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AN EMBEDDING OF BIRGET-RHODES EXPANSION OF GROUPS INTO A SEMIDIRECT PRODUCT

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Abstract. In this paper, we prove that the Birget-Rhodes expansion $\tilde{G}^{\mathcal{R}}$ of a group G is not a semidirect product of a semilattice by a group but it can be nicely embedded into such a semidirect product.

1. Introduction

An *inverse semigroup* S is a semigroup in which for every $s \in S$ there exists a unique element s^{-1}, called the inverse of s, satisfying $ss^{-1}s = s, s^{-1}ss^{-1} = s^{-1}$. The Wagner-Preston representation theorem states that every inverse monoid can be embedded in a symmetric inverse monoid $I(X)$ on a set X, which consists of all partial bijections on the set X under the usual operation of composition of partial functions.

In [6], Exel constructed, in a canonical way, an inverse monoid $\mathcal{S}(G)$ associated to a group G defined via generators and relations. One of main results of Exel is the one-to-one correspondence between actions of $\mathcal{S}(G)$ (An action of an inverse semigroup S on the set X is a homomorphism from S to the symmetric inverse monoid $I(X)$) and the partial actions of G, with its applications on graded C^*-algebras. Moreover in what sense $\mathcal{S}(G)$ is a universial inverse monoid. But in [7], Kellendonk and Lawson realized that the inverse monoid $\mathcal{S}(G)$ is nothing other than a semigroup known as the Birget-Rhodes expansion $\tilde{G}^{\mathcal{R}}$ of the group G, hence all algebraic information of $\mathcal{S}(G)$ read off the Birget-Rhodes expansion of the group G. In [4], the authors proved that if the group G acts faithfully on a Hausdorff space X as homeomorphisms and acts freely at an non-isolated point $x_0 \in X$ then $\tilde{G}^{\mathcal{R}}$ is isomorphic to the inverse monoid of Möbius type of the form $\langle \mathrm{Part}(G, X \setminus \{x_0\}) \rangle$. Also in

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[3], the authors proved that an inverse monoid of Möbius type can be embedded into a semidirect product of a semilattice by a group.

It is natural to enquire that $\tilde{G}^{\mathcal{R}}$ has a semidirect product of a semilattice by a group or it can be embedded into such a semidirect product. In this paper, we construct such a semidirect product containing an isomorphic copy of $\tilde{G}^{\mathcal{R}}$.

2. The inverse monoid $\tilde{G}^{\mathcal{R}}$

Let S be a semigroup. For any finite sequence $(s_1, s_2, \ldots, s_n)$ of elements $s_1, s_2, \ldots, s_n$ in S. Put

$$P(s_1, s_2, \ldots, s_n) := \{1, s_1, s_1s_2, \ldots, s_1s_2\cdots s_n\},$$

where 1 is the identity of S^1. Define

$$\tilde{S}^{\mathcal{R}} := \{(P(s_1, s_2, \ldots, s_n), s_1s_2\cdots s_n) : s_1, s_2, \ldots, s_n \in S, n \geq 1\}$$

with multiplication

$$\begin{aligned}&(P(s_1, s_2, \ldots, s_n), s_1s_2\cdots s_n)(P(t_1, t_2, \ldots, t_m), t_1t_2\cdots t_m)\\ &= (P(s_1, s_2, \ldots, s_n) \cup (s_1s_2\cdots s_n)\cdot P(t_1, t_2, \ldots, t_m), s_1s_2\cdots s_nt_1t_2\cdots t_m)\end{aligned}$$

where $s \cdot U = \{su : U \in U\}$ for every $s \in S$ and $U \subset S$. Then $\tilde{S}^{\mathcal{R}}$ is a semigroup, which is called the *Birget-Rhodes expansion* of the semigroup S (See [1], [2], and [10]).

For an arbitrary group G, denote by $\mathcal{P}_1(G)$ the set of all finite subsets of G containing the identity 1. Let $\imath : G \to \tilde{G}^{\mathcal{R}}$ be defined by $\imath(g) = (\{1, g\}, g)$.

The following proposition is appeared in [2], [7], and [10]

PROPOSITION 2.1. *For any group G, we have*

(i) $\tilde{G}^{\mathcal{R}} = \{(A, g) \in \mathcal{P}_1(G) \times G : g \in G\}$,
(ii) $\tilde{G}^{\mathcal{R}}$ *is generated by* $\{\imath(g) : g \in G\}$,
(iii) $\tilde{G}^{\mathcal{R}}$ *is an F-inverse monoid whose maximum group image is isomorphic to the group G.*

REMARK. In [6], Exel considered inverse monoid $\mathcal{S}(G)$ associated to the group G. $\mathcal{S}(G)$ is the universal semigroup defined via generators and relations as follows. To each element g in G we take a generator $[g]$ (from any fixed set having as many as G). For every pair of elements g, h in G we consider the relations

(i) $[g^{-1}][g][h] = [g^{-1}][gh]$,

(ii) $[g][h][h^{-1}] = [gh][h^{-1}]$,
(iii) $[g][1] = [g]$,
(iv) $[1][g] = [g]$.

THEOREM 2.2 ([7]). *The Birget-Rhodes expansion $\tilde{G}^{\mathcal{R}}$ is isomorphic to the inverse monoid $\mathcal{S}(G)$. The mapping $G \ni g \mapsto [g] \in \mathcal{S}(G)$ induces an isomorphism from $\tilde{G}^{\mathcal{R}}$ to $\mathcal{S}(G)$.*

For each g in G, the element $\imath(g) = (\{1, g\}, g)$ of $\tilde{G}^{\mathcal{R}}$ is corresponding to the element $[g]$ of $\mathcal{S}(G)$. We set $\epsilon_g := (\{1, g\}, 1)$. Then ϵ_g is an idempotent in $\tilde{G}^{\mathcal{R}}$.

THEOREM 2.3. *Every element α in $\tilde{G}^{\mathcal{R}}$ admits a decomposition*

$$\alpha = \epsilon_{g_1} \cdots \epsilon_{g_n} \imath(h),$$

where $n \geq 0$ and $h, g_1, \ldots, g_n \in G$. In addition, one can assume that

(i) *$g_i \neq g_j$ for $i \neq j$,*
(ii) *$g_i \neq h$ and $g_i \neq 1$ for all i.*

Proof. See Proposition 2.5 in [6]. □

If $\alpha = \epsilon_{g_1} \cdots \epsilon_{g_n} \imath(h)$, in such way that conditions (i) and (ii) of Theorem 2.3 are verified, we say that α is in *standard* form. From Proposition 3.2 in [6], every element α of $\tilde{G}^{\mathcal{R}}$ admits a unique standard decomposition $\alpha = \epsilon_{g_1} \cdots \epsilon_{g_n} \imath(h)$ up to order of the ϵ_g's.

LEMMA 2.4. *Let h and $g_1, \ldots, g_n$ be elements of G. Then*

(i) $\imath(g_1)\imath(g_2) \cdots \imath(g_n) = \epsilon_{g_1}\epsilon_{g_1 g_2} \cdots \epsilon_{g_1 g_2 \cdots g_n} \imath(g_1 g_2 \cdots g_n)$.
(ii) $\imath(h)\epsilon_{g_1} \cdots \epsilon_{g_n} = \epsilon_{hg_1} \cdots \epsilon_{hg_n} \imath(h)$.
(iii) $\epsilon_{g_1} \cdots \epsilon_{g_n} \imath(h) = \imath(h)\epsilon_{h^{-1}g_1} \cdots \epsilon_{h^{-1}g_n}$.
(iv) $\imath(g_1)\imath(g_1^{-1} g_2)\imath(g_2^{-1}) = \epsilon_{g_1}\epsilon_{g_2} = \epsilon_{g_2}\epsilon_{g_1}$.

Proof. Straightforward. □

We give an explicit way obtaining the decomposition of $\alpha \in \tilde{G}^{\mathcal{R}} = \langle \{\imath(g) : g \in G\} \rangle$ in standard form.

PROPOSITION 2.5. *Suppose that $\alpha = \imath(g_1) \cdots \imath(g_n) \in \tilde{G}^{\mathcal{R}}$. Let $h_i = g_1 \cdots g_i$ for $i = 1, \ldots, n$. If we write*

$$\{h_i : 1 \leq i \leq n\} \setminus \{1, h_n\} = \{w_1, \cdots, w_k\}$$

then the decomposition of α in standard form is

$$\alpha = \epsilon_{w_1} \ldots \epsilon_{w_k} \imath(h_n).$$

Proof. By Lemma 2.4, we have

$$\alpha = \epsilon_{h_1} \cdots \epsilon_{h_{n-1}} \imath(h_n).$$

The statement follows from the fact that the commutativity of the idempotents of $\tilde{G}^{\mathcal{R}}$ and the fact that $\epsilon_g \imath(g) = \imath(g)$ for all $g \in G$. □

THEOREM 2.6. *If G is a non-trivial group, then $\tilde{G}^{\mathcal{R}}$ is not a semidirect product of a semilattice by a group.*

Proof. Suppose that $\tilde{G}^{\mathcal{R}}$ is a semidirect product of a semilattice by a group. Then for $(A, g) \in \tilde{G}^{\mathcal{R}}$ and $(B, 1) \in E(\tilde{G}^{\mathcal{R}})$ with $g \notin B$, by Theorem 1 of [8], there exists $(C, h) \in \tilde{G}^{\mathcal{R}}$ such that $(C, h)(h^{-1}C, h^{-1}) = (B, 1)$ and $(g^{-1}A, g^{-1})(C, h)$ is an idempotent. Thus $g = h$ and $C = B$. But $(B, g) \notin \tilde{G}^{\mathcal{R}}$. This contradicts the fact that $(B, g) = (C, h) \in \tilde{G}^{\mathcal{R}}$. □

We remark that although the inverse monoid $\tilde{G}^{\mathcal{R}}$ is not a semidirect product of a semilattice by a group, we will show that it can be nicely embedded in such a semidirect product.

3. An embedding of $\tilde{G}^{\mathcal{R}}$ into a semidirect product

In this section we will devote to construct an inverse monoid, which is a semidirect product of a semilattice by a group, containing isomorphic copies of the inverse semigroup $\tilde{G}^{\mathcal{R}}$ and the group G.

Let G be a non-trivial group and let $\tilde{G}^{\mathcal{R}} \bullet G$ be the free product of the inverse monoid $\tilde{G}^{\mathcal{R}}$ and the group G. For each word $w = a_1 a_2 \cdots a_n$ in $\tilde{G}^{\mathcal{R}} \bullet G$, we define the formal inverse w^{-1} of w by

$$w^{-1} = a_n^{-1} a_{n-1}^{-1} \cdots a_1^{-1}.$$

Then the congruence τ on $\tilde{G}^{\mathcal{R}} \bullet G$ generated by the subset

$$\mathbf{T} = \{(ww^{-1}w, w) : w \in \tilde{G}^{\mathcal{R}} \bullet G\} \cup \{(ww^{-1}zz^{-1}, zz^{-1}ww^{-1}) : w, z \in \tilde{G}^{\mathcal{R}} \bullet G\}$$

defines the free inverse product of $\tilde{G}^{\mathcal{R}}$ and G,

$$\tilde{G}^{\mathcal{R}} * G = (\tilde{G}^{\mathcal{R}} \bullet G)/\tau.$$

Note that $\tilde{G}^{\mathcal{R}} * G$ is the coproduct of $\tilde{G}^{\mathcal{R}}$ and G in the category of inverse semigroups. It is known (Proposition VII.4.5 of [5]) that there are monomorphisms

$$i_{\tilde{G}^{\mathcal{R}}} : \tilde{G}^{\mathcal{R}} \ni \alpha \mapsto \alpha\tau \in \tilde{G}^{\mathcal{R}} * G$$

and

$$i_G : G \ni g \mapsto g\tau \in \tilde{G}^{\mathscr{R}} * G,$$

where $\alpha\tau$ and $g\tau$ are the τ-classes containing the elements α and g, respectively. Thus we may identify $\tilde{G}^{\mathscr{R}}$ and G with the isomorphic copies of their in $\tilde{G}^{\mathscr{R}} * G$.

In the following we regard the inverse monoid $\tilde{G}^{\mathscr{R}}$ and the group G as the images of the maps $i_{\tilde{G}^{\mathscr{R}}}$ and $i_G(G)$, respectively.

Let ρ be the congruence on $\tilde{G}^{\mathscr{R}} * G$ generated by the following relation **R**:

(i) $\left\{ \left(g\epsilon_{g^{-1}}, \imath(g) \right) : g \in G \right\}$,
(ii) $\left\{ \left(\epsilon_g g, \imath(g) \right) : g \in G \right\}$,
(iii) $\left\{ \left(\imath(1)\, g\, \imath(1), \imath(g) \right) : g \in G \right\}$,
(iv) $\left\{ \left(\imath(g)1, \imath(g) \right) : g \in G \right\}$,
(v) $\left\{ \left(1\, \imath(g), \imath(g) \right) : g \in G \right\}$.

Define $\tilde{G}^{\mathscr{R}}_*$ by

$$\tilde{G}^{\mathscr{R}}_* = (\tilde{G}^{\mathscr{R}} * G)/\rho.$$

Then $\tilde{G}^{\mathscr{R}}_*$ is an inverse semigroup under multiplication defined by $\alpha\rho \cdot \beta\rho = (\alpha\beta)\rho$ for $\alpha, \beta \in \tilde{G}^{\mathscr{R}} * G$ because it is surmorphic image of the inverse semigroup $\tilde{G}^{\mathscr{R}} * G$.

The following will be useful for our purpose.

LEMMA 3.1. *Let $\alpha = \epsilon_{g_1}\epsilon_{g_2} \cdots \epsilon_{g_n}\imath(h) \in \tilde{G}^{\mathscr{R}}$ with $h \in G$. Then we have*

(i) $(h\alpha^{-1}\alpha, \alpha) \in \rho$.
(ii) $(\alpha\alpha^{-1}h, \alpha) \in \rho$.
(iii) $(1\,\alpha, \alpha) \in \rho$.
(iv) $(\alpha\,1, \alpha) \in \rho$.

Proof. By Lemma 2.4, we have that

$$\begin{aligned} h\alpha^{-1}\alpha &= h\,\imath(h^{-1})\epsilon_{g_n}\epsilon_{g_{n-1}} \cdots \epsilon_{g_1}\epsilon_{g_1}\epsilon_{g_2} \cdots \epsilon_{g_n}\imath(h) \\ &= h\,\imath(h^{-1})\epsilon_{g_1}\epsilon_{g_2} \cdots \epsilon_{g_n}\imath(h) \\ &= h\epsilon_{h^{-1}g_1}\epsilon_{h^{-1}g_2} \cdots \epsilon_{h^{-1}g_n}\epsilon_{h^{-1}} \\ &= h\epsilon_{h^{-1}}\epsilon_{h^{-1}g_1}\epsilon_{h^{-1}g_2} \cdots \epsilon_{h^{-1}g_n}, \\ \alpha\alpha^{-1}h &= \epsilon_{g_1}\epsilon_{g_2} \cdots \epsilon_{g_n}\epsilon_h\epsilon_{g_n}\epsilon_{g_{n-1}} \cdots \epsilon_{g_1}h \\ &= \epsilon_{g_1}\epsilon_{g_2} \cdots \epsilon_{g_n}\epsilon_h h, \end{aligned}$$

and

$$\begin{aligned}\alpha &= \epsilon_{g_1}\epsilon_{g_2}\cdots\epsilon_{g_n}\imath(h)\\ &= \imath(h)\epsilon_{h^{-1}g_1}\epsilon_{h^{-1}g_2}\cdots\epsilon_{h^{-1}g_n}.\end{aligned}$$

Since $(h\epsilon_{h^{-1}}, \imath(h)) \in \rho$ and $(\epsilon_h h, \imath(h)) \in \rho$, (i) and (ii) follow from the compatibility of ρ.

(iii) and (iv) are straightforward. □

We note that $\tilde{G}^{\mathscr{R}}_*$ is a monoid with the identity 1.

Now we want to show that the inverse monoid $\tilde{G}^{\mathscr{R}}_*$ has all properties what we want to have.

We first prove that the group G and the inverse monoid $\tilde{G}^{\mathscr{R}}$ are embedded into the inverse monoid $\tilde{G}^{\mathscr{R}}_*$.

Let S be a monoid and let R be a relation on S. If $a, b \in S$ are such that

$$a = xpy, \quad b = xqy$$

for some $x, y \in S$, where either $(p, q) \in R$ or $(q, p) \in R$, we say that a is connected to b by an *elementary R-transition.* The following appears at Proposition I.5.10 [5].

PROPOSITION 3.2. *If $a, b \in S$, then (a, b) is in the congruence generated by the relation R if and only if either $a = b$ or for some natural number n there is a sequence*

$$a = z_1 \to z_2 \to \cdots \to z_n = b$$

of elementary R-transitions connecting a to b.

Now we return to our inverse monoid $\tilde{G}^{\mathscr{R}}$. Define a map from $\tilde{G}^{\mathscr{R}} \cup G$ to $\tilde{G}^{\mathscr{R}}$ by

$$\hat{r}(\alpha) = \begin{cases} \alpha & \text{if } \alpha \in \tilde{G}^{\mathscr{R}} \\ \imath(g) & \text{if } \alpha = g \in G. \end{cases}$$

Then $\hat{r}$ induces a map $r^\circ : \tilde{G}^{\mathscr{R}} \bullet G \to \tilde{G}^{\mathscr{R}}$ by

$$r^\circ(\alpha) = \hat{r}(a_1)\cdots\hat{r}(a_n),$$

for a reduced word $\alpha = a_1 \cdots a_n \in \tilde{G}^{\mathscr{R}} \bullet G$. Note that r° acts as the identity on $\tilde{G}^{\mathscr{R}}$ and $r^\circ(\alpha^{-1}) = r^\circ(\alpha)^{-1}$ for all $\alpha \in \tilde{G}^{\mathscr{R}} \bullet G$. However, r° is not a homomorphism since $\imath(gh) \neq \imath(g)\imath(h)$ in general.

PROPOSITION 3.3. *There exists a function*

$$r : \tilde{G}^{\mathscr{R}} * G \to \tilde{G}^{\mathscr{R}}$$

such that

(i) *r acts as the identity on $\tilde{G}^{\mathscr{R}}$,*
(ii) *$r(g\tau) = \imath(g)$ for all $g \in G$,*
(iii) *$r(xpy\tau) = r(x\tau)r(p\tau)r(y\tau) = r(xqy\tau)$*

for all $x, y \in \tilde{G}^{\mathscr{R}} \bullet G$ and all $(p,q) \in \mathbf{R}$.

Proof. We define $r : \tilde{G}^{\mathscr{R}} * G \to \tilde{G}^{\mathscr{R}}$ by

$$r(\alpha\tau) = r^\circ(a_1)r^\circ(a_2)\cdots r^\circ(a_n)$$

for a reduced word $\alpha = a_1a_2\cdots a_n \in \tilde{G}^{\mathscr{R}} \bullet G$. Since the map r° is not a homomorphism, it needs to check that r is well-defined. We first prove that $r(ww^{-1}w\tau) = r(w\tau)$ and $r(ww^{-1}zz^{-1}\tau) = r(zz^{-1}ww^{-1}\tau)$ for any words $w, z \in \tilde{G}^{\mathscr{R}} \bullet G$. It is obvious for words with length one.

Let $w = a_1\cdots a_n, z = b_1\cdots b_m$ be reduced words in $\tilde{G}^{\mathscr{R}} \bullet G$ such that $n, m > 1$.

(i) $r(ww^{-1}w\tau) = r(w\tau)$: Suppose that $a_1 = g$ and $a_n = h$ in G. Then the reduced word of $ww^{-1}w$ is

$$ww^{-1}w = guu^{-1}uh,$$

where $u = a_2\cdots a_{n-1}$. Thus $r(ww^{-1}w\tau) = \imath(g)r^\circ(u)r^\circ(u^{-1})r^\circ(u)\imath(h)$. Since $r^\circ(u^{-1}) = r^\circ(u)^{-1}$, it follows that

$$\begin{aligned} r(ww^{-1}w\tau) &= \imath(g)r^\circ(u)r^\circ(u)^{-1}r^\circ(u)\imath(h) \\ &= \imath(g)r^\circ(u)\imath(h) \\ &= r(w\tau). \end{aligned}$$

By the same argument, the equation $r(ww^{-1}w\tau) = r(w\tau)$ holds for the cases $(a_1, a_n) \in \tilde{G}^{\mathscr{R}} \times \tilde{G}^{\mathscr{R}}, (a_1, a_n) \in \tilde{G}^{\mathscr{R}} \times G$ and $(a_1, a_n) \in G \times \tilde{G}^{\mathscr{R}}$.

(ii) $r(ww^{-1}zz^{-1}\tau) = r(zz^{-1}ww^{-1}\tau)$: To show this equality, we only prove the case when both a_1 and b_1 are in G. The proof of remaining cases follows from the same argument. Let $a_1 = g, b_1 = h$. Then the reduced words of $ww^{-1}zz^{-1}$ and $zz^{-1}ww^{-1}$ are

$$\begin{aligned} ww^{-1}zz^{-1} &= gu(a_na_n^{-1})u^{-1}(g^{-1}h)v(b_mb_m^{-1})v^{-1}h^{-1} \\ zz^{-1}ww^{-1} &= hv(b_mb_m^{-1})v^{-1}(h^{-1}g)u(a_na_n^{-1})u^{-1}g^{-1}, \end{aligned}$$

where $u = a_2\cdots a_{n-1}, v = b_2\cdots b_{m-1}$. Note that

$$r(u(a_na_n^{-1})u^{-1}\tau) = r^\circ(u)r^\circ(a_na_n^{-1})r^\circ(u)^{-1}$$

and

$$r(v(b_mb_m^{-1})v^{-1}\tau) = r^\circ(v)r^\circ(b_mb_m^{-1})r^\circ(v)^{-1}$$

are idempotents of $\tilde{G}^{\mathscr{R}}$, say e and f respectively. Then

$$\begin{aligned} r(ww^{-1}zz^{-1}\tau) &= \imath(g)e\imath(g^{-1}h)f\imath(h^{-1}), \\ r(zz^{-1}ww^{-1}\tau) &= \imath(h)f\imath(h^{-1}g)e\imath(g^{-1}). \end{aligned}$$

By Lemma 2.4, we may write $\imath(g)e = e'\imath(g)$ and $f\imath(h^{-1}) = \imath(h^{-1})f'$ for some idempotents e' and f' of $\tilde{G}^{\mathscr{R}}$. By taking the inverse, we have that $e\imath(g^{-1}) = \imath(g^{-1})e'$ and $\imath(h)f = f'\imath(h)$. It then follows that

$$\begin{aligned} r(ww^{-1}zz^{-1}\tau) &= \imath(g)e\imath(g^{-1}h)f\imath(h^{-1}) \\ &= e'\imath(g)\imath(g^{-1}h)\imath(h^{-1})f' \\ &= e'\epsilon_h\epsilon_g f' && \text{by Lemma 2.4 (iv)} \\ &= f'\imath(h)\imath(h^{-1}g)\imath(g^{-1})e' && \text{by Lemma 2.4 (iv)} \\ &= \imath(h)f\imath(h^{-1}g)e\imath(g^{-1}) \\ &= r(zz^{-1}ww^{-1}\tau). \end{aligned}$$

To complete the proof that r is well-defined, it is enough to show that $r^\circ(xpy) = r^\circ(xqy)$ for any $(p,q) \in \mathbf{T}$ and any $x, y \in \tilde{G}^{\mathscr{R}} \cup G$ by Proposition 3.2. When $(p,q) = (ww^{-1}w, w)$ there is no difficulty to prove the identity $r^\circ(xpy) = r^\circ(xqy)$ for all $x, y \in \tilde{G}^{\mathscr{R}} \cup G$. Suppose that $(p,q) = (ww^{-1}zz^{-1}, zz^{-1}ww^{-1})$. We consider the following two cases (other cases are followed by the same argument):

$$\begin{aligned} (a_1, b_1, x, y) &= (g, h, k, l)\ (g, h, k, l \in G), \\ (a_1, b_1, x, y) &= (g, h, x, l)\ (x \in \tilde{G}^{\mathscr{R}}, g, h, l \in G). \end{aligned}$$

Case 1. $(a_1, b_1, x, y) = (g, h, k, l)$ $(g, h, k, l \in G)$. In this case, the reduced words of $xww^{-1}zz^{-1}y$ and $xzz^{-1}ww^{-1}y$ are

$$\begin{aligned} xww^{-1}zz^{-1}y &= (kg)u(a_n a_n^{-1})u^{-1}(g^{-1}h)v(b_m b_m^{-1})v^{-1}(h^{-1}l), \\ xzz^{-1}ww^{-1}y &= (kh)v(b_m b_m^{-1})v^{-1}(h^{-1}g)u(a_n a_n^{-1})u^{-1}(g^{-1}l). \end{aligned}$$

Thus we have

$$\begin{aligned} r^\circ(xww^{-1}zz^{-1}y) &= \imath(kg)e\imath(g^{-1}h)f\imath(h^{-1}l), \\ r^\circ(xzz^{-1}ww^{-1}y) &= \imath(kh)f\imath(h^{-1}g)e\imath(g^{-1}l), \end{aligned}$$

where u, v, e and f are elements in previous proof. By Lemma 2.4 (iii), we may write $e\imath(g^{-1}h) = \imath(g^{-1}h)e'$ and $e'f\imath(h^{-1}l) = \imath(h^{-1}l)e''$ for some idempotents e' and e'' of $\tilde{G}^{\mathscr{R}}$. Then

$$\begin{aligned}
r^\circ(xww^{-1}zz^{-1}y) &= \imath(kg)e\imath(g^{-1}h)f\imath(h^{-1}l)\\
&= \imath(kg)\imath(g^{-1}h)e'f\imath(h^{-1}l)\\
&= \imath(kg)\imath(g^{-1}h)\imath(h^{-1}l)e''\\
&= \epsilon_{kg}\epsilon_{kh}\imath(kl)e'' && \text{by Lemma 2.4 (i)}\\
&= \epsilon_{kg}\imath(kh)\imath(h^{-1}l)e'' && \text{by Lemma 2.4 (i)}\\
&= \imath(kh)\epsilon_{h^{-1}g}\imath(h^{-1}l)e''\\
&= \imath(kh)\epsilon_{h^{-1}g}e'f\imath(h^{-1}l)\\
&= \imath(kh)e'f\epsilon_{h^{-1}g}\imath(h^{-1}l)\\
&= \imath(kh)fe'\imath(h^{-1}g)\imath(g^{-1}l) && \text{by Lemma 2.4 (i)}\\
&= \imath(kh)f\imath(h^{-1}g)e\imath(g^{-1}l)\\
&= r^\circ(xzz^{-1}ww^{-1}y).
\end{aligned}$$

Case 2. $(a_1, b_1, x, y) = (g, h, x, l)$ $(x \in \tilde{G}^{\mathscr{R}}, g, h, l \in G)$. In this case we have

$$\begin{aligned}
r^\circ(xww^{-1}zz^{-1}y) &= x\imath(g)e\imath(g^{-1}h)f\imath(h^{-1}l)\\
&= x\imath(g)\imath(g^{-1}h)e'f\imath(h^{-1}l)\\
&= x\imath(g)\imath(g^{-1})\imath(h)\imath(h^{-1}l)e''\\
&= x\epsilon_h\epsilon_g\imath(l)e''(= x\epsilon_h\epsilon_g\epsilon_h\imath(l)e'')\\
&= x\epsilon_h\epsilon_g\imath(h)\imath(h^{-1}l)e''\\
&= x\epsilon_h\imath(g)\imath(g^{-1}h)\imath(h^{-1}l)e''\\
&= x\imath(h)\imath(h^{-1}g)\imath(g^{-1}h)\imath(h^{-1}l)e''\\
&= x\imath(h)\imath(h^{-1}g)\imath(g^{-1}h)e'f\imath(h^{-1}l)\\
&= x\imath(h)fe'\imath(h^{-1}g)\imath(g^{-1}h)\imath(h^{-1}l)\\
&= x\imath(h)fe'\imath(h^{-1}g)\imath(g^{-1}l)\\
&= x\imath(h)f\imath(h^{-1}g)e\imath(g^{-1}l)\\
&= r^\circ(xzz^{-1}ww^{-1}y).
\end{aligned}$$

Here e, f, e' and e'' are idempotents of $\tilde{G}^{\mathscr{R}}$ in case 1. Therefore, we conclude that the map r is well-defined.

It is easy to see that the mapping r acts as the identity on $\tilde{G}^{\mathscr{R}}$ and $r(g\tau) = \imath(g)$ for all $g \in G$.

Finally, to prove the last statement of the Theorem, it suffices to restrict our attention to the case $x, y \in \tilde{G}^{\mathscr{R}} \cup G$. We only prove the case $(p, q) = (g\epsilon_{g^{-1}}, \imath(g))$. The other cases are similar.

Case 1. $x, y \in \tilde{G}^{\mathscr{R}}$. The reduced word of xpy is $xg\epsilon_{g^{-1}}y$ and hence $r(xpy\tau) = x\imath(g)\epsilon_{g^{-1}}y = x\imath(g)y = r(x\tau)r(p\tau)r(y\tau)$.

Case 2. $x \in \tilde{G}^{\mathscr{R}}, y = h \in G$. Then $xpy = xg\epsilon_{g^{-1}}h$ and hence $r(xpy\tau) = x\imath(g)\epsilon_{g^{-1}}\imath(h) = x\imath(g)\imath(h) = r(x\tau)r(p\tau)r(y\tau)$.

Case 3. $x = h \in G, y \in \tilde{G}^{\mathscr{R}}$. The reduced word of xpy is $hg\epsilon_{g^{-1}}y$ and hence $r(xpy\tau) = \imath(hg)\epsilon_{g^{-1}}y = \imath(h)\imath(g)y = r(x\tau)r(p\tau)r(y\tau)$.

Case 4. $x = h, y = k \in G$. In this case $xpy = hg\epsilon_{g^{-1}}k$ and $r(xpy\tau) = \imath(hg)\epsilon_{g^{-1}}\imath(k) = \imath(h)\imath(g)\imath(k) = r(x\tau)r(p\tau)r(y\tau)$. □

Using the (retractive) function r on $\tilde{G}^{\mathscr{R}} * G$, we have

PROPOSITION 3.4. *The inverse monoid $\tilde{G}^{\mathscr{R}}_*$ contains isomorphic copies of G and $\tilde{G}^{\mathscr{R}}$.*

Proof. Let $\psi : \tilde{G}^{\mathscr{R}} * G \to (\tilde{G}^{\mathscr{R}} * G)/\rho = \tilde{G}^{\mathscr{R}}_*$ be the natural map. We prove that ψ is injective on G and $\tilde{G}^{\mathscr{R}}$. Suppose that $g_1\rho = g_2\rho$ for $g_1, g_2 \in G$. Then one may easily show that for any $(p, q) \in \mathbf{R} \cup \mathbf{R}^{-1}$ and any $x, y \in \tilde{G}^{\mathscr{R}} * G$, $xpy \notin G$ and $xqy \notin G$. By Proposition 3.2, $g_1 = g_2$.

Next, suppose that α and β are elements of $\tilde{G}^{\mathscr{R}}$ such that $(\alpha\tau)\rho = (\beta\tau)\rho$. Then there exist $x_1, \ldots, x_n, y_1, \ldots, y_n \in \tilde{G}^{\mathscr{R}} \bullet G$ and

$$(p_1, q_1), \ldots, (p_n, q_n) \in \mathbf{R} \cup \mathbf{R}^{-1}$$

giving a sequence

$$\alpha\tau = x_1p_1y_1\tau \to x_1q_1x_2\tau = x_2p_2y_2\tau \to \cdots \to x_nq_ny_n\tau = \beta\tau$$

of elementary $\mathbf{R}$-transitions connecting $\alpha\tau$ to $\beta\tau$. Since $r(\alpha\tau) = \alpha$ and $r(\beta\tau) = \beta$, by Proposition 3.3, it follows that

$$\begin{aligned} \alpha = r(\alpha\tau) &= r(x_1p_1y_1\tau) = r(x_1q_1y_1\tau) \\ &= r(x_2p_2y_2\tau) = \cdots = r(x_nq_ny_n\tau) = r(\beta\tau) = \beta. \end{aligned}$$

□

We note that the inverse monoid $\tilde{G}^{\mathscr{R}}$ and the group G are embedded into the inverse monoid $\tilde{G}^{\mathscr{R}}_*$ via the maps

$$\begin{aligned} &\tilde{G}^{\mathscr{R}} \to i_{\tilde{G}^{\mathscr{R}}}(\tilde{G}^{\mathscr{R}}) \to \psi(i_{\tilde{G}^{\mathscr{R}}}(\tilde{G}^{\mathscr{R}})) \subset \tilde{G}^{\mathscr{R}}_*, \\ &G \to i_G(G) \to \psi(i_G(G)) \subset \tilde{G}^{\mathscr{R}}_*. \end{aligned}$$

Next, we will show that the inverse monoid $\tilde{G}^{\mathscr{R}}_*$ admits a semidirect product of a semilattice by a group.

LEMMA 3.5. *Let $\leq_S$ and $\leq_T$ be the natural partial orders in inverse semigroups S and T, and let f be a homomorphism from S to T. If $a \leq_S b$ implies $f(a) \leq_T f(b)$. If f is injective and $f(a) \leq_T f(b)$ then $a \leq_S b$.*

Proof. Notice that f preserves idempotent elements. If $a \leq_S b$ in S, then there exists an idempotent e in S such that $a = eb$. Now $f(a) = f(e)f(b)$ and $f(e)$ is an idempotent in T. This implies $f(a) \leq_T f(b)$.

Suppose that f is a monomorphism and suppose that $f(a) \leq_T f(b)$. Then $f(a) = ef(b)$ for some idempotent e in T. This implies that $f(a) = f(a)f(a^{-1})f(b) = f(aa^{-1}b)$. Since f is injective, $a = (aa^{-1})b$ and hence $a \leq_S b$. □

PROPOSITION 3.6. *The inverse monoid $\tilde{G}^{\mathscr{R}}_*$ is E-unitary, and every element of $\tilde{G}^{\mathscr{R}}_*$ is beneath a unique element of G.*

Proof. We first observe that every element $\alpha\rho$ of $\tilde{G}^{\mathscr{R}}_*$ with $\alpha \in \tilde{G}^{\mathscr{R}}$ is beneath a unique element $g\rho$ of $\tilde{G}^{\mathscr{R}}_*$ with $g \in G$. Suppose that $\alpha = \epsilon_{g_1}\epsilon_{g_2}\cdots\epsilon_{g_n}\imath(g)$ is an element of $\tilde{G}^{\mathscr{R}}$. By Lemma 3.1,

$$(\alpha\alpha^{-1})\rho \cdot g\rho = (\alpha\alpha^{-1}g)\rho = \alpha\rho.$$

Since $(\alpha\alpha^{-1})\rho$ is an idempotent, we have $\alpha\rho \leq g\rho$. Now suppose that $\alpha\rho$ is bounded above by an another element $h\rho \in \tilde{G}^{\mathscr{R}}_*$ with $h \in G$. Then clearly $\epsilon_h\alpha$ is bounded above by $\imath(g)$ in the semigroup $\tilde{G}^{\mathscr{R}}$. Since $\alpha\rho \leq h\rho$, $(\epsilon_h\alpha)\rho \leq (\epsilon_h h)\rho = \imath(h)\rho$. By Lemma 3.5, $\epsilon_h\alpha \leq \imath(h)$ in the semigroup $\tilde{G}^{\mathscr{R}}$. Thus $\imath(g) = \imath(h)$ from Proposition 2.1 (iii).

Let $\alpha\rho = (\alpha_1\alpha_2\cdots\alpha_n)\rho = \alpha_1\rho\cdot\alpha_2\rho\cdots\alpha_n\rho \in \tilde{G}^{\mathscr{R}}_*$ with $\alpha_i \in \tilde{G}^{\mathscr{R}} \cup G$. Then by the remarks of the first paragraph, for each i there exists $g_i \in G$ such that $\alpha_i\rho \leq g_i\rho$. By the compatibility of the order, we have $\alpha\rho \leq (g_1\cdots g_n)\rho$. This shows that every element in $\tilde{G}^{\mathscr{R}}_*$ is bounded above by an element of G.

Now suppose that $\alpha\rho$ is an element of $\tilde{G}^{\mathscr{R}}_*$ and is bounded above by the elements $g\rho, h\rho$ where $g, h \in G$. Then $r(\alpha)\rho \leq \alpha\rho \leq g\rho, h\rho$, where r is the retractive function on $\tilde{G}^{\mathscr{R}} * G$. Since $r(\alpha) \in \tilde{G}^{\mathscr{R}}$, by the remarks of the first paragraph, we conclude that $g\rho = h\rho$. This implies that every element in $\tilde{G}^{\mathscr{R}}_*$ is bounded above by a unique element of G.

Finally, we show that $\tilde{G}^{\mathscr{R}}_*$ is an E-unitary semigroup. Let $e\rho$ be an idempotent in $\tilde{G}^{\mathscr{R}}_*$ and $\alpha\rho \in \tilde{G}^{\mathscr{R}}_*$ such that $e\rho \leq \alpha\rho$. Then $(e\alpha)\rho$ is an idempotent $\tilde{G}^{\mathscr{R}}_*$. Pick an element g in G such that the element $\alpha\rho$

is bounded above by $g\rho$. Then $(e\alpha)\rho \le \alpha\rho \le g\rho$. Since $(e\alpha)\rho$ is an idempotent element, $(e\alpha)\rho \le 1\rho$. Hence $g\rho = 1\rho$. This implies that $\alpha\rho$ is an idempotent element of $\tilde{G}_*^{\mathscr{R}}$. Therefore $\tilde{G}_*^{\mathscr{R}}$ is E-unitary. □

THEOREM 3.7. *The inverse monoid $\tilde{G}_*^{\mathscr{R}}$ is isomorphic to a semidirect product of the semilattice of idempotents of $\tilde{G}^{\mathscr{R}}$ by the group G.*

Proof. Let E^* be the semilattice of idempotents of $\tilde{G}_*^{\mathscr{R}}$. Then the mapping defined by

$$G \times E^* \ni (g\rho, f\rho) \mapsto g\rho \cdot f\rho \cdot (g^{-1})\rho = (gfg^{-1})\rho \in E^*$$

is an action of G on E^*. Set $S = E^* \times G$. Then S becomes an inverse semigroup under the multiplication

$$(e\rho, g\rho)(f\rho, h\rho) = ((egfg^{-1})\rho, (gh)\rho).$$

Now, we establish that the mapping

$$\Phi : \tilde{G}_*^{\mathscr{R}} \ni \alpha\rho \mapsto ((\alpha\alpha^{-1})\rho, g\rho) \in S,$$

where $g\rho$ is the (unique) element of G bounding $\alpha\rho \in \tilde{G}_*^{\mathscr{R}}$, is an isomorphism between inverse semigroups. Suppose that

$$\Phi(\alpha\rho) = ((\alpha\alpha^{-1})\rho, g\rho) = ((\beta\beta^{-1})\rho, h\rho) = \Phi(\beta\rho)$$

for $\alpha\rho, \beta\rho \in \tilde{G}_*^{\mathscr{R}}$. Since $\alpha\rho \le g\rho$ and $\beta\rho \le h\rho$, we have

$$\alpha\rho = \alpha\rho \cdot \alpha^{-1}\rho \cdot g\rho = (\alpha\alpha^{-1})\rho \cdot g\rho = (\beta\beta^{-1})\rho \cdot h\rho = \beta\rho \cdot \beta^{-1}\rho \cdot h\rho = \beta\rho.$$

Thus Φ is injective. Let $(e\rho, g\rho) \in S$. Consider the element $(eg)\rho = e\rho \cdot g\rho$ in $\tilde{G}_*^{\mathscr{R}}$. Since $(eg)\rho = e\rho \cdot g\rho \le 1\rho \cdot g\rho = g\rho$ and since $((eg)\rho) \cdot ((eg)\rho)^{-1} = e\rho$, the map Φ maps $(eg)\rho$ to $(e\rho, g\rho)$. Thus Φ is surjective.

Finally, we show that Φ is a homomorphism. Let $\alpha\rho$ and $\beta\rho$ be elements of $\tilde{G}_*^{\mathscr{R}}$, and let $g, h \in G$ such that $\alpha\rho \le g\rho$ and $\beta\rho \le h\rho$. Then $(\alpha\beta)\rho = \alpha\rho \cdot \beta\rho$ is bounded above by the (unique) element $(gh)\rho = g\rho \cdot h\rho$. Now we also have

$$\begin{aligned}
(\alpha\beta\beta^{-1}\alpha^{-1})\rho &= \alpha\rho \cdot (\beta\beta^{-1})\rho \cdot \alpha^{-1}\rho \\
&= \alpha\rho \cdot (\beta\beta^{-1})\rho \cdot (\alpha^{-1}\alpha)\rho \cdot g^{-1}\rho \\
&= \alpha\rho \cdot (\alpha^{-1}\alpha)\rho \cdot (\beta\beta^{-1})\rho \cdot g^{-1}\rho \\
&= (\alpha\alpha^{-1})\rho \cdot \alpha\rho \cdot (\beta\beta^{-1})\rho \cdot g^{-1}\rho \\
&= (\alpha\alpha^{-1})\rho \cdot (\alpha\alpha^{-1})\rho \cdot g\rho \cdot (\beta\beta^{-1})\rho \cdot g^{-1}\rho \\
&= (\alpha\alpha^{-1})\rho \cdot g\rho \cdot (\beta\beta^{-1})\rho \cdot g^{-1}\rho \\
&= (\alpha\alpha^{-1}g\beta\beta^{-1}g^{-1})\rho.
\end{aligned}$$

This shows that Φ is a homomorphism. □

Set

$$P := \{(e\rho, g\rho) \in E(\tilde{G}^{\mathscr{R}}) \times G : (g^{-1}eg)\rho \in E(\tilde{G}^{\mathscr{R}})\}.$$

COROLLARY 3.8. *The map $\Phi \circ \psi$ maps $\tilde{G}^{\mathscr{R}}$ onto P, where Φ is the isomorphism in Theorem 3.7 and ψ is the natural embedding of $\tilde{G}^{\mathscr{R}}$ into $\tilde{G}^{\mathscr{R}}_*$ in Proposition 3.4. In particular, P is an inverse subsemigroup of $S = E^* \times G$.*

Proof. Let $\alpha \in \tilde{G}^{\mathscr{R}}$ with $\alpha \leq \imath(g)$. Then $\alpha^{-1}\alpha \in E(\tilde{G}^{\mathscr{R}})$. By Lemma 3.1, it follows that

$$\begin{aligned}(g^{-1}\alpha\alpha^{-1}g)\rho &= g^{-1}\rho \cdot (\alpha\alpha^{-1}g)\rho \\ &= g^{-1}\rho \cdot \alpha\rho \\ &= (g^{-1}\alpha)\rho \\ &= (\alpha^{-1}\alpha)\rho \in E(\tilde{G}^{\mathscr{R}}).\end{aligned}$$

Therefore $\Phi \circ \psi(\tilde{G}^{\mathscr{R}}) \subset P$.

Conversely, suppose that $(e\rho, g\rho) \in P$. Then $e \in E(\tilde{G}^{\mathscr{R}})$ and $(g^{-1}eg)\rho = f\rho$ for some $f \in E(\tilde{G}^{\mathscr{R}})$. Since

$$(g^{-1}e\imath(1)g\imath(1))\rho = (g^{-1}eg\imath(1))\rho = f\rho \cdot \imath(1)\rho = f\rho = (g^{-1}eg)\rho,$$

we have $(e\imath(g))\rho = (e\imath(1)g\imath(1))\rho = (eg)\rho$. This implies that

$$\Phi \circ \psi(e\imath(g)) = \Phi(e\imath(g)\rho) = \Phi((eg)\rho) = (e\rho, g\rho).$$

This completes the proof. □

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