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THE INDEX FORM ON THE MULTIPLY WARPED SPACETIME

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ABSTRACT. In this paper we find index form of the multiply warped product manifolds to investigate the physical properties from spacetime.

1. Introduction

The conditions of spacelike boundaries in the multiply warped product spacetimes [4] were also studied and the curvature of the multiply warped product with C^0-warping functions was later investigated [1]. From a physical point of view, these warped product spacetimes are interesting since they include classical examples of spacetime such as the Robertson-Walker manifold and the intermediate zone of RN manifold [3, 7]. Recently, the warped product scheme has been applied to higher dimensional theories such as the Randall-Sundrum model [6, 8] in five dimensions and the non-singular warped Kaluza-Klein embeddings [2] in five to seven dimensional gauged super gravity theories. Moreover, J. Choi, S. Hong and Y. Park [1] studied the multiply warped product manifold associated with the charged black holes such as the Banados-Teitelboim-Zanelli (BTZ) and de Sitter (dS) metrics to investigate the physical properties inside the event horizons.

The Lorentzian index form should be related to the absence of conjugate points in the same way that the positive definiteness of the Riemannian index form is guaranteed by the nonexistence of conjugate points [5]. In order to give a proof of the negative semi-definiteness of the Lorentzian index form in the absence of conjugate points, we find

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index form of the multiply warped products manifold to investigate the physical properties from spacetime.

2. Index form of the multiply warped product

One can also generalize warped product to multiply warped products. Briefly, a multiply warped product (M, g) is a product manifold of the form $M = B \times_{f_1} F_1 \times \cdots \times_{f_n} F_n$ with the metric $g = g_B + f_1^2 g_{F_1} + f_2^2 g_{F_2} + \cdots + f_n^2 g_{F_n}$, where for each $i = 1, \cdots, n$, $f_i : B \to \mathbb{R}^+$ is smooth and $(B, g_B), (F_i, g_{F_i})$ are semi-Riemannian manifolds. In particular, when B is a interval with $g_B = -dt^2$, the corresponding multiply warped product $M = B \times_{f_1} F_1 \times \cdots \times_{f_n} F_n$ is called a multiply warped spacetime, where (F_i, g_{F_i}) is a Riemannian manifold.

We suppose that γ is a timelike curve. Then the warping functions f_i satisfies

$$\sum_{i=1}^{n} f_i^2 < \sum_{i=1}^{n} \frac{\| \alpha' \|_I^2}{\| \beta_i' \|_{F_i}^2}.$$

PROPOSITION 2.1. *Let $M = I \times_{f_1} F_1 \times \cdots \times_{f_n} F_n$ be a multiply warped product with $g_I = -1$ and warping function $f_i : I \to \mathbb{R}^+$, $i = 1, \cdots, n$. Let x be a variation of a timelike curve segment $\gamma = (\alpha, \beta_1, \cdots, \beta_n) : [a, b] \to M$. If L is the length function of $x = (x_\alpha, x_{\beta_1}, \cdots, x_{\beta_n}) : [a, b] \times (-\delta, \delta) \to M$ then*

$$L'(0)$$
$$= -\int_a^b \frac{1}{\sqrt{\| \alpha' \|_I^2 - \Sigma f_i^2 \| \beta_i' \|_{F_i}^2}} [g(\gamma', V') + \sum \frac{f_i \partial f_i}{\partial v}\Big|_{v=0} \| \beta_i' \|_{F_i}^2] du,$$

where V is the variation vector field on γ and $\sum = \sum_{i=1}^{n}$.

Proof. $L(v) = \int_b^a \| x_u(u, v) \| du$, if the $v-$interval $(-\delta, \delta)$ is small enough, $\| x_u \|$ is positive, hence differentiable. Then

$$L'(0) = \int_b^a [\frac{d}{dv} \| x_u \|]_{v=0} du.$$

since $\| x_u \| = (-g(x_u, x_u))^{1/2}$, we compute

$$\frac{d}{dv} \| x_u \| = \frac{1}{2}\Big[- g_I((x_\alpha)_u, (x_\alpha)_u) - \sum_{i=1}^{n} f_i^2 g_{F_i}((x_{\beta_i})_u, (x_{\beta_i})_u)\Big]^{-1/2}$$
$$(-2)\Big[g_I((x_\alpha)_u, (x_\alpha)_{uv}) + \sum_{i=1}^{n}(\frac{f_i \partial f_i}{\partial v} g_{F_i}((x_{\beta_i})_u, (x_{\beta_i})_u)$$
$$+ f_i^2 g_{F_i}((x_{\beta_i})_u, (x_{\beta_i})_{uv}))\Big].$$

Setting $v = 0$ gives $x_u(u,0) = ((x_\alpha)_u, (x_{\beta_i})_u)\Big|_{(u,0)} = (\alpha'(u), \beta_i'(u))$ and, by $V(u) = x_v(u,0) = ((x_\alpha)_v, (x_{\beta_i})_v)\Big|_{(u,0)} = (V_I(u), V_{F_i}(u))$.

$$(x_{\beta_i})_{uv} = \frac{\partial^2 x^t}{\partial u \partial v} + \Gamma^t_{mn} \frac{\partial x^m}{\partial u}\frac{\partial x^n}{\partial v}, \; V'_{F_i}(u) = (x_{\beta_i})_{uv}\Big|_{v=0}.$$

The result follows. □

LEMMA 2.2. *Let γ be timelike geodesic curve with unit speed curves α, β_i respectively. If $V \in V^{\perp}(\gamma)$ then*

$$L'(0) = -\int_a^b \frac{1}{\sqrt{1 - \sum f_i^2}} \frac{f_i \partial f_i}{\partial v}\Big|_{v=0} du,$$

where $\sum = \sum_{i=1}^{n}$.

Proof. The result came from $g(\gamma', V') = 0$ and $\| \alpha' \|_I = \| \beta_i' \|_F = 1$ in the Proposition 2.1. □

PROPOSITION 2.3. *Let $M = I \times_{f_1} F_1 \times \cdots \times_{f_n} F_n$ be a multiply warped product with $g_I = -dt^2$ and warping function $f_i : I \to \mathbb{R}^+$, $i = 1, \cdots, n$. Let $\gamma = (\alpha, \beta_i)$ be a timelike curve and $\alpha(\beta_i)$ have unit speed on $I(F_i)$. Then we have the second variation as follows*

$$L''(0) = \int_a^b [1 - \sum_{i=1}^{n} f_i^2]^{-3/2} [g(\gamma', V') + \sum_{i=1}^{n} \frac{f_i \partial f_i}{\partial v}\Big|_{v=0}]^2 du$$
$$- \int_a^b [1 - \sum_{i=1}^{n} f_i^2]^{-1/2} [g(V', V') + g(\gamma', R(V, \gamma')V) + g(\gamma', A')$$
$$+ \sum_{i=1}^{n} (\frac{f_i \partial^2 f_i}{\partial v^2} + (\frac{\partial f_i}{\partial v})^2 + 4 \frac{f_i \partial f_i}{\partial v} g_{F_i}(\beta', V'_{F_i}))\Big|_{v=0}] du.$$

Proof.

$$
\begin{aligned}
&-\frac{\partial}{\partial v}\big[-g_I((x_\alpha)_u,(x_\alpha)_u)-\sum_{i=1}^n f_i^2 g_{F_i}((x_{\beta_i})_u,(x_{\beta_i})_u)\big]^{-1/2}\\
&\big[g_I((x_\alpha)_u,(x_\alpha)_{uv})+\sum_{i=1}^n(\frac{f_i\partial f_i}{\partial v}g_{F_i}((x_{\beta_i})_u,(x_{\beta_i})_u)\\
&+f_i^2 g_{F_i}((x_{\beta_i})_u,(x_{\beta_i})_{uv}))\big]\\
=&-\frac{1}{2}\big[-g_I((x_\alpha)_u,(x_\alpha)_u)-\sum_{i=1}^n f_i^2 g_{F_i}((x_{\beta_i})_u,(x_{\beta_i})_u)\big]^{-3/2}\\
&2\big[g_I((x_\alpha)_u,(x_\alpha)_{uv})+\sum_{i=1}^n(\frac{f_i\partial f_i}{\partial v}g_{F_i}((x_{\beta_i})_u,(x_{\beta_i})_u)\\
&+f_i^2 g_{F_i}((x_{\beta_i})_u,(x_{\beta_i})_{uv}))\big]^2-\big[-g_I((x_\alpha)_u,(x_\alpha)_u)\\
&-\sum_{i=1}^n f_i^2 g_{F_i}((x_{\beta_i})_u,(x_{\beta_i})_u)\big]^{-1/2}\big[g_I((x_\alpha)_{uv},(x_\alpha)_{uv})\\
&+g_I((x_\alpha)_u,(x_\alpha)_{uvv})+\sum_{i=1}^n(\frac{f_i\partial^2 f_i}{\partial v^2}+(\frac{\partial f_i}{\partial v})^2)g_{F_i}((x_{\beta_i})_u,(x_{\beta_i})_u)\\
&+4\frac{f_i\partial f_i}{\partial v}g_{F_i}((x_{\beta_i})_u,(x_{\beta_i})_{uv})+f_i^2(g_{F_i}((x_{\beta_i})_{uv},(x_{\beta_i})_{uv})\\
&+g_{F_i}((x_{\beta_i})_u,(x_{\beta_i})_{uvv}))\big].
\end{aligned}
$$

We get

$$
\begin{aligned}
&L''(0)\\
=&\int_a^b[1-\sum_{i=1}^n f_i^2]^{-3/2}[g_I(\alpha',V_B')+\sum_{i=1}^n(\frac{f_i\partial f_i}{\partial v}\Big|_{v=0}+f_i^2 g_{F_i}(\beta',V_{F_i}')\big]^2du\\
&-\int_a^b[1-\sum_{i=1}^n f_i^2]^{-1/2}\big[g_I(V_I',V_I')+g_I(\alpha',R^I(V,\gamma')V)\\
&+g_I(\alpha',A_I')+\sum_{i=1}^n(\frac{f_i\partial^2 f_i}{\partial v^2}+(\frac{\partial f_i}{\partial v})^2)\Big|_{v=0}+4\frac{f_i\partial f_i}{\partial v}\Big|_{v=0}g_{F_i}(\beta',V_{F_i}')\\
&+f_i^2(g_{F_i}(V_{F_i}',V_{F_i}')+g_{F_i}(\beta',R^{F_i}(V,\gamma')V)+g_{F_i}(\beta',A_{F_i}'))\big]du.
\end{aligned}
$$

□

THEOREM 2.4. *If the timelike geodesic $\gamma = (\alpha, \beta_i)$ satisfies $V \in V^{\perp}(\gamma)$ with unit speed curves α, β_i respectively. Then the second variation*

$$L''(0) = -\int_a^b [1 - \sum_{i=1}^n f_i^2]^{-1/2} [\sum_{i=1}^n \frac{f_i \partial^2 f_i}{\partial v^2}\bigg|_{v=0} - g(V, R(V, \gamma')\gamma') + g(V', V') + g(\gamma', A')] du,$$

where A is smooth and f on the u, v-variation coordinate has a extremal at $v = 0$.

We now obtain definition of the Lorentzian multiply warped product index form I.

DEFINITION 2.5. Let $\gamma = (\alpha, \beta_i)$ be a timelike geodesic in $M = I \times_{f_i} F_i$ and f on the u, v-variation coordinate has a extremal at $v = 0$, then the index form $I : V^{\perp}(\gamma) \times V^{\perp}(\gamma) \to \mathbb{R}$ is the symmetric bilinear form given by

$$I(X, Y) = -\int_a^b [\parallel \alpha' \parallel^2 - \sum_{i=1}^n f_i^2 \parallel \beta_i' \parallel^2]^{-1/2} [\sum_{i=1}^n \frac{f_i \partial^2 f_i}{\partial v^2}\bigg|_{v=0} \parallel \beta_i' \parallel^2 - g(Y, R(X, \gamma')\gamma') + g(X', Y') + g(\gamma', A')] du.$$

If $X \in V^{\perp}(\gamma)$ is smooth, we also have

$$I(X, Y) = -\int_a^b [1 - \sum_{i=1}^n f_i^2]^{-1/2} [\sum_{i=1}^n \frac{f_i \partial^2 f_i}{\partial v^2}\bigg|_{v=0} + \frac{\partial}{\partial u} g(X', Y) - g(R(X, \gamma')\gamma' + X'', Y) - 2\sum f_i \frac{\partial f_i}{\partial u} g_{F_i}(X'_{F_i}, Y_{F_i}) + \frac{\partial}{\partial u} g(\gamma', A) - 2\sum f_i \frac{\partial f_i}{\partial u} g_{F_i}(\beta', A_{F_i})] du.$$

Thus X is a (smooth) Jacobi vector field and there exists f such that

$$0 = \sum_{i=1}^n \frac{f_i \partial^2 f_i}{\partial v^2}\bigg|_{v=0} \parallel \beta_i' \parallel^2 + \frac{\partial}{\partial u} g(X', Y) + \frac{\partial}{\partial u} g(\gamma', A),$$

for all $Y \in V_0^{\perp}(\gamma)$, if and only if $I(X, Y) = 0$.

EXAMPLE 2.6. Let $X = aU + bY = (X^t, X^r, X^\theta, X^\varphi) \in V^{\perp}(\gamma)$ be a timelike vector field and $\gamma' = U + Z$. Since $g(X, X) = -a^2 + b^2 g(Y, Y) < 0$, $g(Y, Y) < \frac{a^2}{b^2}$. Then

(a) $\frac{dX^t}{dt} = \frac{d(aU)}{dt} + 2\Gamma^t_{\theta\theta} X^\theta \frac{d\gamma^\theta}{dt} + 2\Gamma^t_{\varphi\varphi} X^\varphi \frac{d\gamma^\varphi}{dt} = 0.$

(b) $0 \le g(Z, Z) < 1,$

(c) Since $X \in V^\perp(\gamma)$, $g(Y, Z) = \frac{a}{b}$,

(d)
$$\begin{aligned} & g(X, \mathrm{R}(X, \gamma')\gamma') \\ &= 2\frac{a^2 f''}{f} - \frac{a^2 f'' g(Z, Z)}{f} - \frac{b^2 f'' g(Y, Y)}{f} \\ &\quad - \frac{(f')^2}{f^2}(a^2 - b^2 g(Z, Z) g(Y, Y)). \end{aligned}$$

Let $f(u)$ be defined by $f(u) = u$, on $u \in [1, \infty)$, and $f(u, v) = u \cosh v$, on $v \in [0, \delta) \subset (-\delta, \delta)$. Then $f(u, 0) = f(u)$ and $\left.\frac{\partial^2 f}{\partial v^2}\right|_{v=0} > 0$. We compute the index form

$$\begin{aligned} & I(X, X) \\ &= -\int_a^b \sqrt{1 - \frac{c^2}{f^2 + c^2}} \Big[\left.\frac{f \partial^2 f}{\partial v^2}\right|_{v=0} g(Z, Z) \\ &\quad - g(X, R(X, \gamma')\gamma') + g(X', X') + g(\gamma', A') \Big] dt, \\ &= -\int_a^b \sqrt{1 - \frac{c^2}{f^2 + c^2}} \Big[\left.\frac{f \partial^2 f}{\partial v^2}\right|_{v=0} g(Z, Z) - 2\frac{a^2 f''}{f} + \frac{a^2 f'' g(Z, Z)}{f} \\ &\quad + \frac{b^2 f'' g(Y, Y)}{f} + \frac{(f')^2 + k}{f^2}(a^2 - b^2 g(Z, Z) g(Y, Y)) \\ &\quad + g(X', X') + g(\gamma', A') \Big] dt. \end{aligned}$$

For $f'' = 0$, then $I(X, X) < 0$.

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