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ON SOME TYPES OF CONTINUOUS FUZZY MULTIFUNCTIONS

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Abstract. In this paper, by using operations, some characterizations and some properties of fuzzy lower and upper continuous multifunctions and its weaker and stronger forms including fuzzy lower and upper weakly continuous, fuzzy lower and upper θ-continuous, fuzzy lower and upper strongly θ-continuous, fuzzy lower and upper almost strongly θ-continuous, fuzzy lower and upper weakly θ-continuous, fuzzy lower and upper almost continuous, fuzzy lower and upper super continuous, fuzzy lower and upper δ-continuous, are presented.

1. Introduction

It is well known that several types of fuzzy lower and upper continuous multifunctions and its weaker and stronger forms are given in literature. Aim of this paper, by using operations, give some characterizations and some properties of fuzzy lower and upper continuous multifunctions and its weaker and stronger forms including fuzzy lower and upper weakly continuous, fuzzy lower and upper θ-continuous, fuzzy lower and upper strongly θ-continuous, fuzzy lower and upper almost strongly θ-continuous, fuzzy lower and upper weakly θ-continuous, fuzzy lower and upper almost continuous, fuzzy lower and upper super continuous, fuzzy lower and upper δ-continuous.

The class of fuzzy sets on a universe X will be denoted by I^X and fuzzy sets on X will be denoted by Greek letters as μ, ρ, η, etc. Fuzzy point will be denoted by x_ε, y_ν, etc. For any fuzzy point x_ε and any fuzzy set μ, we write $x_\varepsilon \in \mu$ if and only if $\varepsilon \leq \mu(x)$. A fuzzy set μ is called quasi-coincident with a fuzzy set ρ, denoted by $\mu q \rho$, if and only if there exists a $x \in X$ such that $\mu(x) + \rho(x) > 1$. In this paper we use the concept of a fuzzy topological space as introduced by [1]. By

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$int(\mu)$, $cl(\mu)$ and $1-\mu$, we mean the interior of μ, the closure of μ and complement of μ.

Let $F : X \to Y$ be a fuzzy multifunction from a fuzzy topological space X to a fuzzy topological space Y. For any fuzzy set $\mu \leq X$, $F^+(\mu)$ and $F^-(\mu)$ are defined by $F^+(\mu) = \{x \in X : F(x) \leq \mu\}$ and $F^-(\mu) = \{x \in X : F(x)q\mu\}$, respectively. We know that $F^-(1-\mu) = 1 - F^+(\mu)$ for any fuzzy set $\mu \leq Y$ [8].

2. Some types of continuous fuzzy multifunctions

DEFINITION 1. [5] Let (X, τ) be a fuzzy topological space. A mapping $\varphi : I^X \to I^X$ is called an operation on I^X if for each $\mu \in I^X \backslash \{\emptyset\}$, $int(\mu) \leq \mu^\varphi$ and $\emptyset^\varphi = \emptyset$ where μ^φ denotes the value of φ in μ. The class of all operations on I^X is denoted by $O_{I(X,\tau)}$.

DEFINITION 2. [5] Let (X, τ) be a fuzzy topological space. A partial order "$\leq$" on $O_{I(X,\tau)}$ is defined by "$\varphi_1 \leq \varphi_2$ if and only if for each $\mu \in I^X$, $\mu^{\varphi_1} \leq \mu^{\varphi_2}$ where φ_1, $\varphi_2 \in O_{I(X,\tau)}$".

DEFINITION 3. [5] Let (X, τ) be a fuzzy topological space and let φ be an operation on I^X. φ is called a monotonous operation if for each μ, $\rho \in I^X$ and $\mu \leq \rho$, then $\mu^\varphi \leq \rho^\varphi$.

DEFINITION 4. Let $F : X \to Y$ be a fuzzy multifunction from a fuzzy topological space X to a fuzzy topological space Y and let φ, ψ be operations on I^X, I^Y, respectively. Then it is said that F is fuzzy lower (upper) $\varphi\psi$-continuous multifunction if for each $x_\varepsilon \in X$ and for each fuzzy open set $\mu \leq Y$ such that $x_\varepsilon \in F^-(\mu)$ ($x_\varepsilon \in F^+(\mu)$), there exists an open fuzzy set $\rho \leq X$ containing x_ε such that $\rho^\varphi \leq F^-(\mu^\psi)$ (respectively, $\rho^\varphi \leq F^+(\mu^\psi)$).

The following table gives us the list of fuzzy lower and upper $\varphi\psi$-continuous multifunctions with operations φ and ψ.

The definitions of fuzzy θ-continuous, fuzzy strongly θ-continuous, fuzzy almost strongly θ-continuous, fuzzy weakly θ-continuous, fuzzy super continuous, fuzzy δ-continuous multifunction are considered for fuzzy multivalued setting from [3, 7], [4, 6], [7], [7], [6] and [2, 10], respectively.

The definitions of fuzzy continuous, fuzzy weakly continuous, fuzzy almost continuous multifunction are considered from [8, 9], [8] and [8], respectively.

Operations	Fuzzy lower $\varphi\psi$-con.	fuzzy upper P-con.
1. $\varphi = \psi = i$	f. l. con.	f. u. con.
2. $\varphi = i$, $\psi = cl$	f. l. weakly con.	f. u. weakly con.
3. $\varphi = \psi = cl$	f. l. θ-con.	f. u. θ-con.
4. $\varphi = cl$, $\psi = i$	f. l. str. θ-con.	f. u. str. θ-con.
5. $\varphi = cl$, $\psi = int \circ cl$	f. l. al. str. θ-con.	f. u. al. str. θ-con.
6. $\varphi = int \circ cl$, $\psi = cl$	f. l. weakly θ-con.	f. u. weakly θ-con.
7. $\varphi = i$, $\psi = int \circ cl$	f. l. al. con.	f. u. al. con.
8. $\varphi = int \circ cl$, $\psi = i$	f. l. super con.	f. u. super con.
9. $\varphi = \psi = int \circ cl$	f. l. δ-con.	f. u. δ-con.

DEFINITION 5. Let (X, τ) be a fuzzy topological space and let $(x^{\alpha}_{\varepsilon_\alpha})$ be a net in X. $(x^{\alpha}_{\varepsilon_\alpha})$ is called φ-converges to x_ε if for each open set μ containing x_ε, there exists an index $\alpha_0 \in J$ such that $x^{\alpha}_{\varepsilon_\alpha} \in \mu^{\varphi}$ for all $\alpha \geq \alpha_0$. We will denote by $x^{\alpha}_{\varepsilon_\alpha} \xrightarrow{\varphi} x_\varepsilon$.

We know that a net $(x^{\alpha}_{\varepsilon_\alpha})$ in a fuzzy topological space (X, τ) is called *eventually* in the fuzzy set $\mu \leq X$ if there exists an index $\alpha_0 \in J$ such that $x^{\alpha}_{\varepsilon_\alpha} \in \mu$ for all $\alpha \geq \alpha_0$.

The following theorem give us the characterizations of fuzzy lower and upper $\varphi\psi$-continuous multifunction.

THEOREM 6. *Suppose that $F : X \to Y$ is a fuzzy multifunction from a fuzzy topological space X to a fuzzy topological space Y and φ, ψ are operations on I^X, I^Y, respectively. The following statements are equivalent.*

i-) *F is fuzzy lower (upper) $\varphi\psi$-continuous,*

ii-) *For each $x_\varepsilon \in X$ and for each fuzzy open set $\mu \leq Y$ such that $x_\varepsilon \in F^-(\mu)$ ($x_\varepsilon \in F^+(\mu)$), there exists an open fuzzy set $\rho \leq X$ containing x_ε such that $F^+(1-\mu^{\psi}) \leq 1-\rho^{\varphi}$ (respectively, $F^-(1-\mu^{\psi}) \leq 1-\rho^{\varphi}$),*

iii-) *For each $x_\varepsilon \in X$ and for each net $(x^{\alpha}_{\varepsilon_\alpha})$ which φ-converges to x_ε in X and for each fuzzy open set ρ in Y with $x_\varepsilon \in F^-(\rho)$ ($x_\varepsilon \in F^+(\rho)$), the net $(x^{\alpha}_{\varepsilon_\alpha})$ is eventually in $F^-(\rho^{\psi})$ (respectively, $F^+(\rho^{\psi})$).*

Proof. (i)$\Leftrightarrow$(ii). We know that $F^-(1-\mu) = 1 - F^+(\mu)$ for any fuzzy set $\mu \leq Y$. From here, we obtain it.

(i)$\Rightarrow$(iii). Let $x^{\alpha}_{\varepsilon_\alpha}$ be a net which φ-converges to x_ε in X and let ρ be any fuzzy open set in Y with $x_\varepsilon \in F^-(\rho)$ ($x_\varepsilon \in F^+(\rho)$). Since F is lower (upper) $\varphi\psi$-continuous, it follows that there exists an open set $\mu \leq X$ containing x_ε such that $\mu^{\varphi} \leq F^-(\rho^{\psi})$ (respectively, $\mu^{\varphi} \leq F^+(\rho^{\psi})$).

Since $x^{\alpha}_{\varepsilon_\alpha} \xrightarrow{\varphi} x_\varepsilon$, it follows that there exists an index $\alpha_0 \in J$ such that $x^{\alpha}_{\varepsilon_\alpha} \in \mu^{\varphi}$ for all $\alpha \geq \alpha_0$. From here, we obtain that $x^{\alpha}_{\varepsilon_\alpha} \in \mu^{\varphi} \leq F^{-}(\rho^{\psi})$ (respectively, $x^{\alpha}_{\varepsilon_\alpha} \in \mu^{\varphi} \leq F^{+}(\rho^{\psi})$) for all $\alpha \geq \alpha_0$. Thus, the net $(x^{\alpha}_{\varepsilon_\alpha})$ is eventually in $F^{-}(\rho^{\psi})$ (respectively, $F^{+}(\rho^{\psi})$).

(iii)⇒(i). Suppose that (i) is not true. There exists a point x_ε and an open set μ containing x_ε with $x_\varepsilon \in F^{-}(\mu)$ (respectively, $x_\varepsilon \in F^{+}(\mu)$) such that $\rho^{\varphi} \nleq F^{-}(\mu^{\psi})$ (respectively, $\rho^{\varphi} \nleq F^{+}(\mu^{\psi})$) for each open fuzzy set $\rho \leq X$ containing x_ε. Let $x_{\varepsilon_\rho} \in \rho^{\varphi}$ and $x_{\varepsilon_\rho} \notin F^{-}(\mu^{\psi})$(respectively, $x_{\varepsilon_\rho} \notin F^{+}(\mu^{\psi})$) for each open fuzzy set $\rho \leq X$ containing x_ε. Then for the neighborhood net (x_{ε_ρ}), $x_{\varepsilon_\rho} \xrightarrow{\varphi} x_\varepsilon$, but (x_{ε_ρ}) is not eventually in $F^{-}(\mu^{\psi})$(respectively, $F^{+}(\mu^{\psi})$). This is a contradiction. Thus, F is fuzzy lower (upper) $\varphi\psi$-continuous multifunction. □

EXAMPLE 7. Let $F : X \to Y$ be a fuzzy multifunction from fuzzy topological space (X, τ) to fuzzy topological space (Y, υ). The following statements are equivalent with operations $\varphi = i$, $\psi = cl$.

i-) F is fuzzy lower (upper) weakly continuous,

ii-) For each $x_\varepsilon \in X$ and for each fuzzy open set $\mu \leq Y$ such that $x_\varepsilon \in F^{-}(\mu)$ ($x_\varepsilon \in F^{+}(\mu)$), there exists an open fuzzy set $\rho \leq X$ containing x_ε such that $F^{+}(1 - cl(\mu)) \leq 1 - \rho$ (respectively, $F^{-}(1 - cl(\mu)) \leq 1 - \rho$),

iii-) For each $x_\varepsilon \in X$ and for each net $(x^{\alpha}_{\varepsilon_\alpha})$ which converges to x_ε in X and for each fuzzy open set ρ in Y with $x_\varepsilon \in F^{-}(\rho)$ ($x_\varepsilon \in F^{+}(\rho)$), the net $(x^{\alpha}_{\varepsilon_\alpha})$ is eventually in $F^{-}(cl(\rho))$(respectively, $F^{+}(cl(\rho))$).

THEOREM 8. *Suppose that $F : X \to Y$ is a fuzzy multifunction from fuzzy topological space (X, τ) to fuzzy topological space (Y, υ). Let φ_1, φ_2 be operations on I^X and let ψ_1, ψ_2 be operations on I^Y. If $\varphi_1 \geq \varphi_2$, $\psi_1 \leq \psi_2$ and F is fuzzy lower (upper) $\varphi_1\psi_1$-continuous, then F is fuzzy lower (respectively, upper) $\varphi_2\psi_2$-continuous.*

Proof. Clear. □

By using the above theorem, we obtain that the following diagram for any $F : X \to Y$ is a fuzzy multifunction from fuzzy topological space (X, τ) to fuzzy topological space (Y, υ).

f. l. strongly θ-con.	⇒	f. l. super con.	⇒	f. l. con.
⇓		⇓		⇓
f. l. almost str. θ-con.	⇒	f. l. δ-con.	⇒	f. l. almost con.
⇓		⇓		⇓
f. l. θ-con.	⇒	f. l. weakly θ-con.	⇒	f. l. weakly con.

The above diagram is true for the fuzzy upper continuity.

REMARK 9. The following examples show that the above implications are not reversible.

EXAMPLE 10. Let $X = \{x, y\}$ with topologies $\tau_1 = \{X, \emptyset, \rho\}$ and $\tau_2 = \{X, \emptyset, \mu\}$ where the fuzzy sets ρ, μ are defined as

$$\begin{aligned} \rho(x) &= 0,3, \quad & \rho(y) &= 0,6 \\ \mu(x) &= 0,7, \quad & \mu(y) &= 0,4. \end{aligned}$$

A fuzzy multifunction $F : (X, \tau_1) \to (X, \tau_2)$ given by $x_\varepsilon \to F(x_\varepsilon) = \{x_\varepsilon\}$ is upper (lower) almost continuous, upper (lower) almost strongly θ-continuous and upper (lower) δ-continuous, but it is not upper (resp. lower) continuous, upper (resp. lower) strongly θ-continuous and upper (resp. lower) super continuous.

If the fuzzy sets ρ and μ are defined as $\rho(x) = 0,6 = \rho(y)$ and $\mu(x) = 0,4 = \mu(y)$, then the fuzzy multifunction F is upper weakly continuous, but it is not upper weakly θ-continuous.

EXAMPLE 11. Let $X = \{x, y\}$ with topologies $\tau_1 = \{X, \emptyset, \rho, \eta, \rho \wedge \eta, \rho \vee \eta\}$ and $\tau_2 = \{X, \emptyset, \mu\}$ where the fuzzy sets ρ, η, μ are defined as

$$\begin{aligned} \rho(x) &= 0,3, \quad & \rho(y) &= 0,6 \\ \eta(x) &= 0,5, \quad & \eta(y) &= 0,5 \\ \mu(x) &= 0,3, \quad & \mu(y) &= 0,6. \end{aligned}$$

A fuzzy multifunction $F : (X, \tau_1) \to (X, \tau_2)$ given by $x_\varepsilon \to F(x_\varepsilon) = \{x_\varepsilon\}$ is upper continuous, but it is not upper super continuous.

If the fuzzy set μ is defined as $\mu(x) = 0,3$ and $\mu(y) = 0,4$, then the fuzzy multifunction F is upper weakly continuous, upper weakly θ-continuous and upper θ-continuous, but it is not upper almost continuous, upper δ-continuous and upper almost strongly θ-continuous.

If the fuzzy set ρ, η and μ are defined as $\rho(x) = 0,5$, $\rho(y) = 0,4$, $\eta(x) = \mu(x) = 0,3$ and $\eta(y) = \mu(y) = 0,5$, then the fuzzy multifunction F is upper almost continuous, but it is not upper δ-continuous.

EXAMPLE 12. Let $X = \{x, y\}$ with topologies $\tau_1 = \{X, \emptyset, \rho, \eta\}$ and $\tau_2 = \{X, \emptyset, \mu\}$ where the fuzzy sets ρ, η, μ are defined as

$$\begin{aligned} \rho(x) &= 0,4, \quad & \rho(y) &= 0,4 \\ \eta(x) &= 0,2, \quad & \eta(y) &= 0,1 \\ \mu(x) &= 0,4, \quad & \mu(y) &= 0,4. \end{aligned}$$

A fuzzy multifunction $F : (X, \tau_1) \to (X, \tau_2)$ given by $x_\varepsilon \to F(x_\varepsilon) = \{x_\varepsilon\}$ is upper super continuous and upper δ-continuous, but it is not upper strongly θ-continuous and almost strongly θ-continuous.

If the fuzzy set η and μ are defined as $\eta(x) = 0,5$, $\eta(y) = 0,6$ and $\mu(x) = 0,5$, $\mu(y) = 0,4$, then the fuzzy multifunction F is upper weakly θ-continuous, but it is not upper θ-continuous.

EXAMPLE 13. Let $X = \{x, y\}$ with topologies $\tau_1 = \{X, \emptyset, \rho\}$ and $\tau_2 = \{X, \emptyset, \mu, \eta\}$ where the fuzzy sets ρ, μ, η are defined as

$$\rho(x) = \rho(y) = 0,3,\ \mu(x) = \mu(y) = 0,7,\ \eta(x) = \eta(y) = 0,4.$$

A fuzzy multifunction $F : (X, \tau_1) \to (X, \tau_2)$ given by

$$x_\varepsilon \to F(x_\varepsilon) = \begin{cases} \mu, & 0 < \varepsilon \leq 0,3 \\ \rho, & 0,3 < \varepsilon \leq 1 \end{cases}$$

is lower weakly θ-continuous, lower super continuous and lower δ-continuous, but it is not lower θ-continuous, lower strongly θ-continuous and lower almost strongly θ-continuous.

If the fuzzy set ρ is defined as $\rho(x) = 0,6 = \rho(y)$, then the fuzzy multifunction F is lower θ-continuous, lower weakly continuous and lower weakly θ-continuous, but it is not lower almost strongly θ-continuous, lower almost continuous and lower δ-continuous.

EXAMPLE 14. Let $X = \{x, y\}$ with topologies $\tau_1 = \{X, \emptyset, \rho, \gamma, \xi\}$ and $\tau_2 = \{X, \emptyset, \mu, \eta\}$ where the fuzzy sets ρ, γ, ξ, μ, η are defined as

$$\begin{aligned} &\rho(x) = \rho(y) = 0,6, \quad \gamma(x) = \gamma(y) = 0,4, \\ &\xi(x) = \xi(y) = 0,3, \quad \mu(x) = \mu(y) = 0,7, \\ &\eta(x) = \eta(y) = 0,4. \end{aligned}$$

A fuzzy multifunction $F : (X, \tau_1) \to (X, \tau_2)$ given by

$$x_\varepsilon \to F(x_\varepsilon) = \begin{cases} \mu, & 0 < \varepsilon \leq 0,3 \\ \rho, & 0,3 < \varepsilon \leq 1 \end{cases}$$

is lower continuous and lower almost continuous, but it is not lower super continuous and lower δ-continuous.

If the fuzzy set ρ and η are defined as $\rho(x) = 0,5 = \rho(y)$ and $\eta(x) = 0,5 = \eta(y)$, then the fuzzy multifunction F is lower weakly continuous, but it is not lower weakly θ-continuous.

THEOREM 15. *Suppose that $F : X \to Y$ is a fuzzy multifunction from a fuzzy topological space X to a fuzzy topological space Y and φ, ψ are operations on I^X, I^Y, respectively. If $\{\eta_\alpha : \alpha \in \Phi\}$ is an open cover for X and $F_\alpha = F\mid_{\eta_\alpha}$ is a fuzzy lower (upper) $\varphi\psi$-continuous multifunction for all $\alpha \in \Phi$, then F is a fuzzy lower (upper) $\varphi\psi$-continuous multifunction.*

Proof. Let $x_\varepsilon \in X$ and let $\mu \leq Y$ be a fuzzy open set such that $x_\varepsilon \in F^-(\mu)$. Since $\{\eta_\alpha : \alpha \in \Phi\}$ is an open cover for X, it follows that $x_\varepsilon \in \eta_{\alpha_0}$ for a $\alpha_0 \in \Phi$ and then $x_\varepsilon \in F^-(\mu) \wedge \eta_{\alpha_0} = F \mid_{\eta_\alpha}^- (\mu)$. Since $F \mid_{\eta_\alpha}^- (\mu)$ is a fuzzy lower $\varphi\psi$-continuous multifunction, it follows that there exists an open fuzzy set ρ such that $x_\varepsilon \in \rho \wedge \eta_{\alpha_0}$ and $(\rho \wedge \eta_{\alpha_0})^\varphi \leq F \mid_{\eta_{\alpha_0}}^- (\mu^\psi)$. We obtain that $(\rho \wedge \eta_{\alpha_0})^\varphi \leq F \mid_{\eta_{\alpha_0}}^- (\mu^\psi) = F^-(\mu^\psi) \wedge \eta_{\alpha_0} \leq F^-(\mu^\psi)$. Thus, F is a fuzzy lower $\varphi\psi$-continuous multifunction.

The proof of the fuzzy upper $\varphi\psi$-continuity of F is similar to the above. □

DEFINITION 16. Suppose that (X,τ), (Y,υ) and (Z,ω) are fuzzy topological spaces. If $F_1 : X \to Y$ and $F_2 : Y \to Z$ are fuzzy multifunctions, then the fuzzy multifunction $F_2 \circ F_1 : X \to Z$ is defined by $(F_2 \circ F_1)(x_\varepsilon) = F_2(F_1(x_\varepsilon))$.

THEOREM 17. *Let (X,τ), (Y,υ), (Z,ω) be fuzzy topological spaces and let φ, ψ, ϕ are operations on I^X, I^Y, I^Z respectively. If $F_1 : X \to Y$ is fuzzy lower $\varphi\psi$-continuous multifunction and $F_2 : Y \to Z$ is fuzzy lower $\psi\phi$-continuous multifunction, then $F_2 \circ F_1$ is a fuzzy lower $\varphi\phi$-continuous multifunction.*

Proof. Let $x_\varepsilon \in X$ and let $\mu \leq Z$ be a fuzzy open set such that $x_\varepsilon \in (F_2 \circ F_1)^-(\mu)$. From the definition of $F_2 \circ F_1$, we have $x_\varepsilon \in (F_2 \circ F_1)^-(\mu) = F_1^-(F_2^-(\mu))$. Since $x_\varepsilon \in F_1^-(F_2^-(\mu))$, it follows that there exists a $y_\alpha \in F_2^-(\mu)$ such that $x_\varepsilon \in F_1^-(y_\alpha)$. Since F_2 is fuzzy lower $\psi\phi$-continuous multifunction, it follows that there exists an open fuzzy set $\rho \leq Y$ containing y_α such that $\rho^\psi \leq F_2^-(\mu^\phi)$. From here we obtain that $x_\varepsilon \in F_1^-(\rho)$ and $F_1^-(\rho^\psi) \leq F_1^-(F_2^-(\mu^\phi)) = (F_2 \circ F_1)^-(\mu^\phi)$. Since $x_\varepsilon \in F_1^-(\rho)$ and F_1 is fuzzy lower $\varphi\psi$-continuous multifunction, it follows that there exists an open fuzzy set $\eta \leq X$ containing x_ε such that $\eta^\varphi \leq F_1^-(\rho^\psi)$. Thus, from $F_1^-(\rho^\psi) \leq (F_2 \circ F_1)^-(\mu^\phi)$ we obtain that $\eta^\varphi \leq (F_2 \circ F_1)^-(\mu^\phi)$.

It shows that $F_2 \circ F_1$ is the fuzzy lower $\varphi\phi$-continuous multifunction.

DEFINITION 18. Let (Y,υ) be a fuzzy topological space and let ψ be an operation on I^Y. (Y,υ) is called fuzzy ψ-hyperconnected space if $\mu^\psi = Y$ for all fuzzy open set $\mu \neq \emptyset$.

THEOREM 19. *Suppose that $F : X \to Y$ is a fuzzy multifunction from a fuzzy topological space X to a fuzzy topological space Y and φ, ψ are operations on I^X, I^Y, respectively. If Y is fuzzy ψ-hyperconnected space, then F is fuzzy lower (upper) $\varphi\psi$-continuous multifunction.*

Proof. Let $x_\varepsilon \in X$ and let $\mu \leq Y$ be a fuzzy open set such that $x_\varepsilon \in F^-(\mu)$. Since Y is fuzzy ψ-hyperconnected space, it follows that $\mu^\psi = Y$. Hence for all open fuzzy set ρ containing x_ε, $\rho^\varphi \leq F^-(\mu^\psi) = F^-(Y) = X$. Thus, F is fuzzy lower $\varphi\psi$-continuous multifunction.

The proof of the fuzzy upper $\varphi\psi$-continuity of F is similar to the above. □

DEFINITION 20. Suppose that $F : X \to Y$ is a fuzzy multifunction from a fuzzy topological space X to a fuzzy topological space Y. The fuzzy graph multifunction $G_F : X \to X \times Y$ of F is defined as $G_F(x_\varepsilon) = \{x_\varepsilon\} \times F(x_\varepsilon)$ [8].

DEFINITION 21. Take $X = \prod_{\alpha\in J} X_\alpha$. Let $(\prod_{\alpha\in J} X_\alpha, \tau)$ be a product space and let ψ be an operation on I^X and on I^{X_α} for all $\alpha \in J$. ψ is called a productive operation if $(\prod_{\alpha\in J} \mu_\alpha)^\psi \leq \prod_{\alpha\in J} \mu_\alpha^\psi$ for all $\prod_{\alpha\in J} \mu_\alpha \leq \prod_{\alpha\in J} X_\alpha$, $\mu_\alpha \leq X_\alpha$ for $\alpha \in J$.

THEOREM 22. *Suppose that (X, τ) and (Y, υ) are fuzzy topological spaces and φ, ψ are operations on I^X, I^Y, respectively. Let $F : X \to Y$ be any fuzzy multifunction and let $G_F : (X, \tau) \to (X \times Y, \tau')$ be graph function of F and let ψ be a productive operation on $I^{X\times Y}$. If G_F is fuzzy lower (upper) $\varphi\psi$-continuous, then F is fuzzy lower (upper) $\varphi\psi$-continuous.*

Proof. For the fuzzy sets $\gamma \leq X$, $\eta \leq Y$, we take

$$(\gamma \times \eta)(z, y) = \begin{cases} 0, & \text{if } z \notin \gamma \\ \eta(y), & \text{if } z \in \gamma. \end{cases}$$

Let $x_\varepsilon \in X$ and let $\mu \leq Y$ be a fuzzy open set such that $x_\varepsilon \in F^-(\mu)$. We obtain that $x_\varepsilon \in G_F^-(X \times \mu)$. Since fuzzy graph multifunction G_F is fuzzy lower $\varphi\psi$-continuous and ψ is a productive operation, it follows that there exists an open fuzzy set $\rho \leq X$ containing x_ε such that $\rho^\varphi \leq G_F^-((X \times \mu)^\psi) \leq G_F^-(X^\psi \times \mu^\psi) = G_F^-(X \times \mu^\psi)$. From here, we obtain that $\rho^\varphi \leq F^-(\mu^\psi)$. Thus, F is fuzzy lower $\varphi\psi$-continuous multifunction.

The proof of the fuzzy upper $\varphi\psi$-continuity of F is similar to the above. □

THEOREM 23. *Suppose that $F : X \to Y$ is a fuzzy multifunction from fuzzy topological space (X, τ) to fuzzy topological space (Y, υ) and φ, ψ are operations on I^X, I^Y, respectively. Let $G_F : (X, \tau) \to (X \times Y, \tau \times \upsilon)$*

be graph function of F and ψ be a productive operation on $I^{X\times Y}$ and let $X \times Y$ be fuzzy ψ-hyperconnected space. G_F is fuzzy lower (upper) $\varphi\psi$-continuous multifunction if and only if F is fuzzy lower (upper) $\varphi\psi$-continuous multifunction.

Proof. The proof is obvious from Theorem 19 and Theorem 22. □

THEOREM 24. *Suppose that (X, τ) and (X_α, τ_α) are fuzzy topological spaces where $\alpha \in J$ and φ is operation on I^X and ψ is an operation on I^{X_α} for each $\alpha \in J$. Let $F : X \to \prod_{\alpha\in J} X_\alpha$* be a multifunction from X to *the product space* $\prod_{\alpha\in J} X_\alpha$ and let $P_\alpha : \prod_{\alpha\in J} X_\alpha \to X_\alpha$ *be the projection multifunction for each $\alpha \in J$ which is defined by $P_\alpha((x_\alpha)) = \{x_\alpha\}$ and ψ be a productive operation on $I^{\prod_{\alpha\in J} X_\alpha}$. If F is fuzzy upper $\varphi\psi$-continuous multifunction, then $P_\alpha \circ F$ is fuzzy* upper *$\varphi\psi$-continuous multifunction for each $\alpha \in J$.*

Proof. Take any $\alpha_0 \in J$. Let $x_\varepsilon \in X$ and let $\mu_{\alpha_0} \le X_{\alpha_0}$ be a fuzzy open set such that $x_\varepsilon \in (P_{\alpha_0} \circ F)^+(\mu_{\alpha_0})$. We know that $x_\varepsilon \in (P_{\alpha_0} \circ F)^+(\mu_{\alpha_0}) = F^+(P^+_{\alpha_0}(\mu_{\alpha_0})) = F^+(\mu_{\alpha_0} \times \prod_{\alpha\neq\alpha_0} X_\alpha)$. Since F is fuzzy upper $\varphi\psi$-continuous multifunction, it follows that there exists an open fuzzy set $\rho \le X$ containing x_ε such that $\rho^\varphi \le F^+((\mu_{\alpha_0} \times \prod_{\alpha\neq\alpha_0} X_\alpha)^\psi) \le F^+(\mu^\psi_{\alpha_0} \times \prod_{\alpha\neq\alpha_0} X^\psi_\alpha) = F^+(\mu^\psi_{\alpha_0} \times \prod_{\alpha\neq\alpha_0} X_\alpha) = F^+(P^+_{\alpha_0}(\mu^\psi_{\alpha_0})) = (P_{\alpha_0} \circ F)^+(\mu^\psi_{\alpha_0})$. It shows that $P_{\alpha_0} \circ F$ is fuzzy upper $\varphi\psi$-continuous multifunction.

Thus we obtain that $P_\alpha \circ F$ is fuzzy upper $\varphi\psi$-continuous multifunction for each $\alpha \in J$. □

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