q\ \lambda \circ f$$

for all $x \in G$ and $f \in \mathfrak{D}$.

LEMMA 3.2. (Kim [9]) *For an operator domain $\mathfrak{D}$ of G, a fuzzy subset λ of G is $\mathfrak{D}$–admissible if and only if*

$$\lambda(f(x)) \geq M(\lambda(x), 0.5) \quad \textit{for all} \quad f \in \mathfrak{D} \quad \textit{and} \quad x \in G.$$

For a fuzzy subset λ of G and an operator domain $\mathfrak{D}$ on G, $\mathfrak{D}$–admissibility in the sense of [8] implies that in the sense of Definition 3.1. However, the converse may not be true as follows.

Let S_3 be the symmetric group of order 3 whose elements are

$$\sigma_0 = (\ 1\ ), \qquad \sigma_1 = (1,\ 2,\ 3), \qquad \sigma_2 = (1,\ 3,\ 2)$$
$$\tau_1 = (2,\ 3), \qquad \tau_2 = (1,\ 3), \qquad \tau_3 = (1,\ 2)$$

and let $\mathfrak{D} = Aut(S_3)$. Consider the fuzzy subset $\lambda : S_3 \to [0,1]$ defined by

$$\lambda(\sigma_0) = \lambda(\sigma_1) = 0.7, \quad \lambda(\sigma_2) = 0.6, \quad \lambda(\tau_1) = \lambda(\tau_2) = \lambda(\tau_3) = 0.4.$$

Since every automorphism f of S_3 is of the form $f(\sigma_0) = \sigma_0$, $f(\sigma_i) = \sigma_{i'}$, $(i,\ i' \in \{1,\ 2\})$, and $f(\tau_j) = \tau_{j'}$ $(j,\ j' \in \{1,\ 2,\ 3\})$, it follows

$$\lambda(f(x)) \geq M(\lambda(x), 0.5) \quad \text{for all} \quad x \in S_3,\ f \in Aut(S_3).$$

Hence the fuzzy subset λ is $\mathfrak{D}$–admissible by Lemma 3.2 in the sense of Definition 3.1. However, the fuzzy subset λ is not $\mathfrak{D}$–admissible in the sense of [8] because the map $g : S_3 \to S_3$ defined by

$$g(\sigma_0) = \sigma_0,\ g(\sigma_1) = \sigma_2, \cdot g(\sigma_2) = \sigma_1,\ g(\tau_1) = \tau_1,\ g(\tau_2) = \tau_3,\ g(\tau_3) = \tau_2$$

is an automorphism and $\lambda(g(\sigma_1)) = \lambda(\sigma_2) < \lambda(\sigma_1)$.

As mentioned earlier, level subsets of a fuzzy subset have played a fundamental role in the study of a fuzzy subgroup. In fact, $(\in \vee q)$–level subsets of a fuzzy subset are very closely connected with a $\mathfrak{D}$–admissible fuzzy subset as shown in the following theorem.

THEOREM 3.3. (Kim [9]) *Let $\mathfrak{D}$ be an operator domain on G. Then a fuzzy subset λ of G is $\mathfrak{D}$–admissible if and only if $(\in \vee q)$–level subsets λ_t is $\mathfrak{D}$–admissible for all $t \in (0,1]$.*

DEFINITION 3.4. Let $Inn(G)$, $Aut(G)$, $End(G)$ denote the classes of all inner automorphisms, all automorphisms, all endomorphisms of a group G, respectively. An $(\in,\ \in \vee q)$–fuzzy subgroup λ is an *$(\in,\ \in \vee q)$–fuzzy normal* (resp, an *$(\in,\ \in \vee q)$–fuzzy characteristic*, an *$(\in,\ \in \vee q)$–fuzzy fully invariant*) subgroup of G if λ is $Inn(G)$–admissible (resp, $Aut(G)$–admissible, $End(G)$–admissible).

As usual in Group Theory, there are $(\in,\ \in \vee q)$–fuzzy characteristic subgroup of G which is not $(\in,\ \in \vee q)$–fuzzy fully invariant and $(\in, \in \vee q)$–fuzzy normal subgroup of G which is not $(\in,\ \in \vee q)$–fuzzy characteristic.

EXAMPLE 3.5. (1) Let A_4 be the alternating group of order 4 and let $Z(G)$ be the center of the group $G = A_4 \times Z_2$. Consider a fuzzy subset $\lambda : G \to [0,1]$ defined by

$$\lambda(x) = \begin{cases} 0.8 & \text{if } x \text{ is the identity } e, \\ 0.7 & \text{if } x \in Z(G) - \{e\}, \\ 0.3 & \text{if } x \in G - Z(G). \end{cases}$$

Then by Theorem 3.3, λ is an $(\in,\ \in\vee q)$–fuzzy characteristic subgroup of G which in not an $(\in,\ \in\vee q)$–fuzzy fully invariant subgroup.

(2) Let $G=\{e,a,b,c\}$ be the Klein's 4-group and λ be defined by

$$\lambda(e)=0.8, \lambda(a)=0.9, \lambda(b)=\lambda(c)=0.3.$$

Then λ is an $(\in,\ \in\vee q)$–fuzzy subgroup because $(\in\vee q)$–level subsets are given as follows;

$$\lambda_t=\begin{cases} G & \text{if } t\leq 0.3,\\ \{e,a\} & \text{if } 0.3<t\leq 0.7,\\ G & \text{if } t>0.7.\end{cases}$$

Hence, by Theorem 3.3, λ is an $(\in,\ \in\vee q)$–fuzzy normal subgroup of G but not an $(\in,\ \in\vee q)$–fuzzy characteristic subgroup of G.

THEOREM 3.6. *Let λ, μ, η be $(\in,\ \in\vee q)$–fuzzy subgroups of G such that $\lambda\subseteq\mu\subseteq\eta$. If λ is an $(\in,\ \in\vee q)$–fuzzy characteristic subgroup of $Supp(\mu)$ and μ is an $(\in,\ \in\vee q)$–fuzzy characteristic subgroup of $Supp(\eta)$ then λ is an $(\in,\ \in\vee q)$–fuzzy characteristic subgroup of $Supp(\eta)$.*

Proof. Let f be any automorphism of $Supp(\eta)$. If $x\in Supp(\mu)$, we have

$$\mu(f(x))\geq M(\mu(x),0.5)>0$$

because μ is an $(\in,\ \in\vee q)$–fuzzy characteristic subgroup of $Supp(\eta)$. This means that $f(Supp(\mu))\subseteq Supp(\mu)$. Since f^{-1} is also an automorphism of $Supp(\eta)$, we have

$$f^{-1}(Supp(\mu))\subseteq Supp(\mu)$$

in the same way. Thus the restriction g of f to $Supp(\mu)$ is an automorphism of $Supp(\mu)$. Since λ is an $(\in,\ \in\vee q)$–fuzzy characteristic subgroup of $Supp(\mu)$ we get

$$\lambda(f(x))=\lambda(g(x))\geq M(\lambda(x),0.5) \text{ for all } x\in Supp(\mu).$$

On the other hand, it follows that, for $x\in Supp(\eta)-Supp(\mu)$,

$$\lambda(f(x))\geq M(\lambda(x),0.5)=M(0,0.5)=0$$

because $Supp(\lambda)\subseteq Supp(\mu)$. Therefore we have $\lambda(f(x))\geq M(\lambda(x),0.5)$ for all $x\in Supp(\eta)$. □

THEOREM 3.7. *Let λ, μ, η be $(\in, \in \vee q)$–fuzzy groups of G such that $\lambda \subseteq \mu \subseteq \eta$. If λ is an $(\in, \in \vee q)$–fuzzy characteristic subgroup of Supp(μ) and μ is an $(\in, \in \vee q)$–fuzzy normal subgroup of Supp(η) then λ is an $(\in, \in \vee q)$–fuzzy normal subgroup of Supp(η).*

Proof. The proof is similar to that of Theorem 3.6. □

DEFINITION 3.8. (Bhakat and Das [4]) Let λ be a $(\in, \in \vee q)$–fuzzy subgroup of G. For any $x \in G$, $\hat{\lambda}_x$(resp. $\check{\lambda}_x$) $: G \to [0,1]$ defined by

$$\hat{\lambda}_x(g) = M(\lambda(gx^{-1}), 0.5) \quad (\text{resp. } \check{\lambda}_x(g) = M(\lambda(x^{-1}g), 0.5))$$

for all $g \in G$ is called the $(\in, \in \vee q)$*–fuzzy left coset* (resp. *right coset*) of G determined by x and G.

It follows from Theorem 4.5 of [4] that λ is an $(\in, \in \vee q)$–fuzzy normal subgroup of G if and only if $\hat{\lambda}_x = \check{\lambda}_x, \forall x \in G$. When λ is an $(\in, \in \vee q)$–fuzzy normal subgroup, the $(\in, \in \vee q)$–fuzzy coset of G determined by λ and x is denoted by $\hat{\lambda}_x$. Let λ be an $(\in, \in \vee q)$–fuzzy normal subgroup of G and $\mathcal{F}$ be the set of all $(\in, \in \vee q)$–fuzzy cosets of λ in G. Then $\mathcal{F}$ is a group with the identity $\hat{\lambda}_e$ under the multiplication defined by

$$\hat{\lambda}_x\hat{\lambda}_y = \hat{\lambda}_{xy}, \quad \forall x, y \in G.$$

THEOREM 3.9. *If λ is an Inn(G)–admissible $(\in, \in \vee q)$–fuzzy subgroup of G, the fuzzy subset $\bar{\lambda} : \mathcal{F} \to [0,1]$ defined by*

$$\bar{\lambda}(\hat{\lambda}_x) = \lambda(x), \quad \forall x \in G$$

is also an Inn($\mathcal{F}$)–admissible $(\in, \in \vee q)$–fuzzy subgroup of $\mathcal{F}$.

Proof. It follows from Theorem 4.5 of [4] that $\bar{\lambda}$ is an $(\in, \in \vee q)$–fuzzy subgroup of $\mathcal{F}$. Let $i_{\hat{\lambda}_g}$ be an inner automorphism of $\mathcal{F}$ and $\hat{\lambda}_x \in \mathcal{F}$. Then we have $\bar{\lambda}(i_{\hat{\lambda}_g}(\hat{\lambda}_x)) = \bar{\lambda}(\hat{\lambda}_g\hat{\lambda}_x\hat{\lambda}_{g^{-1}}) = \bar{\lambda}(\hat{\lambda}_{gxg^{-1}}) = \lambda(gxg^{-1}) = \lambda(i_g(x))$ where $i_g \in Inn(G)$. It follows from Lemma 3.2 that $\lambda(i_g(x)) \geq M(\lambda(x),\ 0.5) = M(\bar{\lambda}(\hat{\lambda}_x),\ 0.5)$. Therefore $\bar{\lambda}$ is an $Inn(\mathcal{F})$–admissible $(\in, \in \vee q)$–fuzzy subgroup of $\mathcal{F}$. □

DEFINITION 3.10. A fuzzy subgroup λ of G is said to be *of an isolated tip* if $\lambda^{-1}(\lambda(e)) = \{e\}$. In this case, $\lambda(x) = \lambda(e) \Rightarrow x = e$.

THEOREM 3.11. *Let λ be an $(\in, \in \vee q)$–fuzzy normal subgroup of G of an isolated tip such that $\lambda(x) \leq \frac{1}{2}$ for all $x \in G$. Then λ is an*

$Aut(G)$–admissible ($\in$, $\in \vee q$)–fuzzy subgroup of G if and only if $\bar{\lambda}$ is an $Aut(\mathcal{F})$–admissible ($\in$, $\in \vee q$)–fuzzy subgroup of $\mathcal{F}$.

Proof. For $f \in Aut(G)$, we define a map $\phi_f : \mathcal{F} \to \mathcal{F}$ by $\phi_f(\hat{\lambda}_x) = \hat{\lambda}_{f(x)}$. Then we have

$$\begin{aligned}\phi_f(\hat{\lambda}_x\hat{\lambda}_y) &= \phi_f(\hat{\lambda}_{xy}) = \hat{\lambda}_{f(xy)} = \hat{\lambda}_{f(x)f(y)} \\ &= \hat{\lambda}_{f(x)}\hat{\lambda}_{f(y)} = \phi_f(\hat{\lambda}_x)\phi_f(\hat{\lambda}_y).\end{aligned}$$

Since λ is of an isolated tip such that $\lambda(x) \leq \frac{1}{2}$ for all $x \in G$, it follows that

$$\begin{aligned}Ker(\phi_f) &= \{\hat{\lambda}_x \in \mathcal{F} | \phi_f(\hat{\lambda}_x) = \hat{\lambda}_{f(x)} = \hat{\lambda}_e\} \\ &= \{\hat{\lambda}_e\}.\end{aligned}$$

These facts yield ϕ_f is an automorphism of a group $\mathcal{F}$. On the other hand, let $g \in Aut(\mathcal{F})$. Then, for every $x \in G$, there exists uniquely $g_x \in G$ such that $g(\hat{\lambda}_x) = \hat{\lambda}_{g_x}$. Hence we can define, for every $g \in Aut(\mathcal{F})$, a map $\psi_g : G \to G$ by $\psi_g(x) = g_x$. Since

$$\begin{aligned}\hat{\lambda}_{g_{xy}} &= g(\hat{\lambda}_{xy}) = g(\hat{\lambda}_x\hat{\lambda}_y) \\ &= g(\hat{\lambda}_x)g(\hat{\lambda}_y) = \hat{\lambda}_{g_x}\hat{\lambda}_{g_y} = \hat{\lambda}_{g_x g_y},\end{aligned}$$

the fact λ is of an isolated tip implies that $g_{xy} = g_x g_y$. Hence we have $\psi_g(xy) = g_{xy} = g_x g_y = \psi_g(x)\psi_g(y)$, and $Ker(\psi_g) = \{x \in G | g(\hat{\lambda}_x) = \hat{\lambda}_{g_x} = \hat{\lambda}_e\} = \{x \in G | \hat{\lambda}_x = \hat{\lambda}_e\}$, since g is a monomorphism. Thus $Ker(\psi_g) = \{e\}$ holds because λ is of an isolated tip. Clearly ψ_g is surjective. Hence we can deduce an automorphism ψ_g of a group G from an automorphism g of a group $\mathcal{F}$.

Now we suppose that λ is an $Aut(G)$–admissible ($\in$, $\in \vee q$)–fuzzy subgroup of G. Then, for every $g \in Aut(\mathcal{F})$ and $\hat{\lambda}_x \in \mathcal{F}$, we have $\bar{\lambda}(g(\hat{\lambda}_x)) = \bar{\lambda}(\hat{\lambda}_{g_x}) = \lambda(g_x) = \lambda(\psi_g(x)) \geq M(\lambda(x), 0.5) = M(\bar{\lambda}(\hat{\lambda}_x), 0.5)$ which implies that $\bar{\lambda}$ is an $Aut(G)$–admissible ($\in$, $\in \vee q$)–fuzzy subgroup of a group $\mathcal{F}$.

Conversely, we suppose that $\bar{\lambda}$ is an $Aut(G)$–admissible ($\in$, $\in \vee q$)–fuzzy subgroup of $\mathcal{F}$. By Lemma 3.2, for every $f \in Aut(G)$ and $x \in G$, we have $\lambda(f(x)) = \bar{\lambda}(\hat{\lambda}_{f(x)}) = \bar{\lambda}(\phi_f(\hat{\lambda}_x)) \geq M(\bar{\lambda}(\hat{\lambda}_x), 0.5) = M(\lambda(x), 0.5)$, which proves that λ is an $Aut(G)$–admissible ($\in$, $\in \vee q$)–fuzzy subgroup of a group G. □

References

[1] S. K. Bhakat, $(\in \vee q)$*-level subset*, Fuzzy Sets and Systems **103** (1999), 529–533.
[2] ———, $(\in, \in \vee q)$*-fuzzy normal, quasinormal and maximal subgroups*, Fuzzy Sets and Systems **112** (2000), 299–312.
[3] S. K. Bhakat and P. Das, *On the definition of a fuzzy subgroup*, Fuzzy Sets and Systems **51** (1992), 235–241.
[4] ———, $(\in, \in \vee q)$*-fuzzy subgroup*, Fuzzy Sets and Systems **80** (1996), 359–368.
[5] ———, *Fuzzy subrings and ideals redefined*, Fuzzy Sets and Systems **81** (1996), 383–393.
[6] P. S. Das, *Fuzzy groups and level subgroups*, J. Math. Anal. Appl. **85** (1981), 264–269.
[7] V. N. Dixit, R. Kumar and N. Ajmal, *Level subgroups and union of fuzzy subgroups*, Fuzzy Sets and Systems **37** (1990), 359–371.
[8] K. C. Gupta and B. K. Sarma, *Operator domains on fuzzy groups*, Inform. Sci. **84** (1995), 247–259.
[9] D. S. Kim, $(\in, \in \vee q)$*-fuzzy subgroups*, Far East J. Math. Sci. **8** (2003), 161–172.
[10] P. P. Ming and L. Y. Ming, *Fuzzy topology 1: Neighborhood structure of a fuzzy point and Moore-Smith convergence*, J. Math. Anal. Appl. **76** (1980), 571–599.
[11] N. P. Mukherjee and P. Bhattacharya, *Fuzzy normal subgroups and fuzzy cosets*, Inform. Sci. **34** (1984), 225–239.
[12] A. Rosenfeld, *Fuzzy groups*, J. Math. Anal. Appl. **35** (1971), 512–517.
[13] F. I. Sidky and M. Atif Mishref, *Fully invariant, characteristic, and S-fuzzy subgroups*, Inform. Sci. **55** (1991), 27–33.

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