

ON ZEROS OF CERTAIN SUMS OF POLYNOMIALS

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ABSTRACT. A convex combination of two products with same degree of finitely many finite geometric series with each having even degree does not always have all its zeros on the unit circle. However, in this paper, we show that a polynomial obtained by just adding a finite geometric series multiplied by a large constant to such a convex combination has all its zeros on the unit circle.

1. Introduction

Let u be a positive integer. Define, for positive integers a_1, a_2, \dots, a_u ,

$$F_{a_1, a_2, \dots, a_u}(z) = \prod_{j=1}^u \frac{z^{a_j} - 1}{z - 1}.$$

Suppose that, for all j with $1 \leq j \leq u$, a_j and b_j are positive integers such that $\sum_{j=1}^u a_j = \sum_{j=1}^u b_j = n$. If $u = 1, 2$, all zeros of

$$\Phi_u(z) = F_{a_1, a_2, \dots, a_u}(z) + F_{b_1, b_2, \dots, b_u}(z)$$

lie on the unit circle. However, an example for $u = 3$: The polynomial equation

$$(z - 1)(z^3 - 1)(z^{23} - 1) + (z^2 - 1)(z^{11} - 1)(z^{14} - 1) = 0$$

has four nonreal zeros with modulus $\neq 1$, tells us that, for some a_j 's and b_j 's with $\sum_{j=1}^u a_j = \sum_{j=1}^u b_j = n$, not all zeros of Φ_u ($u \geq 3$) lie on

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the unit circle. In this paper, we consider not just above sums of two polynomials but convex combinations of them. Above example implies that, for $u \geq 3$, not every convex combination

$$(1) \quad (1 - r)F_{a_1, \dots, a_u}(z) + r F_{b_1, \dots, b_u}(z)$$

has all its zeros on the unit circle. However, in Section 2 of this paper, we show that a polynomial obtained by adding a finite geometric series $t(z^{n-u} + z^{n-u-1} + \dots + 1)$ (with the same degree with both $F_{a_1, \dots, a_u}(z)$ and $F_{b_1, \dots, b_u}(z)$) to (1) has all its zeros on the unit circle provided that, for all j , a_j and b_j are positive odd integers and t is large enough. Here we note that $F_{a_1, \dots, a_u}(z)$ and $F_{b_1, \dots, b_u}(z)$ are products of u finite geometric series. Basic tool for our proof is the Chebyshev transformation. We state the definition of the Chebyshev transformation and its properties in Section 2 without proof. For their proofs, see [1].

2. Preliminaries

We denote by \mathcal{R}_{2n} the set of all real semi-reciprocal polynomials

$$p(z) = \sum_{j=0}^{2n} a_j z^j, \quad a_j = a_{2n-j} \quad (0 \leq j \leq n - 1)$$

of degree at most $2n$, and o by the zero polynomial. Let T_j and U_j be the j th Chebyshev polynomials of the first kind and of the second kind, respectively.

PROPOSITION 2.1. *Let $p(z) = \sum_{j=0}^{2n} a_j z^j \in \mathcal{R}_{2n}$ and $p \neq o$. Suppose that*

$$a_{2n} = a_{2n-1} = \dots = a_{n+k+1} = 0 = a_{n-k-1} = \dots = a_0, \\ \text{but } a_{n+k} = a_{n-k} \neq 0$$

for some k , $0 \leq k \leq n$. Then $p(z)$ has the decomposition

$$(2) \quad p(z) = a_{n+k} z^{n-k} \prod_{j=1}^k (z^2 - \alpha_j z + 1),$$

where $\alpha_j \in \mathbb{C}$ ($1 \leq j \leq k$) are the zeros of the polynomial $a_n + \sum_{j=1}^k a_{n+j} T_j(\frac{x}{2})$ and the convention $\prod_{j=1}^0 b_j = 1$ (for $p(z) = z^n$) is adopted. If $p \in \mathcal{R}_{2n}$ is a reciprocal polynomial of degree $2n$, then (2) holds with $k = n$.

DEFINITION 2.2. The Chebyshev transform of a polynomial $p \in \mathcal{R}_{2n} - \{o\}$ having the decomposition (2) is defined by

$$\mathcal{T}p(x) = a_{n+k} \prod_{j=1}^k (x - \alpha_j)$$

(with $\prod_{j=1}^0 b_j = 1$ (for $p(z) = z^n$) adopted) while, for the zero polynomial $p = o$, let

$$\mathcal{T}o(x) = 0.$$

PROPOSITION 2.3. *The Chebyshev transform \mathcal{T} is an isomorphism of the real vector space \mathcal{R}_{2n} onto the set of all polynomials of degree $\leq n$ with real coefficients.*

LEMMA 2.4. *Let p be a real reciprocal polynomial of degree $2n$. Then all zeros of p are on the unit circle if and only if all zeros of its Chebyshev transform $\mathcal{T}p$ are in the closed interval $[-2, 2]$.*

3. Results and proofs

The following lemma will be used in the proof of Theorem 3.2.

LEMMA 3.1. *For $p_1, p_2, \dots, p_n \in \mathcal{R}_{2n} - \{o\}$, we have*

$$\mathcal{T}p_1 p_2 \cdots p_n = \mathcal{T}p_1 \mathcal{T}p_2 \cdots \mathcal{T}p_n.$$

Proof. It is enough to show that, for $p, q \in \mathcal{R}_{2n} - \{o\}$, we have $\mathcal{T}pq = \mathcal{T}p\mathcal{T}q$. Suppose that

$$p(z) = \sum_{j=0}^{2n} a_j z^j = a_{n+k} z^{n-k} \prod_{j=1}^k (z^2 - \alpha_j z + 1) \in \mathcal{R}_{2n} - \{o\}$$

and

$$q(z) = \sum_{j=0}^{2n} b_j z^j = b_{n+h} z^{n-h} \prod_{j=1}^h (z^2 - \beta_j z + 1) \in \mathcal{R}_{2n} - \{o\},$$

where $\alpha_j, \beta_j \in \mathbb{C}$, $a_{n+k} \neq 0$ and $b_{n+h} \neq 0$ for some k, h with $0 \leq k, h \leq n$. Then

$$\mathcal{T}p(x) = a_{n+k} \prod_{j=1}^k (x - \alpha_j)$$

and

$$\mathcal{T}q(x) = b_{n+h} \prod_{j=1}^h (x - \beta_j).$$

Hence

$$\mathcal{T}p(x) \cdot \mathcal{T}q(x) = a_{n+k} b_{n+h} \prod_{j=1}^{k+h} (x - \gamma_j),$$

where $\gamma_j = \alpha_j$ for $1 \leq j \leq k$ and $\gamma_j = \beta_{j-k}$ for $k+1 \leq j \leq k+h$. Now

$$\begin{aligned} p(z)q(z) &= a_{n+k} b_{n+h} z^{2n-k-h} \prod_{j=1}^k (z^2 - \alpha_j z + 1) \prod_{j=1}^h (z^2 - \beta_j z + 1) \\ &= a_{n+k} b_{n+h} z^{2n-k-h} \prod_{j=1}^{k+h} (z^2 - \gamma_j z + 1). \end{aligned}$$

Hence

$$\mathcal{T}p q = \mathcal{T}p \mathcal{T}q.$$

□

Now we prove our main theorem.

THEOREM 3.2. *Let u be an integer ≥ 2 . If, for all j with $1 \leq j \leq u$, a_j and b_j are positive odd integers such that $\sum_{j=1}^u a_j = \sum_{j=1}^u b_j = n$, then, for*

$$|t| \geq \frac{1}{\sin^{u-1} \frac{\pi}{2(n-u+1)}},$$

all zeros of

$$G_r(z) = (1-r)F_{a_1, \dots, a_u}(z) + rF_{b_1, \dots, b_u}(z) + t(z^{n-u} + z^{n-u-1} + \dots + 1)$$

lie on the unit circle.

Proof. With the notation $v_j(z) = z^j + z^{j-1} + \dots + 1$, $j \geq 0$, we have

$$G_r(z) = (1-r) \prod_{j=1}^u v_{a_j-1}(z) + r \prod_{j=1}^u v_{b_j-1}(z) + t v_{n-u}(z),$$

and, by Proposition 2.3 and Lemma 3.1, we have

$$TG_r(x) = (1 - r) \prod_{j=1}^u \mathcal{T}v_{a_j-1}(x) + r \prod_{j=1}^u \mathcal{T}v_{b_j-1}(x) + t \mathcal{T}v_{n-u}(x).$$

But

$$(3) \quad \mathcal{T}v_{n-u}(x) = U_{\frac{n-u}{2}}\left(\frac{x}{2}\right) + U_{\frac{n-u}{2}-1}\left(\frac{x}{2}\right).$$

In fact,

$$\begin{aligned} v_{n-u}(z) &= \prod_{j=1}^{n-u} \left(z - e^{\frac{2j\pi i}{n-u+1}} \right) \\ &= \prod_{j=1}^{\frac{n-u}{2}} \left(z - e^{\frac{2j\pi i}{n-u+1}} \right) \left(z - e^{-\frac{2j\pi i}{n-u+1}} \right) \\ &= \prod_{j=1}^{\frac{n-u}{2}} \left(z^2 - 2 \cos \frac{2j\pi}{n-u+1} z + 1 \right) \end{aligned}$$

and so $\mathcal{T}v_{n-u}(x) = \prod_{j=1}^{\frac{n-u}{2}} \left(x - 2 \cos \frac{2j\pi}{n-u+1} \right)$. On the other hand, we observe that

$$(4) \quad U_{\frac{n-u}{2}}(\cos y) + U_{\frac{n-u}{2}-1}(\cos y) = \frac{\sin \frac{n-u+1}{2} y}{\sin \frac{y}{2}},$$

and the right side is zero if and only if $y = \frac{2j\pi}{n-u+1}$ ($j \in \mathbb{Z} - \{0\}$), so all zeros of $U_{\frac{n-u}{2}}\left(\frac{x}{2}\right) + U_{\frac{n-u}{2}-1}\left(\frac{x}{2}\right)$ are $2 \cos \frac{2j\pi}{n-u+1}$, $1 \leq j \leq \frac{n-u}{2}$. Hence both sides of (3) are monics which have the same zeros, and so they are identical. Now, for $x = x_j = 2 \cos y_j$, where $y_j = \frac{j+\frac{1}{2}}{n-u+1} 2\pi$, $0 \leq j \leq \frac{n-u}{2}$, we have, by (3) and (4),

$$\mathcal{T}v_{n-u}(x_j) = \frac{\sin \left(j + \frac{1}{2} \right) \pi}{\sin \frac{y_j}{2}} = \frac{(-1)^j}{\sin \frac{y_j}{2}}$$

and $0 < \sin \frac{y_j}{2} \leq 1$ with

$$\min_{0 \leq j \leq \frac{n-u}{2}} \sin \frac{y_j}{2} = \sin \frac{\pi}{2(n-u+1)}.$$

On the other hand, for

$$|t| \geq \frac{1}{\sin^{u-1} \frac{\pi}{2(n-u+1)}},$$

we have

$$\begin{aligned} & \left| (1-r) \prod_{k=1}^u \mathcal{T}v_{a_k-1}(x_j) + r \prod_{k=1}^u \mathcal{T}v_{b_k-1}(x_j) \right| \\ & \leq (1-r) \prod_{k=1}^u \frac{1}{\sin \frac{y_j}{2}} + r \prod_{k=1}^u \frac{1}{\sin \frac{y_j}{2}} \\ & = \frac{1}{\sin^{u-1} \frac{y_j}{2}} \frac{1}{\sin \frac{y_j}{2}} \leq \frac{1}{\sin^{u-1} \frac{\pi}{2(n-u+1)}} \frac{1}{\sin \frac{y_j}{2}} \\ & \leq |t| \frac{1}{\sin \frac{y_j}{2}} = |t| |\mathcal{T}v_{n-u}(x_j)|. \end{aligned}$$

Hence the sign of $\mathcal{T}G_r(x)$ is $\operatorname{sgn}(-1)^j \operatorname{sgn} t$ for $x = x_j$, where $0 \leq j \leq \frac{n-u}{2}$, which means that $\mathcal{T}G_r$ has $\frac{n-u}{2}$ distinct zeros in the interval $[-2, 2]$. Applying Lemma 2.4 completes the proof. \square

REMARK 3.3. In Theorem 3.2, without restriction of all a_j and b_j odd, $G_r(z)$ still seems to have all its zeros on the unit circle for some large t . But, for the case other than all a_j 's and b_j 's odd, our tool "Chebyshev transformation" does not seem to be suitable for proof.

References

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