Bull. Korean Math. Soc. **41** (2004), No. 4, pp. 665–676

ASYMPTOTIC STABILITY IN GENERAL DYNAMICAL SYSTEMS

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ABSTRACT. In this paper we characterize asymptotic stability via Lyapunov function in general dynamical systems on c-first countable space. We give a family of examples which have first countable but not c-first countable, also c-first countable and locally compact space but not metric space. We obtain several necessary and sufficient conditions for a compact subset M of the phase space X to be asymptotic stability.

1. Introduction

In [1], Bhatia and Szegö verified several necessary and sufficient conditions for a compact subset M of the metric space X to be asymptotically stability . The purpose of this paper is to extend this result to a general dynamical systems on c-first countable and locally compact space. The basic feature of the stability theory in dynamical systems is that we find several necessary and sufficient conditions for a compact subset M of the phase space X to be asymptotically stable.

2. C-first countable spaces

In the sequel, we denote by $\overline{M}$ and ∂M, respectively, the closure and the boundary of the set M.

DEFINITION. A space X is said to be *c-first countable* if for each compact subset K of X, the quotient space X/K is first countable.

Received September 22, 2003.
2000 Mathematics Subject Classification: 37B25.
Key words and phrases: C- first countable space, asymptotic stability, general dynamical systems, Lyapunov function and strict Lyapunov function.
The second author was supported by the Hoseo University Research Fund in 2003.

Let X be a c-first countable space. Given any compact subset K of X, there exists a family $\mathcal{U}$ consisting of countably many neighborhoods of K such that every neighborhood of K contains some member of $\mathcal{U}$. Such a family $\mathcal{U}$ will be called *countable neighborhood base of* K.

THEOREM 2.1. *Every second countable space is c-first countable.*

Proof. Let X be a second countable space. There exists a countable basis β for X. Given any compact subset K of X, let $\mathcal{U}$ be the family neighborhoods of K which are finite unions of members of β. Thus $\mathcal{U}$ is a countable neighborhood base of K. Then X is c-first countable. □

The converse of the above theorem is not true as shown by uncountable discrete spaces. Clearly every c-first countable space is first countable space. But its converse does not hold.

EXAMPLE 2.1. Let $X_0 = \{(x, 0) : x \in \mathbb{R}\}$ and $X_1 = \{(x, 1) : x \in \mathbb{R}\}$ be two subsets of the plane $\mathbb{R}^2$. We take a basis β for the topology on the set $X = X_0 \cup X_1$ as follows ;

$$\beta = \{\{(x, 1)\} : x \in \mathbb{R}\} \cup \{B(x, r) : x \in \mathbb{R}, r > 0\},$$

where $B(x, y) = \{(y, 0) : |x - y| < r\} \cup \{(y, 1) : 0 < |x - y| < r\}$. It is clear that X is first countable. But X is not c-first countable.

Proof. Let us choose a compact subset $K = \{(x, 0) : x \in I\}$ of X, where I is the unit interval. For each neighborhood U of K, Let $I(U) = \{x \in I : (x, 1) \notin U\}$. Suppose that $I(U)$ is infinite for some neighborhood U of K. $I(U)$ has a cluster point, say y, in I. Since $(y, 0) \in K \subset U$, there exists a number $r > 0$ such that $B(y, r) \subset U$. Since y is a cluster point of $I(U)$, there is a number $z \in I(U)$ such that $0 < |y - z| < r$. Since $(z, 1) \in B(y, r) \subset U$, we have a contradicition.Thus $I(U)$ is finite for all neighborhoods U of K. Let $U_1, U_2, U_3, \ldots$ be neighborhoods of K. Since $I(U_n)$ is finite for all n, $A = \cup_{n=1}^{\infty} I(U_n)$ is countable. Thus there is a number $w \in I - A$. Let $V = X_0 \cup \{(x, 1) : x \neq w\}$. Then V is a neighborhood of K and $U_n \not\subset V$ for all n. Thus there is no countable neighborhood base of K. Hence X is not c-first countable. □

THEOREM 2.2. *Every metric space is c-first countable.*

Proof. Let (X, d) be a metric space. Given any compact subset K of X, it is easy to show that the family $\{B(K, 1/n) : n = 1, 2, 3, \ldots\}$ is a countable neighborhood base of K, where $B(K, 1/n) = \{x \in X : d(x, K) < 1/n\}$. Thus X is c-first countable space. □

The converse of the Theorem 2.2 is not true. The following example shows that there exists a c-first countable and locally compact space which is not a metric space.

EXAMPLE 2.2. For each irrational x, we choose a sequence (x_n) of rationals converging to it in the Euclidean topology. The rational sequence topology $\mathcal{T}$ on $\mathbb{R}$ is then defined by declaring each rational open and selecting the sets

$$U_n(x) = \{x_i : i = n, n+1, n+2, \ldots\} \cup \{x\}$$

as a basis for the irrational point x. The space $(\mathbb{R}, \mathcal{T})$ is Hausdorff, locally compact and not metrizable [3]. But, the space $(\mathbb{R}, \mathcal{T})$ is c-first countable space.

Proof. Let K be a compact subset of $\mathbb{R}$. If $K - \mathbb{Q}$ is infinite, where $\mathbb{Q}$ is the set of rationals, then the open cover $\{U_1(x) : x \in K - \mathbb{Q}\} \cup \{\mathbb{Q}\}$ of K has no finite subcover. We have a contradiction. Thus $K - \mathbb{Q}$ is finite, say, $K - \mathbb{Q} = \{x^1, x^2, x^3, \ldots, x^m\}$. Let U be a neighborhood of K. For each $i = 1, 2, 3, \ldots, m$, since $x^i \in K - \mathbb{Q} \subset U - \mathbb{Q}$, there is an n_i such that $U_{n_i}(x^i) \subset U$. Let $N = \max n_i$. Then $\cup_{i=1}^m U_N(x^i) \cup (K \cap \mathbb{Q}) \subset U$. Thus $\{\cup_{i=1}^m U_N(x^i) \cup (K \cap \mathbb{Q}) : n = 1, 2, 3, \ldots\}$ is a countable neighborhood base of K. Hence $(\mathbb{R}, \mathcal{T})$ is c-first countable. □

LEMMA 2.1. *Let X be a c-first countable and locally compact space, and let K be a compact subset of X. For each neighborhood U of K, there exists a countable neighborhood base $\{U(r) : r \in D\}$ of K such that*

(1) *$U(1) = U$ and*
(2) *if $r_1 < r_2$, then $\overline{U(r_1)} \subset U(r_2)$,*

where D is the set of all rationals of form $k/2^n$, $0 < k/2^n \leq 1$.

Proof. Let us show that for each $r \in D$ we can associate a neighborhood $U(r)$ of K satisfying the above conditions (1) and (2). We proceed by induction on exponent of dyadic fractions, letting $\mathcal{U}_n = \{U(k/2^n) : k = 1, 2, 3, \ldots, 2^n\}$. There exists a countable neighborhood base $\{V_m : m = 1, 2, 3, \dots\}$ of K. We assume that $V_1 \supset V_2 \supset \dots$ and $\overline{V_1}$ is compact.

There is an m_1 such that $\overline{V_{m_1}} \subset U$. $\mathcal{U}_1$ consists of $U(1/2) = V_{m_1}$ and $U(1) = U$. Assume $\mathcal{U}_{n-1}$ constructed. Note that only $U(k/2^n)$ for odd number k requires. There is an $m_n > m_{n-1}$ such that $\overline{V_{m_n}} \subset U(1/2^{n-1})$. We define $U(1/2^n) = V_{m_n}$. For odd $k \neq 1$, we have from $\mathcal{U}_{n-1}$ that $\overline{U(k-1/2^n)} \subset U(k+1/2^n)$. So we define $U(k/2^n)$ to be an open set V satisfying $\overline{U(k-1/2^n)} \subset V \subset \overline{V} \subset U(k+1/2^n)$ and $\overline{V}$ is compact. This completes inductive step. Given any neighborhood W of K, there is an n such that $V_{m_n} = U(1/2^n) \subset W$. Thus the family $\{U(r) : r \in D\}$ is a countable neighborhood base of K. □

THEOREM 2.3. *Let X be a locally compact space. Then X is c-first countable if and only if for any compact subset K of X there exists a continuous nonnegative real valued function on X vanishes exactly on K.*

Proof. By Lemma 2.1, there exists a countable neighborhood base $\{U(r) : r \in D\}$ such that if $U(1) = X$ and that if $r_1 < r_2$, then $\overline{U(r_1)} \subset U(r_2)$. Define a function $h : X \to \mathbb{R}^+$ by $h(x) = \inf\{x \in D : x \in U(r)\}$. Clearly $0 \leq h \leq 1$. It is easy to show that h vanishes exactly on K. Given any $\varepsilon > 0$, we can choose an $r \in D$ such that $r < \varepsilon$. Since $h(U(r)) \subset (-\varepsilon, \varepsilon)$, h is continuous on K. We will show that h is continuous at $x \in X - K$. There are two possibilities;

(1) $h(x) < 1$; Given any $\varepsilon > 0$, we choose r_1 and r_2 in D such that $h(x) - \varepsilon < r_1 < h(x) < r_2 < h(x) + \varepsilon$. Then $U(r_2) - \overline{U(r_1)}$ is a neighborhood of x and $h(U(r_2) - \overline{U(r_1)}) \subset (h(x) - \varepsilon, h(x) + \varepsilon)$.

(2) $h(x) = 1$; Given any $\varepsilon > 0$, there exists a number $r \in D$ such that $1 - \varepsilon < r < 1$. Then $X - \overline{U(r)}$ is a neighborhood of x and $h(X - \overline{U(r)}) \subset (1 - \varepsilon, 1 + \varepsilon)$. Thus h is continuous.

On the other hand, there exists a neighborhood U of K such that $\overline{U}$ is compact. For each positive integer n, the set $U_n = h^{-1}[0, 1/n) \cap U$ is a neighborhood of K. Given any neighborhood V of K, suppose that $U_n \not\subset V$ for all n. For each n, we can choose an $x_n \in U_n - V$. Since $\overline{U}$ is compact, the sequence (x_n) in $\overline{U}$ has a convergent subsequence. Let $x_n \to x$. It is clear that $x \in X - V$ and $h(x_n) \to h(x)$. Since $h(x_n) < 1/n$ for all n, $h(x_n) \to 0$. Thus $h(x) = 0$ and $x \in K$. This is a contradiction. So $U_n \subset V$ for some n. Hence the family $\{U_n : n = 1, 2, 3, \dots\}$ is a countable neighborhood base of K. □

3. Asymptotic stability in general dynamical systems

DEFINITION 3.1. Let $S : X \to 2^Y$ be a function. Then S is called

(1) *upper semicontinuous* at $x \in X$ if for any neighborhood U of $S(x)$, there exists a neighborhood V of x such that $y \in V$ implies $S(y) \subset U$.
(2) *lower semicontinuous* at $x \in X$ if for any neighborhood U of x with $S(x) \cap U \neq \emptyset$, there exists a neighborhood V of x such that $y \in V$ implies $S(y) \cap U \neq \emptyset$.
(3) *continuous* at $x \in X$ if it is upper semicontinuous at x and lower semicontinuous at x.

Let $C(X)$ be the set of all nonempty closed subsets of X.

DEFINITION 3.2. A continuous mapping $f : X \times \mathbb{R}^+ \to C(X)$ is said to be a *general dynamical system* if the following axioms hold:

(1) $f(x, 0) = \{x\}$ for all $x \in X$
(2) if $t_1 t_2 > 0$, then $f(f(x, t_1), t_2) = f(x, t_1 + t_2)$
(3) if $y \in f(x, t)$, then $x \in f(y, -t)$.

Throughout this section, let $f : X \times \mathbb{R}^+ \to C(X)$ be a general dynamical system on c-first countable and locally compact space X.

DEFINITION 3.3. A *trajectory* of f on $[a, b] \subset \mathbb{R}$ ia a continuous mapping $\mathcal{V} : [a, b] \to X$ satisfying $\mathcal{V}(t_2) \in f(\mathcal{V}(t_1), t_2 - t_1)$ for any $t_1, t_2 \in [a, b]$ with $t_1 \leq t_2$.

PROPOSITION 3.1. *Let $y \in f(x, t_2 - t_1)$ with $t_1 \leq t_2$. Then there is a trajectory $\mathcal{V}$ of f on $[t_1, t_2]$ such that $\mathcal{V}(t_1) = x$, $\mathcal{V}(t_2) = y$* [2].

COROLLARY. *$f(x, [a, b])$ is path connected.*

DEFINITION 3.4. Let $x \in X$. The *limit set* $L^+(x)$ of x is defined by

$$L^+(x) = \cap \{\overline{f(x, [t, \infty))} : t \in \mathbb{R}^+\}.$$

PROPOSITION 3.2. *$y \in L^+(x)$ if and only if there exist $t_n \to \infty$ and $y_n \in f(x, t_n)$ such that $y_n \to y$* [2].

DEFINITION 3.5. A compact subset M of X is said to *stable* if for any neighborhood U of M, there is a neighborhood V of M such that $f(V \times \mathbb{R}^+) \subset U$.

PROPOSITION 3.3. *A necessary and sufficient condition for a compact subset M of X to be stable is that there exists a positively invariant neighborhood V of M with $V \subset U$ for any neighborhood U of M.*

Proof. The necessity is obvious. We shall prove the sufficiency. Since X is locally compact, there exists a neighborhood W of M such that $\overline{W} \subset U$ and $\overline{W}$ is compact. Also, there is a neighborhood V of M such that $f(V \times \mathbb{R}^+) \subset W$.

Then $\overline{f(V \times \mathbb{R}^+)} \subset U$ is a positively invariant compact neighborhood of M. □

DEFINITION 3.6. Suppose the set $M \subset X$ is compact. The *region of attraction* $A(M)$ of the set M is defined

$$A(M) = \{x \in X : L^+(x) \neq \emptyset, L^+(x) \subset M\}.$$

PROPOSITION 3.4. *Let M be a compact subset of X. $x \in A(M)$ if and only if there exists $t \in \mathbb{R}^+$ with $f(x, [t, \infty)) \subset U$ for any neighborhood U of M.*

Proof. Necessity: There exists a neighborhood V of M such that $\overline{V} \subset U$ and $\overline{V}$ is compact. Let $y \in L^+(x)$. Then $y_n \to y$ for some $t_n \to \infty$, $y_n \in f(x, t_n)$. Suppose that there is an $s \geq t$ such that $f(x, s) \not\subset \overline{V}$ for each $t \in \mathbb{R}^+$. Since V is a neighborhood of y, we may assume that $y_n \in V$ for all n. There exists $s_n \geq t_n$ such that $f(x, s_n) \not\subset \overline{V}$. Since $f(x, [t_n, s_n])$ is connected, $f(x, [t_n, s_n]) \cap \partial V \neq \emptyset$. Let $z_n \in f(x, r_n) \cap \partial V, t_n \leq r_n \leq s_n$. Since ∂V is compact, (z_n) has a convergent subsequence. Let $z_n \to z \in \partial V$. We have $z \in L^+(x) \subset M$ using the fact $r_n \to \infty$. This is a contradiction. Thus $f(x, [t, \infty)) \subset \overline{V} \subset U$ for some $t \in \mathbb{R}^+$.

Sufficiency: There exists a neighborhood U of M such that $\overline{U}$ is compact. We can choose a $t \in \mathbb{R}^+$ so that $f(x, [t, \infty)) \subset U$. Since $\overline{f(x, \mathbb{R}^+)} = f(x, [0, t]) \cup \overline{f(x, [t, \infty))} \subset f(x, [0, t]) \cup \overline{U}$, $\overline{f(x, \mathbb{R}^+)}$ is compact. Thus $L^+(x) \neq \emptyset$. To show $L^+(x) \subset M$, suppose that there exists an $y \in L^+(x) - M$. There are neighborhoods V of M and W of y such that $V \cap W = \emptyset$. We can choose a $t \in R^+$ so that $f(x, [t, \infty)) \subset V$. Since $W \cap f(x, [t, \infty)) = \emptyset$, $y \notin \overline{f(x, [t, \infty))}$ and so $y \notin L^+(x)$. This is a contradiction. Thus $L^+(x) \subset M$. Hence $x \in A(M)$. □

DEFINITION 3.7.

(1) A compact subset M of X is *attractor* if $A(M)$ is a neighborhood of M.
(2) A compact subset M of X is said to be *asymptotically stable* if M is stable and attractor.

DEFINITION 3.8. Let M be a compact subset of X. Let W be a positively invariant neighborhood of M. A continuous functions $\Phi : W \to \mathbb{R}^+$ is called *Lyapunov function* for M if the following two conditions are satisfied:

(1) $\Phi(x) = 0$ if and only if $x \in M$
(2) If $t > 0$ and $y \in f(x,t)$, then $\Phi(y) \le \Phi(x)$.

DEFINITION 3.9. Let M be a compact subset of X. A Lyapunov function $\Phi : W \to \mathbb{R}^+$ is called *strict Lyapunov function* for M if the following two conditions are satisfied:

(1) If $y \in f(x,t)$, $x \notin M$ and $t > 0$, then $\Phi(y) < \Phi(x)$
(2) For all $y, z \in L^+(x)$, $\Phi(y) = \Phi(z)$.

PROPOSITION 3.5. *Let a compact subset M of X be asymptotically stable and U be a neighborhood of M. Let $x \in A(M)$. If $f(x, \mathbb{R}^+) \subset U$, then $f(V \times \mathbb{R}^+) \subset U$ for some neighborhood V of x.*

Proof. By the hypothesis, there is a neighborhood U_1 of M such that $f(U_1 \times \mathbb{R}^+) \subset U$. By Proposition 3.4, $f(x, [s, \infty)) \subset U$ for some $s \in \mathbb{R}^+$. We can choose a neighborhood W_1 of x such that $f(W_1, s) \subset U_1$, using the fact from $f(x, s) \subset U_1$ and f is upper semicontinuous. Let $t \in [0, s]$. Then $f(x,t) \subset U$. Since f is upper semicontinuous at (x,t), there exist neighborhoods V_t of x and I_t of t, respectively, such that $f(V_t \times I_t) \subset U$. There are finitely many $0 \le t_1 \ \ldots t_n \le s$ such that $[0, s] \subset \cup_{i=1}^n I_{t_i}$. Put $W_2 = \cap_{i=1}^n V_{t_i}$. Then W_2 is an neighborhood of x. Let $y \in W_2$ and $t \in [0, s]$. We have $f(y,t) \subset f(V_{t_i} \times I_{t_i}) \subset U$ using $t \in I_{t_i}$ for some i. Thus $f(W_2 \times [0, s]) \subset U$. Also, $V = W_1 \cap W_2$ is a neighborhood of x. From the fact that $f(V \times [0, s]) \subset f(W_2 \times [0.s]) \subset U$ and $f(V \times [s, \infty)) \subset f(W_1 \times [s, \infty)) = f(f(W_1, s), \mathbb{R}^+) \subset f(U_1 \times \mathbb{R}^+) \subset U$, we have $f(V \times \mathbb{R}^+) = f(V \times [0, s]) \cup f(V \times [s, \infty)) \subset U$. □

THEOREM 3.6. *Let a compact subset M of X be asymptotically stable. Then there exists a Lyapunov function $\Phi : A(M) \to [0,1]$ for M.*

Proof. Let D be the set of all rationals r of form $k/2^n$ with $0 < k/2^n \leq 1$. By Lemma 2.1, there exists a countable neighborhood base $\{U(r) : r \in D\}$ of M satisfying

(1) $U(1) = A(M)$ and

(2) if $r_1 < r_2$, then $\overline{U(r_1)} \subset U(r_2)$.

Define a function $l : A(M) \times \mathbb{R}^+ \to [0,1]$ by $l(x,t) = \inf\{r \in D : f(x,t) \subset U(r)\}$. Let us show that l is continuous. Let $(x,t) \in A(M) \times \mathbb{R}^+$. There are two possibilities:

(1) $l(x,t) = 0$; Give any $\varepsilon > 0$ we can choose an $r \in D$ such that $r < \varepsilon$ and $f(x,t) \subset U(r)$. Since f is upper semicontinuous at (x,t), there exists a neighborhood A of (x,t) such that $f(y,s) \subset U(r)$ for all $(y,s) \in A$. We have $l(A) \subset (-\varepsilon, \varepsilon)$. Thus l is continuous at (x,t).

(2) $l(x,t) > 0$; Give any $\varepsilon > 0$ we can choose an $r_1, r_2 \in D$ such that $l(x,t) - \varepsilon < r_1 < l(x,t) < r_2 < l(x,t) + \varepsilon$ and $f(x,t) \subset U(r_2)$. We have $f(x,t) \not\subset \overline{U(r_1)}$, that is, $f(x,t) \cap (X - \overline{U(r_1)}) \neq \emptyset$. Since f is lower semicontinuous at (x,t) there exists a neighborhood A of (x,t) such that $f(y,s) \cap (X - \overline{U(r_1)}) \neq \emptyset$, that is, $f(y,s) \not\subset U(r_1)$ for all $(y,s) \in A$. Also, there exists a neighborhood B of (x,t) such that $f(y,s) \subset U(r_2)$ for all $(y,s) \subset B$, using the fact from that f is upper semicontinuous.Let $V = A \cap B$. Then V is a neighborhood of (x,t) and $l(V) \subset (l(x,t) - \varepsilon, l(x,t) + \varepsilon)$. Thus l is continuous at (x,t). Hence l is a continuous function.

Define a function $\Phi : A(M) \to [0,1]$ by $\Phi(x) = \sup\{l(x,t) : t \in \mathbb{R}^+\}$. Let $x \in M$. Given any $r \in D$, since $f(x, \mathbb{R}^+) \subset M \subset U(r)$, we have $\Phi(x) \leq r$. Thus $\Phi(x) = 0$. Let $x \in A(M) - M$. There exists $r \in D$ such that $x \notin U(r)$. Then $0 < r \leq l(x,0) \leq \Phi(x)$. Let $y \in f(x,t)$ for $t \in \mathbb{R}^+$. Suppose that $\Phi(y) > \Phi(x)$. We can choose $r \in D$ such that $\Phi(y) > r > \Phi(x)$. There exists $s \in \mathbb{R}^+$ such that $r < l(y,s)$. We have $f(y,s) \not\subset U(r)$. Since $l(x, t+s) \leq \Phi(x) < r$, we have $f(y,s) \subset f(f(x,t),s) = f(x,t+s) \subset U(r)$. This is a contradiction, thus $\Phi(y) \leq \Phi(x)$. Let us show that Φ is continuous. There are three possibilities;

(1) $\Phi(x) = 0$; Given any $\varepsilon > 0$, we can choose $r \in D$ such that $r < \varepsilon$. Since M is stable, there exists a neighborhood V of M such that $f(V \times \mathbb{R}^+) \subset U(r)$. Since V is a neighborhood of x

and $\Phi(V) \subset (-\varepsilon, \varepsilon)$, Φ is continuous at $x \in X$

(2) $0 < \Phi(x) < 1$; Given any $\varepsilon > 0$, we can choose $r_1, r_2 \in D$ such that $\Phi(x) - \varepsilon < r_1 < \Phi(x) < r_2 < \Phi(x) + \varepsilon$. There exists $t \in \mathbb{R}^+$ such that $r_1 < l(x,t)$. We have $f(x,t) \not\subset \overline{U(r_1}$, that is, $f(x,t) \cap (X - \overline{U(r_1)}) \neq \emptyset$. Since f is lower semicontinuous at (x,t), there exists a neighborhood V_1 of x such that $f(y,t) \cap (X - \overline{U(r_1)}) \neq \emptyset$, that is, $f(y,t) \not\subset \overline{U(r_1)}$ for all $y \in V_1$. Then $\Phi(y) \geq l(y,t) \geq r_1 > \Phi(x) - \varepsilon$ for all $y \in V_1$. Since $f(x, \mathbb{R}^2) \subset U(r_2)$, by Proposition 3.6, there exists a neighborhood V_2 of x such that $f(V \times \mathbb{R}^+) \subset U(r_2)$. Then we have $\Phi(y) \leq r_2 < \Phi(x) + \varepsilon$ for all $y \in V_2$. Let $V = V_1 \cap V_2$. Then V is a neighborhood of x and $\Phi(V) \subset (\Phi(x) - \varepsilon, \Phi(x) + \varepsilon)$. Φ is continuous at x.

(3) $\Phi(x) = 1$; Given any $\varepsilon > 0$, there exists $t \in \mathbb{R}^+$ such that $1 - \varepsilon < l(x,t)$. We can choose $r \in D$ such that $1 - \varepsilon < r < l(x,t)$. Then we have $f(x,t) \not\subset \overline{U(r)}$, that is, $f(x,t) \cap (X - \overline{U(r)}) \neq \emptyset$. Since f is lower semicontinuous at (x,t), there exists a neighborhood V of x such that $f(y,t) \cap (X - \overline{U(r)}) \neq \emptyset$, that is, $f(y,t) \not\subset \overline{U(r)}$ for all $y \in V$. Thus we have $1 - \varepsilon < r \leq l(x,t) \leq \Phi(y) \leq 1 < 1 + \varepsilon$ for all $y \in V$, that is, $\Phi(V) \subset (1-\varepsilon, 1+\varepsilon)$. Hence Φ is continuous at x.

Therefore Φ is a Lyapunov function for M. □

THEOREM 3.7. *A necessary and sufficient condition for a compact subset M of X to be asymptotically stable is that there exists a strict Lyapunov function for M.*

Proof. Necessity: Let the set M be asymptotically stable. By Theorem 3.7, there exists a Lyapunov function $\Phi : A(M) \to [0,1]$ for M. We define a function $h : A(M) \times \mathbb{R}^+ \to [0,1]$ by $h(x,t) = \max\{\Phi(y) : y \in f(x,t)\}$. We shall prove that the following conditions are satisfied:

(1) h is continuous.
(2) $h(x,t) = 0$ for all $(x,t) \in A(M) \times \mathbb{R}^+$.
(3) $h(x,0) > 0$ for all $x \in A(M) - M$.
(4) $h(y,t) \leq h(x, t+s)$ for any $y \in f(x,s)$ and $t \in \mathbb{R}^+$.
(5) For all $t \geq s$ and $x \in A(M)$, $h(x,t) \leq h(x,s$.
(6) if $t \to \infty$, then $h(x,t) \to 0$.

To verify condition (1), let $(x,t) \in A(M) \times \mathbb{R}^+$. Then there are two possibilities;

(a) If $h(x,t) = 0$, given any $\varepsilon > 0$, we can choose $r \in D$ so that $r < \varepsilon$. Since M is stable, there exists a neighborhood V of M such that

$f(V \times \mathbb{R}^+) \subset U(r)$. For each $y \in f(x,t)$, since $0 \leq \Phi(y) \leq h(x,t) = 0$, we have $\Phi(y) = 0$ and so $y \in M$. Thus $f(x,t) \subset M$. Since $f(x,t) \subset V$ and f is upper semicontinuous at (x,t), there exists a neighborhood A of (x,t) such that $f(y,s) \subset V$ for all $(y,s) \in A$. For any $z \in f(y,s)$, since $f(z, \mathbb{R}^+) \subset f(V \times \mathbb{R}^+) \subset U(r)$, we have $\Phi(z) \leq r$. Thus $h(y,s) \leq r < \varepsilon$. Hence h is continuous at (x,t).

(b) If $0 < h(x,t) < 1$, given any $\varepsilon > 0$, there exists $z \in f(x,t)$ such that $h(x,t) - \varepsilon < \Phi(z)$. Since Φ is a continuous function, there exists a neighborhood V of z such that $\Phi(w) > h(x,t) - \varepsilon$ for all $w \in V$. Also, there is a neighborhood A_1 of (x,t) such that $f(y,s) \cap V \neq \emptyset$ for all $(y,s) \in A_1$, from that fact that $f(x,t) \cap V \neq \emptyset$ and f is lower semicontinuous at (x,t). For each $(y,s) \in A_1$, we have $f(y,s) \cap V \neq \emptyset$. Let $w \in f(y,s) \cap V$. Then $h(x,t) - \varepsilon < \Phi(w) \leq h(y,s)$. For each $a \in f(x,t)$, since $\Phi(a) \leq h(x,t) < h(x,t) + \varepsilon/2$, there is a neighborhood B_a of a such that $\Phi(z) < h(x,t) + \varepsilon/2$ for all $z \in B_a$. $\cup_{a \in f(x,t)} V_a$ is a neighborhood of $f(x,t)$. Since $f(x,t)$ is upper semicontinuous at (x,t), there exists a neighborhood A_2 of (x,t) such that $f(y,s) \subset \cup_{a \in f(x,t)} V_a$ for all $(y,s) \in A_2$.

Let $(y,s) \in A_2$. For any $z \in f(y,s)$ we can choose $a \in f(x,t)$ so that $z \in V_a$. Then $\Phi(z) < h(x,t) + \varepsilon/2$. Thus $h(y,s) \leq h(x,t) + \varepsilon/2 < h(x,t) + \varepsilon$. Let $A = A_1 \cap A_2$. Then A is a neighborhood of (x,t) and $h(x,t) - \varepsilon < h(y,s) < h(x,t) + \varepsilon$ for all $(y,s) \in A$. Thus h is continuous at (x,t).

To show condition (2), let $x \in M$ and $t \in \mathbb{R}^+$. By virtue of Lyapunov function Φ, $\Phi(x) = 0$. Also, $\Phi(y) = 0$ by $0 \leq \Phi(y) \leq \Phi(x) = 0$ for $y \in f(x,t)$. We conclude that $h(x,t) = 0$.

To prove condition (3), choose $x \in A(M) - M$, then $\Phi(x) > 0$. Hence, $h(x,0) = \Phi(x) > 0$ using $f(x,0) = \{x\}$.

To prove condition (4), let $y \in f(x,s)$ and $t \in \mathbb{R}^+$. Then $h(y,t) = \sup\{\Phi(z) : z \in f(y,t)\} \leq \sup\{\Phi(z) : z \in f(f(x,s),t)\} = \sup\{\Phi(z) : z \in f(x, s+t)\} = h(x, s+t)$.

To verify condition (5), let $t \geq s$ and $x \in A(M)$. There exists a $z \in f(x,s)$ such that $y \in f(z, t-s)$for $y \in f(x,t) = f(f(x,s), t-s)$. Then $h(x,t) \leq h(x,s)$ by $\Phi(y) \leq \Phi(z) \leq h(x,s)$.

Finally, to verify condition (6), let $\varepsilon > 0$ and $U = \{x \in A(M) : \Phi(x) < \varepsilon\}$. For $x \in A(M)$ and a neighborhood U of M, there exists an $s \in \mathbb{R}^+$ such that $f(x, [s, \infty)) \subset U$.

Also, $y \in f(x,t) \subset f(x,[s,\infty)) \subset U$ for $y \in f(x,t)$ and $t \geq s$. Hence $h(x,t) \leq \varepsilon$. We conclude that $h(x,t) \to 0$ if $t \to \infty$.

Next, define function $\Psi : A(M) \to \mathbb{R}^+$ by $\Psi(x) = \int_0^\infty e^{-t}h(x,t)\,dt$.

We verify that Ψ is strict Lyapunov function. The continuity of Ψ is obvious. Let $x \in M$. By (1), $h(x,t) = 0$ for $t \in \mathbb{R}^+$. Hence, $\Psi(x) = 0$. Let $x \in A(M) - M$. By virtue to (3), $h(x,0) > 0$. Hence $\Psi(x) > 0$. Let $y \in f(x,s)$. Then

$$\Psi(y) = \int_0^\infty e^{-t}h(y,t)\,dt \leq \int_0^\infty e^{-t}h(x,s+t)\,dt \leq \int_0^\infty e^{-t}h(x,t)\,dt$$
$$= \Psi(x).$$

Let $x \in A(M)-M$ and $y \in f(x,s)$ for $s > 0$. To show that $\Psi(y) < \Psi(x)$, assume that $\Psi(y) = \Psi(x)$. Then $h(x,s+t) = h(x,t)$ for any $t \in \mathbb{R}^+$. Put $t = ns$, where $n = 0,1,2,\ldots$. Then $h(x,0) = h(x,ns)$ for given n. We have $\lim_{n\to\infty} h(x,ns) = 0$ and $h(x,0) = 0$, contradicting the condition (3) that $h(x,0) > 0$. Hence $\Psi(y) < \Psi(x)$. Let $x \in A(M)$. By definition, $L^+(x) \neq \emptyset$ and $L^+(x) \subset M$. Also, $f(y,\mathbb{R}^+) \subset M$ for $y \in L^+(x)$. Hence $h(y,t) = 0$ for $t \in \mathbb{R}^+$. By virtue to the definition of Ψ, $\Psi(y) = 0$.

Sufficiency: Let a function $\Psi : W \subset A(M) \to \mathbb{R}^+$ be strict Lyapunov function. To show that M is stable, assume that there exists a neighborhood U_0 of M such that $f(V \times \mathbb{R}^+) \not\subset U_0$ for any neighborhood V of M.

There exists a neighborhood U of M such that $\overline{U} \subset U_0$ and $\overline{U}$ is compact.

Since $f(U(1/2^n) \times \mathbb{R}^+) \not\subset \overline{U}$ for n, there exists $x_n \in U(1/2^n)$ and $t_n \in \mathbb{R}^+$ such that $f(x_n,t_n) \not\subset \overline{U}$. Let $x_n \to x \in M$. Since $f(x_n,[0,t_n])$ is connected, $f(x_n,[0,t_n]) \cap \partial U \neq \emptyset$. Let $y_n \in f(x_n,[0,t_n]) \cap \partial U$. From the fact that ∂U is compact, let $y_n \to y \in \partial U$. Since $\Psi(y_n) \leq \Psi(x_n)$,

$$\Psi(y) = \Psi(\lim_{n\to\infty} y_n) = \lim_{n\to\infty} \Psi(y_n) \leq \lim_{n\to\infty} \Psi(x_n) = \Psi(\lim_{n\to\infty} x_n) = \Psi(x).$$

$\Psi(x) = 0$ since $x \in M$. We obtain that $y \in M$, from the fact that $\Psi(y) = 0$, contradicting the fact that $y \notin M$. Therefore, M is stable. Next, let us verify that M is attractor. By Proposition 3.3, there exists positively invariant compact neighborhood U of M such that $U \subset W$. We obtain that $f(x,\mathbb{R}^+) \subset U$ for any $x \in U$. Since U is compact and $\overline{f(x,\mathbb{R}^+)}$ is compact, $L^+(x) \neq \emptyset$. Now we prove that $L^+(x) \subset M$. We shall prove this fact by assuming the opposite and arriving at a contradiction. Suppose that $L^+(x)$ is not subset of M. So, let $y \in L^+(x) - M$ and $z \in f(y,t)$ for $t > 0$.

From the fact that Ψ is strict Lyapunov function and $z \in f(y,t) \subset f(L^+(x),t) \subset L^+(x)$, we obtain that $\Psi(z) = \Psi(y)$, contradicting the condition that $\Psi(z) \neq \Psi(y)$. Therefore $L^+(x) \subset M$. Also, $x \in A(M)$ and $U \subset A(M)$. Hence, M is attractor. Consequently, M is asymptotically stable. □

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