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# NONEXISTENCE OF RICCI-PARALLEL REAL HYPERSURFACES IN $P_2\mathbb{C}$ OR $H_2\mathbb{C}$

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Abstract. Niebergall and Ryan posed many open problems on real hypersurfaces in complex space forms. One of them is "Are there any Ricci-parallel real hypersurfaces in complex projective space $P_2\mathbb{C}$ or complex hyperbolic space $H_2\mathbb{C}$?" The purpose of present paper is to prove the nonexistence of such hypersurfaces.

## 1. Introduction

A complex 2-dimensional Kaehlerian manifold of constant holomorphic sectional curvature $4c$ is called a *complex space form*, which is denoted by $M_2(c)$. A complete and simply connected complex space form consists of a complex projective space $P_2\mathbb{C}$, a complex Euclidean space $\mathbb{C}^2$ or a complex hyperbolic space $H_2\mathbb{C}$, according to $c > 0$, $c = 0$ or $c < 0$.

In [2], R. Niebergall and P. J. Ryan gave the necessary background material to access the study of real hypersurfaces in complex space forms and gave a survey of this field of the study. Also they posed many open problems. One of them is the following:

"Are there any Ricci-parallel real hypersurfaces in $P_2\mathbb{C}$ or $H_2\mathbb{C}$?"

The purpose of the present paper is to give a negative answer for this open problem.

## 2. Preliminaries

Let $(M_2(c), <, >, J)$ be a complex space form with constant holomorphic sectional curvature $4c(\neq 0)$ and with Levi-Civita connection $\tilde{\nabla}$. Let $M$ be a real hypersurface immersed in $M_2(c)$. Then, denoting

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the Riemannian metric on $M$ induced from the metric on $M_2(c)$ by the same symbol $<,>$, the Levi-Civita connection $\nabla$ of the induced metric $<,>$ and the shape operator $A$ of the immersion are characterized respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + < AX, Y > \xi, \quad \tilde{\nabla}_X \xi = -AX,$$

where $\xi$ is a local choice of unit normal. Define the structure vector $W = -J\xi$. Then $W \in TM$ and $< W, W >= 1$. Denote $\alpha =< AW, W >$.

Define a skew-symmetric $(1,1)$-tensor $\phi$ from the tangential projection of $J$ by

$$JX = \phi X + < X, W > \xi.$$

Then we have

$$\begin{aligned} &\phi^2 X = -X + < X, W > W, \quad \phi W = 0, \\ &< \phi X, \phi Y >=< X, Y > - < X, W >< Y, W >, \end{aligned} \tag{2.1}$$

that is, $(\phi, W, <,>)$ determines an almost contact metric structure ([1]). The Gauss and Codazzi equations are given by

$$\begin{aligned} &R(X,Y)Z \\ &= c[< Y, Z > X - < X, Z > Y + < \phi Y, Z > \phi X - < \phi X, Z > \phi Y \\ &\quad - 2 < \phi X, Y > \phi Z] + < AY, Z > AX - < AX, Z > AY, \end{aligned} \tag{2.2}$$

$$\begin{aligned} &(\nabla_X A)Y - (\nabla_Y A)X \\ &= c(< X, W > \phi Y - < Y, W > \phi X + 2 < X, \phi Y > W). \end{aligned} \tag{2.3}$$

From equation (2.2) we get the Ricci tensor $S$ of type $(1,1)$ as

$$SX = 5cX - 3c < X, W > W + mAX - A^2 X, \tag{2.4}$$

where $m = traceA$ is the mean curvature of $M$.

It is well known ([2]) that

$$\nabla_X W = \phi AX, \tag{2.5}$$

$$(\nabla_X \phi)Y =< Y, W > AX - < AX, Y > W. \tag{2.6}$$

If $W$ is a principal vector, then $M$ is called a *Hopf hypersurface* ([2]). In a 3-dimensional Hopf hypersurface, the present author proved the following

THEOREM 2.1 ([3]). *Let $M$ be a 3-dimensional Hopf hypersurface in a complex space form of constant holomorphic sectional curvature $4c \neq 0$. Then $M$ cannot have harmonic curvature, that is, $(\nabla_Y S)Z - (\nabla_Z S)Y$ cannot vanish identically.*

## 3. Nonexistence of Ricci-parallel real hyperfaces in complex space forms $P_2\mathbb{C}$ or $H_2\mathbb{C}$

Let $M$ be a 3-dimensional real hypersurface in complex space form $M_2(c)$ with constant holomorphic sectional curvature $4c \neq 0$. Suppose that $M$ is a Ricci-parallel real hypersurface. Then, by Theorem 2.1, $M$ is not a Hopf hypersurface. We choose a local orthonormal frame field $W$, $X$, $\phi X$ of $M$ and put

$$\begin{aligned} AW &= \alpha W + bX + e\phi X, \\ AX &= bW + \beta X + \delta\phi X, \\ A\phi X &= eW + \delta X + \gamma\phi X, \end{aligned} \tag{3.1}$$

where we have used the property $< AY, Z >=< Y, AZ >$.

Since $M$ is not a Hopf hypersurface, the structure vector field $W$ is not principal at some point $P$ of $M$. Hence $b \neq 0$ in an open neighborhood $\mathcal{U}_1$ of $P$ or $e \neq 0$ in an open neighborhood $\mathcal{U}_2$ of $P$. Thus, $b \neq 0$ in an open neighborhood $\mathcal{U}$ of $P$ or $e \neq 0$ in an open neighborhood $\mathcal{U}$ of $P$. Hereafter we consider the local frame field $W$, $X$, $\phi X$ in $\mathcal{U}$ only.

Case A. Assume that the function $\delta$ is always zero in $\mathcal{U}$.

Then the equation (3.1) can be written as follows.

$$\begin{aligned} AW &= \alpha W + bX + e\phi X, \\ AX &= bW + \beta X, \\ A\phi X &= eW + \gamma\phi X. \end{aligned} \tag{3.2}$$

From (3.2), $m$ is given by $m = \alpha + \beta + \gamma$. From (2.4) and (3.2), we have

$$\begin{aligned} SW &= (2c + \alpha\beta + \alpha\gamma - b^2 - e^2)W + b\gamma X + e\beta\phi X, \\ SX &= b\gamma W + (5c + \alpha\beta + \beta\gamma - b^2)X - be\phi X, \\ S\phi X &= e\beta W - beX + (5c + \alpha\gamma + \beta\gamma - e^2)\phi X. \end{aligned} \tag{3.3}$$

Since $< \nabla_W X, W >= - < X, \nabla_W W >= - < X, \phi AW >= - < X, b\phi X - eX >= e$, we can put

$$\nabla_W X = eW + f\phi X, \tag{3.4}$$

where we have put $f =< \nabla_W X, \phi X >$.

Since $< \nabla_X X, W >= - < X, \nabla_X W >= - < X, \beta\phi X >= 0$, we can put

$$\nabla_X X = h\phi X, \tag{3.5}$$

where we have put $h =< \nabla_X X, \phi X >$. Similarly, we can put

$$\nabla_{\phi X} X = \gamma W + i\phi X, \tag{3.6}$$

where we have put $i =< \nabla_{\phi X} X, \phi X >$. From (3.4) and (3.5) we obtain, by the help of (2.6), (3.2), (3.4) and (3.5),

$$\begin{aligned} \nabla_X \nabla_W X &= (Xe - f\beta)W - fhX + (e\beta + Xf)\phi X, \\ \nabla_W \nabla_X X &= -bhW - fhX + (Wh)\phi X, \\ \nabla_X W - \nabla_W X &= -eW + (\beta - f)\phi X. \end{aligned}$$

Since

$$< \nabla_X \nabla_W X - \nabla_W \nabla_X X - \nabla_{\nabla_X W - \nabla_W X} X, W >=< R(X,W)X, W >$$

and

$$< \nabla_X \nabla_W X - \nabla_W \nabla_X X - \nabla_{\nabla_X W - \nabla_W X} X, \phi X >=< R(X,W)X, \phi X >,$$

we have from (2.2), (3.2), (3.4), and (3.6),

$$Xe = f\beta - bh - (f - \beta)\gamma - e^2 - c + b^2 - \alpha\beta, \tag{3.7}$$

$$Wh - Xf = 2e\beta + ef - (\beta - f)i. \tag{3.8}$$

Since $M$ has the parallel Ricci tensor $S$, we have from the first equation of (3.3)

$$\begin{aligned} S\nabla_W W = {} & W(\alpha\beta + \alpha\gamma - b^2 - e^2)W + (2c + \alpha\beta + \alpha\gamma - b^2 - e^2)\nabla_W W \\ & + W(b\gamma)X + b\gamma\nabla_W X + W(e\beta)\phi X \\ & + e\beta(\nabla_W \phi)X + e\beta\phi\nabla_W X, \end{aligned}$$

from which and (2.5) and (2.6) we get, by the help of (3.2) and (3.3),

$$\begin{aligned} & b\{e\beta W - beX + (5c + \alpha\gamma + \beta\gamma - e^2)\phi X\} \\ & - e\{b\gamma W + (5c + \alpha\beta + \beta\gamma - b^2)X - be\phi X\} \\ = {} & W(\alpha\beta + \alpha\gamma - b^2 - e^2)W + b(2c + \alpha\beta + \alpha\gamma - b^2 - e^2)\phi X \\ & - e(2c + \alpha\beta + \alpha\gamma - b^2 - e^2)X \\ & + W(b\gamma)X + b\gamma\nabla_W X + W(e\beta)\phi X - be\beta W + e\beta\phi\nabla_W X. \end{aligned} \tag{3.9}$$

Taking inner product (3.9) with $W$, $X$ and $\phi X$, respectively, we have

$$W(\alpha\beta + \alpha\gamma - b^2 - e^2) = 2be(\beta - \gamma), \tag{3.10}$$

$$W(b\gamma) = e(-3c + \alpha\gamma - \beta\gamma - b^2 - e^2 + f\beta), \tag{3.11}$$

$$W(e\beta) = b(3c - \alpha\beta + \beta\gamma + b^2 + e^2 - f\gamma). \tag{3.12}$$

Differentiating the first equation of (3.3) with respect to $X$ covariantly, we obtain, by the help of (2.5), (2.6), (3.2), (3.3) and (3.5),

$$X(\alpha\beta + \alpha\gamma - b^2 - e^2) = 2e\beta^2, \tag{3.13}$$

$$X(b\gamma) = (h - b)e\beta, \tag{3.14}$$

$$X(e\beta) = \beta(3c + \beta\gamma - \alpha\beta + b^2) - bh\gamma. \tag{3.15}$$

Differentiating the first equation of (3.3) with respect to $\phi X$ and taking account of (2.5), (2.6), (3.2), (3.3) and (3.6), we have

$$\phi X(\alpha\beta + \alpha\gamma - b^2 - e^2) = -2b\gamma^2, \tag{3.16}$$

$$\phi X(b\gamma) = -\gamma(3c + \beta\gamma - \alpha\gamma + e^2) + ie\beta, \tag{3.17}$$

$$\phi X(e\beta) = b\gamma(e - i). \tag{3.18}$$

Differentiating the second equation of (3.3) with respect to $W$, $X$ and $\phi X$, respectively and taking inner product the resulting equations with $X$ and $\phi X$, respectively, we obtain

$$W(\alpha\beta + \beta\gamma - b^2) = 2be(\gamma - f), \tag{3.19}$$

$$W(be) = -f(\alpha\gamma - \alpha\beta + b^2 - e^2) - e^2\beta + b^2\gamma, \tag{3.20}$$

$$X(\alpha\beta + \beta\gamma - b^2) = -2beh, \tag{3.21}$$

$$X(be) = -h(\alpha\gamma - \alpha\beta + b^2 - e^2) + b\beta\gamma, \tag{3.22}$$

$$\phi X(\alpha\beta + \beta\gamma - b^2) = 2b\gamma^2 - 2bei, \tag{3.23}$$

$$\phi X(be) = -i(\alpha\gamma - \alpha\beta + b^2 - e^2) - e\beta\gamma. \tag{3.24}$$

Differentiating the third equation of (3.3) with respect to $W$, $X$ and $\phi X$, respectively and taking inner product the resulting equations with respect to $\phi X$, we have

$$W(\alpha\gamma + \beta\gamma - e^2) = 2be(f - \beta), \tag{3.25}$$

$$X(\alpha\gamma + \beta\gamma - e^2) = 2e(bh - \beta^2), \tag{3.26}$$

$$\phi X(\alpha\gamma + \beta\gamma - e^2) = 2bei. \tag{3.27}$$

From (3.3), the scalar curvature $s$ of $M$ is given by

$$s = 12c + 2(\alpha\beta + \alpha\gamma + \beta\gamma - b^2 - e^2). \tag{3.28}$$

Since the Ricci tensor $S$ is parallel, we have from (3.28)

$$Z(\alpha\beta + \alpha\gamma + \beta\gamma - b^2 - e^2) = 0 \tag{3.29}$$

for every vector field $Z$. Hence we have from (3.10), (3.13), (3.16), (3.19), (3.21), (3.23), (3.25), (3.26), (3.27) and (3.29)

$$W(\beta\gamma) = 2be(\gamma - \beta), \tag{3.30}$$

$$X(\beta\gamma) = -2e\beta^2, \tag{3.31}$$

$$\phi X(\beta\gamma) = 2b\gamma^2, \tag{3.32}$$

$$W(e^2 - \alpha\gamma) = 2be(\gamma - f), \tag{3.33}$$

$$X(e^2 - \alpha\gamma) = -2beh, \tag{3.34}$$

$$\phi X(e^2 - \alpha\gamma) = 2b(\gamma^2 - ei), \tag{3.35}$$

$$W(b^2 - \alpha\beta) = 2be(f - \beta), \tag{3.36}$$

$$X(b^2 - \alpha\beta) = 2e(bh - \beta^2), \tag{3.37}$$

$$\phi X(b^2 - \alpha\beta) = 2bei. \tag{3.38}$$

From $\nabla_X\nabla_W W - \nabla_W\nabla_X W - \nabla_{\nabla_X W - \nabla_W X} W = R(X, W)W$, we have

$$Xb - W\beta = -2be + eh. \tag{3.39}$$

Since $\{[X, W] - (\nabla_X W - \nabla_W X)\}(\beta\gamma) = 0$, we have from (2.5), (3.2), (3.4), (3.30), (3.31) and (3.32)

$$\begin{aligned}(W\beta - Xb)e\beta - bX(e\beta) + \beta W(e\beta) + b\gamma X(e)\\ + eX(b\gamma) - (\beta - f)b\gamma^2 + e(-be\beta + be\gamma) = 0.\end{aligned}$$

Substituting (3.7), (3.12), (3.14), (3.15) and (3.39) into the above equation, we find

$$be^2\beta = (c - b^2 + \alpha\beta)b\gamma. \tag{3.40}$$

From $\nabla_{\phi X}\nabla_W X - \nabla_W\nabla_{\phi X} X - \nabla_{\nabla_{\phi X} W - \nabla_W \phi X} X = R(\phi X, W)X$, we obtain

$$(\phi X)e - W\gamma = b(2e - i), \tag{3.41}$$

$$(\phi X)f - Wi = 2b\gamma + bf + h(f - \gamma). \tag{3.42}$$

From $\nabla_{\phi X}\nabla_W W - \nabla_W\nabla_{\phi X} W - \nabla_{\nabla_{\phi X} W - \nabla_{W(\phi X)}} W = R(\phi X, W)W$, we obtain

$$(\phi X)b = ei - f\gamma + b^2 + \beta(f - \gamma) + \alpha\gamma - e^2 + c. \tag{3.43}$$

Since $\{[\phi X, W] - (\nabla_{\phi X} W - \nabla_W(\phi X))\}(\beta\gamma) = 0$, we have

$$\begin{aligned}- e\beta(\phi X)b - b(\phi X)(e\beta) + [(\phi X)e - W\gamma]b\gamma - \gamma W(b\gamma) + e(\phi X)(b\gamma)\\ + (f - \gamma)e\beta^2 + b^2 e(\beta - \gamma) = 0.\end{aligned}$$

Substituting (3.11), (3.17), (3.18), (3.41) and (3.43) into this equation, we find

$$b^2 e\gamma = e\beta(\alpha\gamma - e^2 + c). \tag{3.44}$$

From $\nabla_{\phi X}\nabla_X X - \nabla_X \nabla_{\phi X} X - \nabla_{\nabla_{\phi X}X - \nabla_X(\phi X)} X = R(\phi X, X)X$, we have

$$X\gamma = -2e\beta - (\gamma - \beta)i - e\gamma, \tag{3.45}$$

$$(\phi X)h - Xi = 4c + 2\beta\gamma + (\beta + \gamma)f + h^2 + i^2. \tag{3.46}$$

From $\nabla_{\phi X}\nabla_X W - \nabla_X \nabla_{\phi X} W - \nabla_{\nabla_{\phi X}X - \nabla_X(\phi X)} W = R(\phi X, X)W$, we have

$$(\phi X)\beta = (\beta - \gamma)h + b(\beta + 2\gamma). \tag{3.47}$$

From $\{[\phi X, X] - (\nabla_{\phi X} X - \nabla_X(\phi X))\}(e^2 - \alpha\gamma) = 0$, we obtain, by the help of (3.33), (3.34) and (3.35),

$$\begin{aligned} - be\{(\phi X)h - Xi\} - h(\phi X)(be) - \gamma X(b\gamma) - b\gamma X(\gamma) + iX(be) \\ - be(\beta + \gamma)(\gamma - f) + beh^2 - bi(\gamma^2 - ie) = 0. \end{aligned}$$

Substituting (3.14), (3.22), (3.24), (3.45) and (3.46) into this equation, we find

$$be = 0. \tag{3.48}$$

Now, we shall show that we have a contradiction in each case of $b \neq 0$ and $e \neq 0$ in $\mathcal{U}$.

Case 1. $b \neq 0$ in $\mathcal{U}$.

In this case, we have $e = 0$ in $\mathcal{U}$ from (3.48).

Firstly, we shall show that

$$i = 0 \quad \text{in } \mathcal{U}. \tag{3.49}$$

To show (3.49), assume that $i \neq 0$ at a point $Q \in \mathcal{U}$. Since we have $i(\alpha\gamma - \alpha\beta + b^2) = 0$ in $\mathcal{U}$ from (3.24), we have $\alpha\gamma - \alpha\beta + b^2 = 0$ in an open neighborhood $\mathcal{U}' \subset \mathcal{U}$ of the point $Q$. Hence we have $\gamma = 0$ in $\mathcal{U}'$ from (3.20) and hence we get $c = 0$ from (3.12) and $b^2 - \alpha\beta = 0$. This is absurd.

Secondly, we shall show that

$$b^2 - \alpha\beta = c \quad \text{in } \mathcal{U}. \tag{3.50}$$

To show (3.50), assume that $b^2 - \alpha\beta - c \neq 0$ at a point $Q \in \mathcal{U}$. Then we have $\gamma = 0$ in an open neighborhood $\mathcal{V} \subset \mathcal{U}$ of $Q$ from (3.40). Since $3c - \alpha\beta + b^2 = 0$ in $\mathcal{V}$ from (3.12), we get $f = 0$ in $\mathcal{V}$ from (3.20). Hence we get $bh = -4c$ in $\mathcal{V}$ from (3.7). Differentiating $bh = -4c$ with respect to $\phi X$ covariantly in $\mathcal{V}$, we find, by the help of (3.43), (3.46) and (3.49),

$$\begin{aligned} b(\phi X)h + h(\phi X)b &= b(4c + h^2) + h(b^2 + c) \\ &= 4bc - 4ch - 4bc + ch = -3ch = 0, \end{aligned}$$

which shows that $h = 0$. So, we get $c = -\frac{1}{4}bh = 0$. This is absurd.

Thirdly, we shall show that

$$h = 0 \quad \text{in } \mathcal{U}. \tag{3.51}$$

To show this, we start with the equation $4c + \beta\gamma = f\gamma$ from (3.12) and (3.50). Thus we get

$$\gamma\phi X(\beta - f) + (\beta - f)\phi X(\gamma) = 0,$$

which implies, by the help of (3.42), (3.47) and (3.49),

$$(\beta - f)(\gamma b + \gamma h + \phi X(\gamma)) = 0 \quad \text{in } \mathcal{U}.$$

Since $\beta \neq f$ from $4c + \beta\gamma = f\gamma$, we have $(\phi X)\gamma = -\gamma(b + h)$. From (3.32), we have $\gamma\phi X(\beta) + \beta\phi X(\gamma) = 2b\gamma^2$. Hence we have $h\gamma = 0$ in $\mathcal{U}$ from (3.47).

If $h \neq 0$ at a point $Q$ in $\mathcal{U}$, then we have $\gamma = 0$ in an open neighborhood $\mathcal{V} \subset \mathcal{U}$ of $Q$. Then we have $3c = \alpha\beta - b^2$ from (3.12). This is impossible because of (3.50). Hence $h = 0$ in $\mathcal{U}$.

From (3.22) and (3.51), we have $\beta\gamma = 0$ and hence $\gamma = 0$ in $\mathcal{U}$ from (3.32). Thus we obtain $3c = \alpha\beta - b^2$ from (3.12). The equations $3c = \alpha\beta - b^2$ and (3.50) imply $c = 0$, which contradicts to the hypothesis.

Therefore we have a contradiction in the case of $b \neq 0$ in an open set $\mathcal{U}$ of $M$.

Case 2. $e \neq 0$ in $\mathcal{U}$.

In this case, we have $b = 0$ in $\mathcal{U}$ from (3.48). From (3.4?) (3.14), (3.11), we get in $\mathcal{U}$ respectively

$$\beta(\alpha\gamma - e^2 + c) = 0, \tag{3.52}$$

$$h\beta = 0, \tag{3.53}$$

$$3c + \beta\gamma = \alpha\gamma - e^2 + f\beta. \tag{3.54}$$

From (3.22) and (3.53), we get

$$h(\alpha\gamma - e^2) = 0. \tag{3.55}$$

If $h \neq 0$ at a point $Q \in \mathcal{U}$, then we have from (3.53) and (3.55)

$$\beta = 0 \quad \text{and} \quad \alpha\gamma - e^2 = 0$$

in an open neighborhood $\mathcal{V} \subset \mathcal{U}$ of $Q$. Hence we have $c = 0$ from (3.54). This is impossible. Thus we have in $\mathcal{U}$

$$h = 0. \tag{3.56}$$

We shall show that

$$e^2 - \alpha\gamma - c = 0 \quad \text{in } \mathcal{U}. \tag{3.57}$$

To show this, assume that $e^2 - \alpha\gamma - c \neq 0$ at a point $Q$ in $\mathcal{U}$. Then we obtain $e^2 - \alpha\gamma - c \neq 0$ in open neighborhood $\mathcal{V} \subset \mathcal{U}$ of $Q$. Hence we get $\beta = 0$ in $\mathcal{V}$ from (3.52) and hence $\alpha\gamma - e^2 = 3c$ from (3.54). Moreover we get $f(\alpha\gamma - e^2) = 0$ and $i(\alpha\gamma - e^2) = 0$ in $\mathcal{V}$ from (3.20) and (3.24), respectively, which implies $f = i = 0$ in $\mathcal{V}$. Since $f = i = h = \beta = 0$ in $\mathcal{V}$, we have $c = 0$ from (3.46). This is absurd. Hence we get $e^2 - \alpha\gamma - c = 0$ in $\mathcal{U}$.

From (3.54) and (3.57), we get in $\mathcal{U}$

$$4c + \beta\gamma = f\beta. \tag{3.58}$$

From (3.17) we find, by the help of (3.57) and (3.58), $\beta(ei - f\gamma) = 0$ in $\mathcal{U}$. If $ei - f\gamma \neq 0$ at a point $Q$ in $\mathcal{U}$, then $\beta = 0$ at $Q$. This is impossible from (3.58). Hence we have in $\mathcal{U}$

$$ei = f\gamma. \tag{3.59}$$

Substituting (3.57), (3.58) and (3.59) into (3.43), we have $c = 0$. This contradicts to our hypothesis $c \neq 0$. Thus we have a contradiction in the case of $e \neq 0$ in $\mathcal{U}$.

Summing up, if we assume that $\delta$ is always zero in $\mathcal{U}$, then we can deduce a contradiction.

Case B. Assume that $\delta$ is not zero at some point $Q$ of $\mathcal{U}$.

Then, $\delta$ is not zero in some open neighborhood $\mathcal{W}' \subset \mathcal{U}$ of $Q$. In this case, we can choose another local frame field $W$, $X'$, $\phi X'$ in the open neighborhood $\mathcal{W}'$ by

$$\begin{aligned} X' &= \cos\theta X + \sin\theta\phi X, \\ \phi X' &= -\sin\theta X + \cos\theta\phi X, \end{aligned}$$

where $\theta(0 < \theta < \frac{\pi}{2})$ is determined by $\cot 2\theta = \frac{\beta-\gamma}{2\delta}$. Then $\theta$ is a differentiable function in the open neighborhood and we have

$$\begin{aligned} AW &= aW + (b\cos\theta + e\sin\theta)X' + (-b\sin\theta + e\cos\theta)\phi X', \\ AX' &= (b\cos\theta + e\sin\theta)W + (\beta\cos^2\theta + \gamma\sin^2\theta + 2\delta\sin\theta\cos\theta)X', \\ A\phi X' &= (-b\sin\theta + e\cos\theta)W + (\beta\sin^2\theta + \gamma\cos^2\theta - 2\delta\sin\theta\cos\theta)\phi X'. \end{aligned}$$

Therefore we have the following form of equations in $\mathcal{W}'$ instead of (3.2)

$$(3.60)\qquad \begin{aligned} AW &= \alpha W + b'X' + e'\phi X', \\ AX' &= b'W + \beta' X', \\ A\phi X' &= e'W + \gamma'\phi X'. \end{aligned}$$

Since $W$ is not principal in $\mathcal{U}$, it is also not principal in $\mathcal{W}'$. Hence $b' \neq 0$ in an open neighborhood $\mathcal{W}'_1 \subset \mathcal{U}$ of $Q$ or $e' \neq 0$ in an open neighborhood $\mathcal{W}'_2 \subset \mathcal{U}$ of $Q$. Thus, $b' \neq 0$ in an open neighborhood $\mathcal{W} \subset \mathcal{U}$ of $Q$ or $e' \neq 0$ in an open neighborhood $\mathcal{W} \subset \mathcal{U}$ of $Q$. Since the function $\delta'$ corresponding to $\delta$ is always zero in $\mathcal{W}$, the situation is same to the Case A and we also have a contradiction.

Thus we have the following:

THEOREM 3.1. *There does not exist a Ricci-parallel real hypersurface in $P_2\mathbb{C}$ or $H_2\mathbb{C}$.*

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