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SKEW POWER SERIES EXTENSIONS OF α-RIGID P.P.-RINGS

EBRAHIM HASHEMI AND AHMAD MOUSSAVI

ABSTRACT. We investigate skew power series of α-rigid p.p.-rings, where α is an endomorphism of a ring R which is not assumed to be surjective. For an α-rigid ring R, $R[[x;\alpha]]$ is right p.p., if and only if $R[[x,x^{-1};\alpha]]$ is right p.p., if and only if R is right p.p. and any countable family of idempotents in R has a join in $I(R)$.

1. Introduction

Throughout this paper R denotes an associative ring with identity and $\alpha : R \to R$ is an endomorphism. We denote $C(R)$ the center of R and $S = R[[x;\alpha]]$ the skew power series ring, whose elements are power series of the form $\Sigma_{i=0}^{\infty} r_i x^i$ with coefficients $r_i \in R$, where the addition is defined as usual and the multiplication subject to the condition $xb = \alpha(b)x$, for any $b \in R$. The set $\{x^i\}_{i\geq 0}$ is an Öre subset of $R[[x;\alpha]]$, so that one can localize $R[[x;\alpha]]$ and form the skew Laurent series ring $R[[x,x^{-1};\alpha]]$. Elements of $R[[x,x^{-1};\alpha]]$ are formal combinations of elements of the form $x^{-j}rx^i$, where $r \in R$ and i,j are nonnegative integers.

Recall that R is (quasi-)Baer if the right annihilator of every (right ideal) non-empty subset of R is generated (as a right ideal) by an idempotent of R. These definitions are left-right symmetric. The study of Baer rings has its roots in functional analysis. In [19] Rickart studied C^*-algebras with the property that every right annihilator of any element is generated by a projection (i.e., p is a projection if $p = p^2 = p^*$, where $*$ is the involution on the algebra). Using Rickart's work, Kaplansky [13] defined an AW*-algebra as a C*-algebra with the stronger property that the right annihilator of the nonempty subset is generated by a projection. A ring satisfying a generalization of Rickart's condition

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(i.e., every right annihilator of any element is generated (as a right ideal) by an idempotent) has a homological characterization as a right p.p.-ring. A ring R is called a *right* (*resp. left*) *p.p.*-ring if every principal right (resp. left) ideal is projective (equivalently, if the right (resp. left) annihilator of an element of R is generated (as a right (resp. left) ideal) by an idempotent of R). R is called a p.p.-ring if it is both right and left p.p. In [4] Birkenmeier et al. defined a ring to be called *right* (*resp. left*) *principally quasi-Baer* (or simply *right* (*resp. left*) *p.q.-Baer*) if the right annihilator of a principal right (resp. left) ideal of R is generated by an idempotent. A ring is called p.q.-Baer if it is both right and left p.q.-Baer. Observe that every biregular ring and every quasi-Baer ring is p.q.-Baer. Note that in a *reduced* ring R (i.e. it has no nonzero nilpotent elements), R is p.q.-Baer if and only if R is p.p. For more details and examples of right p.q.-Baer rings, see [4].

In [5], Birkenmeier et al. showed that the quasi-Baer condition is preserved by many polynomial extensions including $R[[x;\alpha]]$ and $R[[x,x^{-1};\alpha]]$. Following Krempa [15], a ring R is said to be *α-rigid* if for each $a \in R$, $a\alpha(a) = 0$ implies that $a = 0$. Note that α-rigid rings are reduced, and hence *abelian* (i.e. every idempotent is central). In [9] Hong et al. showed that, an α-rigid ring R is quasi-Baer if and only if $R[[x;\alpha]]$ is quasi-Baer. Following [18], a ring R is called *Armendariz* if whenever two polynomials $f(x) = \sum_{i=0}^{m} a_i x^i$, $g(x) = \sum_{j=0}^{n} b_j x^j \in R[x]$ satisfy $f(x)g(x) = 0$ we have $a_i b_j = 0$ for every i, j. By [2, Theorem 10], for an Armendariz ring R, R is left p.p. if and only if $R[x]$ is left p.p. Fraser and Nicholson in [7] showed that $R[[x]]$ is reduced p.p. if and only if R is reduced p.p. and any countable family of idempotents of R has a least upper bound in $I(R)$, the set of all idempotents. Z. Liu in [16, Theorem 3], showed that: If R is a ring such that all left semicentral idempotents are central, then $R[[x]]$ is right p.q.-Baer if and only if R is right p.q.-Baer and any countable family of idempotents in R has a generalized join in $I(R)$.

In this paper we show that for an α-rigid ring R, $R[[x;\alpha]]$ is right p.p. if and only if $R[[x,x^{-1};\alpha]]$ is right p.p.if and only if R is right p.p. and any countable family of idempotents in R has a join in $I(R)$. As a consequence, for a reduced ring R, $R[[x,x^{-1}]]$ is right p.p. if and only if $R[[x]]$ is right p.p. if and only if R is right p.p. and any countable family of idempotents in R has a join in $I(R)$. This extends the main result of Fraser and Nicholson [7].

2. Skew power series extensions of α-rigid p.p.-rings

In this section, we give a necessary and sufficient condition for some rings under which the ring $R[[x;\alpha]]$ is right p.p.

For a nonempty subset X of R, $r_R(X)$ and $\ell_R(X)$ denote the right and left annihilators of X in R respectively. We put $rAnn_R(2^R)=\{r_R(V) \mid V \subseteq R\}$ and $\ell Ann_R(2^R) = \{\ell_R(V) \mid V \subseteq R\}$.

Motivated by results in Armendariz [2], Anderson and Camillo [1], Kim and Lee [14], Hong et al. [9] and [10], we introduce conditions (SA1) and (SA2) which are skew power series versions of the Armendariz rings:

DEFINITION 2.1. For a ring R and a monomorphism $\alpha : R \to R$, we say R satisfies the (SA1) condition if for each $f(x) = \sum_{i=0}^{\infty} a_i x^i$ and $g(x) = \sum_{j=0}^{\infty} b_j x^j \in S = R[[x;\alpha]]$, $f(x)g(x) = 0$, implies that $a_i b_j = 0$ for all i, j.

LEMMA 2.2. [9, Lemma 4]. *Let R be α-rigid. Then we have the following:*

(i) *If $ab = 0$, then $a\alpha^n(b) = \alpha^n(a)b = 0$ for each positive integer n.*

(ii) *If $a\alpha^k(b) = 0$ for some positive integer k, then $ab = 0$.*

PROPOSITION 2.3. *Let R be α-rigid and S the skew power series ring $R[[x;\alpha]]$. Then we have the following:*

(i) *R satisfies conditione (SA1);*

(ii) *$\varphi : rAnn_R(2^R) \to rAnn_S(2^S); A \to AS$ is bijective;*

(iii) *$\psi : \ell Ann_R(2^R) \to \ell Ann_S(2^S); B \to SB$ is bijective.*

Proof. (i) It follows from [10, Proposition 17]. (ii) It is clear that φ is a well defined map. Let J be an element of $rAnn_S(2^S)$. There exists a nonempty subset Y of S such that $r_S(Y) = J$. Suppose that X is the set of coefficients of elements of Y. We show that $r_S(Y) = r_R(X)S$. Since R is α-rigid, $r_R(X) \subseteq r_S(Y)$ and hence $r_R(X)S \subseteq r_S(Y)$. Let $f(x) = a_0 + a_1x + \cdots \in r_S(Y)$. Since R satisfies condition (SA1), $Xa_i = 0$ for $i = 0, 1, \cdots$. Hence $f(x) \in r_R(X)S$, thus $r_S(Y) = r_R(X)S$.

Similarly we can prove (iii). □

DEFINITION 2.4. (Z. Liu, [16]). Let $\{e_0, e_1, \cdots\}$ be a countable family of idempotents of R. We say $\{e_0, e_1, \cdots\}$ has a join in $I(R)$ if there exists an idempotent $e \in I(R)$ such that

1. $e_i(1 - e) = 0$, and
2. If $f \in I(R)$ is such that $e_i(1 - f) = 0$, then $e(1 - f) = 0$.

THEOREM 2.5. *Let R be α-rigid. Then the following conditions are equivalent:*

1. $S = R[[x;\alpha]]$ is right p.p.

2. R is right p.p. and any countable family of idempotents in R has a join in $I(R)$.

Proof. 1$\Longrightarrow$2. Let $a \in R$. There exists an idempotent $e(x) = e_0 + e_1x + \cdots \in S$ such that $r_S(a) = e(x)S$. By [9, Corollary 7], $e(x) = e_0$ and thus $r_S(a) = e_0S$. Therefore $r_R(a) = e_0R$. Suppose that $\{e_0, e_1, \cdots\}$ is a countable family of idempotents in R. Set $\phi(x) = e_0 + e_1x + e_2x^2 + \cdots \in S$. Since S is right p.p., there exists an idempotent $e(x) = f_0 + f_1x + \cdots \in S$, such that $r_S(\phi(x)) = e(x)S$. By a similar argument we have, $r_S(\phi(x)) = f_0S$. Hence, by Lemma 2.2, $e_if_0 = 0$ for $i = 0, 1, \cdots$. Let $g = 1 - f_0$. Then $e_i(1-g) = 0$ for each i. Suppose that h is an idempotent of R such that $e_i(1-h) = 0$ for each i. Then by Lemma 2.2, $(1-h) \in r_S(\phi(x))$. Thus $(1-h) = f_0(1-h)$ and $g(1-h) = (1-f_0)(1-h) = 0$. Hence g is a join of the set $\{e_0, e_1, \cdots\}$.

$2 \Longrightarrow 1$. Let $f(x) = a_0 + a_1x + \cdots \in S$. Then there exist idempotents e_i, with $i = 0, 1, \cdots$, such that $r_R(a_i) = e_iR$. Suppose that h is a join of the set $\{1 - e_i | i = 0, 1, \cdots\}$. Thus $(1-e_i)(1-h) = 0$ and hence $(1-h) = e_i(1-h)$. Thus, $a_i(1-h) = a_ie_i(1-h) = 0$ for $i = 0, 1, \cdots$. Hence $(1-h) \in r_S(f(x))$, by Lemma 2.2, which implies that $(1-h)S \subseteq r_S(f(x))$. Suppose that $g(x) = b_0 + b_1x + \cdots \in r_S(f(x))$. Since R satisfies condition (SA1), $a_ib_j = 0$ for all i, j. Then $b_j = e_ib_j$ for all i,j. Now $b_j(1-e_i) = 0$ because $e_i \in C(R)$ for all i, j. Since R is right p.p., $r_R(b_j) = f_jR$ for idempotents $f_j \in R$. Thus $(1-e_i) \in r_R(b_j) = f_jR$, so $(1-e_i) = f_j(1-e_i)$ for all i, j. Hence from $(1-e_i) \in C(R)$, we have $(1-e_i)(1-f_j) = 0$. Since h is a join of $\{1-e_i | i = 0, 1, \cdots\}$, $h(1-f_j) = 0$ for all j. Hence $b_j = b_j - b_jf_j = (1-f_j)b_j = (1-h)(1-f_j)b_j \in (1-h)R$ for all j. So $g(x) \in (1-h)S$. Therefore $r_S(f(x)) = (1-h)S$, and hence S is right p.p. □

COROLLARY 2.6. (Fraser and Nicholson [7, Theorem 3]). *Let R be a reduced ring. Then the following conditions are equivalent:*

1. $R[[x]]$ is right p.p.

2. R is right p.p. and any countable family of idempotents in R has a join in $I(R)$.

3. Skew Laurent power series extensions of α-rigid p.p.-rings

In this section, we give a necessary and sufficient condition for some rings under which the ring $R[[x, x^{-1};\alpha]]$ is right p.p.

Now consider D.A. Jordan's construction of the ring $A(R,\alpha)$ (See [12], for more details). Let $A(R,\alpha)$ or A be the subset $\{x^{-i}rx^i \mid r \in R\ ,\ i \geq 0\}$ of the skew power series ring $R[[x,x^{-1};\alpha]]$. For each $j \geq 0$, $x^{-i}rx^i = x^{-(i+j)}\alpha^j(r)x^{(i+j)}$. It follows that the set of all such elements forms a subring of $R[[x,x^{-1};\alpha]]$ with $x^{-i}rx^i + x^{-j}rx^j = x^{-(i+j)}(\alpha^j(r) + \alpha^i(s))x^{(i+j)}$ and $(x^{-i}rx^i)(x^{-j}sx^j) = x^{-(i+j)}\alpha^j(r)\alpha^i(s)x^{(i+j)}$ for $r,s \in R$ and $i,j \geq 0$. Note that α is actually an automorphism of $A(R,\alpha)$. We have $R[[x,x^{-1};\ \alpha]] \simeq A[[x,x^{-1};\alpha]]$, by way of an isomorphism which maps $x^{-i}rx^j$ to $\alpha^{-i}(r)x^{j-i}$. Also for an automorphism α of R we have $R = A(R,\alpha)$.

DEFINITION 3.1. For a ring R and a monomorphism $\alpha : R \to R$, we say R satisfies the (SA2) condition if for each $f(x) = \sum_{i=m}^{\infty} u_i x^i$ and $g(x) = \sum_{j=n}^{\infty} v_j x^j \in T = A[[x,x^{-1};\alpha]]$, $f(x)g(x) = 0$, implies that $u_i v_j = 0$ for all i,j.

PROPOSITION 3.2. *Let α be an automorphism of R. Let R be α-rigid and T the skew Laurent power series ring $R[[x,x^{-1};\alpha]]$. Then we have the following:*
(i) *R satisfies condition (SA2) ;*
(ii) *$\varphi : rAnn_R(2^R) \to rAnn_T(2^T); A \to AT$ is bijective;*
(iii) *$\psi : \ell Ann_R(2^R) \to \ell Ann_T(2^T); B \to TB$ is bijective.*

Proof. (i) Let $f(x) = \sum_{i=m}^{\infty} u_i x^i$, $g(x) = \sum_{j=n}^{\infty} v_j x^j \in T = A[[x,x^{-1};\alpha]]$ and $f(x)g(x) = 0$ with $m,n \in \mathbb{Z}$. Put $f_1(x) = x^{-m}f(x)$ and $g_1(x) = g(x)x^{-n}$, hence $f_1(x)g_1(x) = (\sum_{i=m}^{\infty} \alpha^m(u_i)x^{i-m})(\sum_{j=n}^{\infty} v_j x^{j-n}) = 0$. By [9, Proposition 17], $\alpha^m(u_i)v_j = 0$ for all i,j. Hence $u_i v_j = 0$ for all i,j, by Lemma 2.2.

In a similar way as in the proof of Propositions 2.2, we can prove (ii) and (iii). □

LEMMA 3.3. *A ring R is α-rigid if and only if $A(R,\alpha)$ is α-rigid.*

Proof. It is clear that any subring of an α-rigid ring is also α-rigid. Suppose that R is α-rigid and $(x^{-i}rx^i)\alpha(x^{-i}rx^i) = 0$, where $i \geq 0$ and $r \in R$. Hence $r\alpha(r) = 0$, and so $r = 0$. □

LEMMA 3.4. *Let R be α-rigid. Then each countable family of idempotents in R has a join in $I(R)$ if and only if each countable family of idempotents in $A(R,\alpha)$ has a join in $I(A(R,\alpha))$.*

Proof. Let $\{e_i^{'} \mid i = 0,1,\cdots\}$ be a countable family of idempotents in A. For each $e_i^{'}$ there exists an idempotent $e_i \in R$ and nonnegative integer j_i such that $e_i^{'} = x^{-j_i}e_i x^{j_i}$. Then $\{e_i \mid i = 0,1,\cdots\}$ has a join e in

$I(R)$. We show that e is a join of $\{e_i' \mid i = 0, 1, \cdots\}$. Since $e_i(1-e) = 0$, $e_i'(1-e) = 0$ for all i, by Lemma 2.2. Suppose that $f' \in I(A)$ is such that $e_i'(1-f') = 0$ for all i. There exist an idempotent $f \in R$ and nonnegative integer n such that $f' = x^{-n}fx^n$. Then $1 - f' = x^{-n}(1-f)x^n$. Since $e_i'(1-f') = 0$, $e_i(1-f) = 0$ for all i by [9, Proposition 5], because $\alpha(e) = e$. Since e is a join of $\{e_i \mid i = 0, 1, \cdots\}$, $e(1-f) = 0$. By Lemma 2.2, $e(1-f') = 0$, and hence e is a join of $\{e_i' \mid i = 0, 1, \cdots\}$. Conversely, suppose that $\{e_i \mid i = 0, 1, \cdots\}$ is a countable family of idempotents in R. Then $\{e_i \mid i = 0, 1, \cdots\}$ has a join e' in $I(A)$. There exist an idempotent $e \in R$ and nonnegative integer n such that $e' = x^{-n}ex^n$. By a similar argument one can show that e is a join of $\{e_i \mid i = 0, 1, \cdots\}$. □

LEMMA 3.5. *Let R be α-rigid. Then R is right p.p. if and only if $A(R, \alpha)$ is right p.p.*

Proof. Assume that R is right p.p. Let $a = x^{-i}tx^i$ be an element of A and $x^{-j}bx^j \in r_A(a)$. By Lemma 2.2, $b \in r_R(t)$. Since R is right p.p., $r_R(t) = eR$ for an idempotent $e \in R$. Thus $eb = b$, so by Lemma 2.2, $\alpha^n(e)b = b$ for each positive integer n. Hence $e(x^{-j}bx^j) = x^{-j}bx^j$, thus $r_A(a) \subseteq eA$. Since R is α-rigid, $eA \subseteq r_A(a)$. Hence $r_A(a) = eA$, thus A is right p.p. Conversely, suppose that A is right p.p. Let $t \in R$. Since R is α-rigid and A is p.p., $r_A(t) = (x^{-j}ex^j)A$, where e is an idempotent of R and j is a nonnegative integer. By Lemma 2.2, $eR \subseteq r_R(t)$. Now let $b \in r_R(t)$. By Lemma 2.2, $b \in r_A(t) = (x^{-j}ex^j)A$, hence $b = (x^{-j}ex^j)b$. Therefore $b = eb$ and so $r_R(t) \subseteq eR$, which implies that R is right p.p. □

THEOREM 3.6. *Let R be α-rigid. Then the following conditions are equivalent:*

1. $R[[x, x^{-1}; \alpha]]$ is right p.p.

2. R is right p.p. and any countable family of idempotents in R has a join in $I(R)$.

Proof. We have $R[[x, x^{-1}; \alpha]] \simeq A[[x, x^{-1}; \alpha]]$ where α is an automorphism of A. By Lemma 3.3, R is α-rigid if and only if A is α-rigid. By Lemma 3.4, any countable family of idempotents in R has a join in $I(R)$ if and only if any countable family of idempotents in A has a join in $I(A)$. By Lemma 3.5, R is right p.p. if and only if A is right p.p. The rest of the proof is similar to the proof of Theorem 2.5. □

LEMMA 3.7. *Every α-rigid ring satisfies condition (SA2).*

Proof. We observe that α is an automorphism of $A(R, \alpha)$ and by Lemma 3.4, A is α-rigid. Now the proof follows from Proposition 3.2. □

The following result is a generalization of Fraser and Nicholson [7]:

COROLLARY 3.8. *For an α-rigid ring R, the following conditions are equivalent:*
1. $R[[x;\alpha]]$ is right p.p.
2. $R[[x,x^{-1};\alpha]]$ is right p.p.
3. R is right p.p. and any countable family of idempotents in R has a join in $I(R)$.

Proof. It follows from Theorems 2.5 and 3.6. □

COROLLARY 3.9. *For a reduced ring R, the following conditions are equivalent:*
1. $R[[x]]$ is right p.p.
2. $R[[x,x^{-1}]]$ is right p.p.
3. R is right p.p. and any countable family of idempotents in R has a join in $I(R)$.

The following example [6, Example 3.6] shows that condition " any countable family of idempotents in R has a join in $I(R)$" is not superfluous.

EXAMPLE 3.10. There is a reduced right p.p.-ring R such that $R[[x;\alpha]]$ is not a right p.p.-ring. For a given field F, let

$$R = \{(a_n)_{n=1}^{\infty} \in \Pi_{n=1}^{\infty} F_n \mid a_n \text{ is eventually constant } \}$$

which is a subring of $\Pi_{n=1}^{\infty} F_n$, where $F_n = F$ for $n = 1, 2, \cdots$. Then the ring R is a commutative von Neumann regular ring and hence it is right p.p. Let α be the identity map on R. Then R is α-rigid, but $R[[x;\alpha]]$ is not right p.p.

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Department of Mathematics, University of Tarbiat Modarres, P.O. Box 14115-170, Tehran, Iran
E-mail: eb_hashemi@yahoo.com
moussava@modares.ac.ir