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NEW INEQUALITIES OF OSTROWSKI AND TRAPEZOID TYPE FOR N-TIME DIFFERENTIABLE FUNCTIONS

B. G. PACHPATTE

ABSTRACT. In this paper using integral representations for n-time differentiable mappings, we establish new generalizations of certain Ostrowski and Trapezoid type inequalities by using fairly elementary analysis.

1. Introduction

In [1] Cerone, Dragomir and Roumeliatis proved the following identity:

$$\begin{aligned}\int_a^b f(t)\,dt = \sum_{k=0}^{n-1}\left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right] f^{(k)}(x) \\ + (-1)^n \int_a^b K_n(x,t)\, f^{(n)}(t)\,dt,\end{aligned} \tag{1.1}$$

for $x \in [a,b]$, where $f : [a,b] \to R$ be a mapping such that $f^{(n-1)}$ is absolutely continuous on $[a,b]$, $K_n : [a,b]^2 \to R$ is given by

$$K_n(x,t) = \begin{cases} \frac{(t-a)^n}{n!} & \text{if} \quad t \in [a,x] \\ \frac{(t-b)^n}{n!} & \text{if} \quad t \in (x,b], \end{cases} \tag{1.2}$$

and $n \geq 1$ is a natural number.

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In another paper [2], Cerone, Dragomir, Roumeliotis and Sunde have proved the following identity:

(1.3)
$$\int_a^b f(t)\,dt = \sum_{k=0}^{n-1} \frac{1}{(k+1)!}\left[(x-a)^{k+1} f^{(k)}(a) + (-1)^k (b-x)^{k+1} f^{(k)}(b)\right] + \frac{1}{n!}\int_a^b (x-t)^n f^{(n)}(t)\,dt,$$

for $x \in [a,b]$, where $f : [a,b] \to R$ be a mapping such that $f^{(n-1)}$ is absolutely continuous on $[a,b]$, and $n \geq 1$ is a natural number.

In the same papers [1] and [2], based on the identities (1.1) and (1.3), the authores pointed out the following inequalities which provide approximation formulae for the integral $\int_a^b f(t)\,dt$ whose error can be estimated as follows:

$$\left|\int_a^b f(t)\,dt - \sum_{k=0}^{n-1}\left[\frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!}\right] f^{(k)}(x)\right| \leq \frac{\|f^{(n)}\|_\infty}{(n+1)!}\left[(x-a)^{n+1} + (b-x)^{n+1}\right] \quad \text{if} \quad f^{(n)} \in L_\infty[a,b], \tag{1.4}$$

where $\|f^{(n)}\|_\infty = \sup_{t \in [a,b]} |f^{(n)}(t)| < \infty$ and

(1.5)
$$\left|\int_a^b f(t)\,dt - \sum_{k=0}^{n-1} \frac{1}{(k+1)!}\left[(x-a)^{k+1} f^{(k)}(a) + (-1)^k (b-x)^{k+1} f^{(k)}(b)\right]\right| \leq \frac{\|f^{(n)}\|_\infty}{(n+1)!}\left[(x-a)^{n+1} + (b-x)^{n+1}\right] \quad \text{if} \quad f^{(n)} \in L_\infty[a,b]$$

respectively.

For similar results, see the book [4] by Mitrinovic, Pecaric and Fink, and the recent papers [1-3, 5, 6], where further references are given. The main aim of this paper is to establish new inequalities which generalize the inequalities (1.4) and (1.5) involving two n-time differentiable mappings. Our proofs are quite elementary and based on the identities given in (1.1) and (1.3).

2. Statement of results

In what follows R denotes the set of real numbers and $[a,b] \subset R$. The n-th derivative of a function $h : [a,b] \to R$ is denoted by $h^{(n)}(x)$. For some suitable functions $f, g, p, q : [a,b] \to R$, we use the following notations to simplify the details of presentation:

$$F_k(x) = \sum_{k=0}^{n-1} \left[\frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!} \right] f^{(k)}(x), \tag{2.1}$$

$$G_k(x) = \sum_{k=0}^{n-1} \left[\frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!} \right] g^{(k)}(x), \tag{2.2}$$

(2.3)
$$P_k(x) = \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[(x-a)^{k+1} p^{(k)}(a) + (-1)^k (b-x)^{k+1} p^{(k)}(b) \right],$$

(2.4)
$$Q_k(x) = \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[(x-a)^{k+1} q^{(k)}(a) + (-1)^k (b-x)^{k+1} q^{(k)}(b) \right],$$

for $x \in [a,b]$.

Our main results are given in the following theorems.

THEOREM 1. *Let $f, g : [a,b] \to R$ be mappings such that $f^{(n-1)}$, $g^{(n-1)}$ are absolutely continuous on $[a,n]$ and $f^{(n)}, g^{(n)} \in L_\infty[a,n], n \geq 1$ is a natural number. Then*

(2.5)
$$\left| 2 \left(\int_a^b f(t)\,dt \right) \left(\int_a^b g(t)\,dt \right) - \left[F_k(x) \int_a^b g(t)\,dt + G_k(x) \int_a^b f(t)\,dt \right] \right|$$
$$\leq \left[\left\| f^{(n)} \right\|_\infty \int_a^b |g(t)|\,dt + \left\| g^{(n)} \right\|_\infty \int_a^b |f(t)|\,dt \right]$$
$$\times \frac{1}{(n+1)!} \left[(x-a)^{n+1} + (b-x)^{n+1} \right],$$

for $x \in [a,b]$, where

(2.6)
$$\left\| f^{(n)} \right\|_\infty = \sup_{x \in [a,b]} \left| f^{(n)}(t) \right| < \infty, \left\| g^{(n)} \right\|_\infty = \sup_{x \in [a,b]} \left| g^{(n)}(t) \right| < \infty.$$

THEOREM 2. *Let $p, q : [a, b] \to R$ be mappings such that $p^{(n-1)}, q^{(n-1)}$ are absolutely continuous on $[a, b]$ and $p^{(n)}, q^{(n)} \in L_\infty [a, b], n \geq 1$ is a natural number. Then*

(2.7)
$$\left| 2 \left(\int_a^b p(t)\, dt \right) \left(\int_a^b q(t)\, dt \right) - \left[P_k(x) \int_a^b q(t)\, dt + Q_k(x) \int_a^b p(t)\, dt \right] \right|$$
$$\leq \left[\left\| p^{(n)} \right\|_\infty \int_a^b |q(t)|\, dt + \left\| q^{(n)} \right\|_\infty \int_a^b |p(t)|\, dt \right]$$
$$\times \frac{1}{(n+1)!} \left[(x-a)^{n+1} + (b-x)^{n+1} \right],$$

for $x \in [a, b]$, where $\left\| p^{(n)} \right\|_\infty$ and $\left\| q^{(n)} \right\|_\infty$ are as defined in (2.6).

REMARK 1. It is easy to observe that, by taking $g(t) = 1$ and hence $g^{(n)}(t) = 0$ in Theorem 1 and $q(t) = 1$ and hence $q^{(n)}(t) = 0$ in Theorem 2, we get respectively the inequalities (1.4) and (1.5) in [1] and [2].

3. Proofs of Theorems 1 and 2

From the hypotheses of Theorem 1, we have the following identities (see [1]):

$$\int_a^b f(t)\, dt = F_k(x) + (-1)^n \int_a^b K_n(x, t) f^{(n)}(t)\, dt, \tag{3.1}$$

$$\int_a^b g(t)\, dt = G_k(x) + (-1)^n \int_a^b K_n(x, t) g^{(n)}(t)\, dt, \tag{3.2}$$

for $x \in [a, b]$, where $K_n(x, t)$ is given by (1.2). Multiplying both sides of (3.1) and (3.2) by $\int_a^b g(t)\, dt$ and $\int_a^b f(t)\, dt$ respectively, and adding we get

$$2 \left(\int_a^b f(t)\, dt \right) \left(\int_a^b g(t)\, dt \right)$$

$$
\begin{aligned}
&= \left[F_k(x)\int_a^b g(t)\,dt + G_k(x)\int_a^b f(t)\,dt\right] \\
&+ \left((-1)^n \int_a^b K_n(x,t)\, f^{(n)}(t)\,dt\right)\int_a^b g(t)\,dt \qquad (3.3)\\
&+ \left((-1)^n \int_a^b K_n(x,t)\, g^{(n)}(t)\,dt\right)\int_a^b f(t)\,dt.
\end{aligned}
$$

From (3.3) and using the properties of modulus we have

(3.4)

$$
\begin{aligned}
&\left|2\left(\int_a^b f(t)\,dt\right)\left(\int_a^b g(t)\,dt\right) - \left[F_k(x)\int_a^b g(t)\,dt + G_k(x)\int_a^b f(t)\,dt\right]\right| \\
&\leq \left(\int_a^b |K_n(x,t)|\left|f^{(n)}(t)\right|dt\right)\int_a^b |g(t)|\,dt \\
&+ \left(\int_a^b |K_n(x,t)|\left|g^{(n)}(t)\right|dt\right)\int_a^b |f(t)|\,dt \\
&\leq \left[\left\|f^{(n)}\right\|_\infty \int_a^b |g(t)|\,dt + \left\|g^{(n)}\right\|_\infty \int_a^b |f(t)|\,dt\right]\int_a^b |K_n(x,t)|\,dt.
\end{aligned}
$$

Now, observe that

$$
\begin{aligned}
\int_a^b |K_n(x,t)|\,dt &= \int_a^x \frac{(t-a)^n}{n!}dt + \int_x^b \frac{(b-t)^n}{n!}dt \qquad (3.5)\\
&= \frac{1}{(n+1)!}\left[(x-a)^{n+1} + (b-x)^{n+1}\right].
\end{aligned}
$$

Using (3.5) in (3.4) we get the required inequality in (2.5).

To prove Theorem 2, from the hypotheses we have the following identities (see [2]):

$$
\int_a^b p(t)\,dt = P_k(x) + \frac{1}{n!}\int_a^b (x-t)^n p^{(n)}(t)\,dt, \qquad (3.6)
$$

$$\int_a^b q(t)\,dt = Q_k(x) + \frac{1}{n!}\int_a^b (x-t)^n q^{(n)}(t)\,dt, \tag{3.7}$$

for $x \in [a,b]$. Multiplying both sides of (3.6) and (3.7) by $\int_a^b q(t)\,dt$ and $\int_a^b p(t)\,dt$ respectively, and adding we get

$$\begin{aligned} &2\left(\int_a^b p(t)\,dt\right)\left(\int_a^b q(t)\,dt\right) \\ &= \left[P_k(x)\int_a^b q(t)\,dt + Q_k(x)\int_a^b p(t)\,dt\right] \\ &+ \left(\frac{1}{n!}\int_a^b (x-t)^n p^{(n)}(t)\,dt\right)\int_a^b q(t)\,dt \\ &+ \left(\frac{1}{n!}\int_a^b (x-t)^n q^{(n)}(t)\,dt\right)\int_a^b p(t)\,dt. \end{aligned} \tag{3.8}$$

From (3.8) and using the properties of modulus, we have

$$\begin{aligned} &\left|2\left(\int_a^b p(t)\,dt\right)\left(\int_a^b q(t)\,dt\right) - \left[P_k(x)\int_a^b q(t)\,dt + Q_k(x)\int_a^b p(t)\,dt\right]\right| \\ &\leq \left[\left\|p^{(n)}\right\|_\infty \int_a^b |q(t)|\,dt + \left\|q^{(n)}\right\|_\infty \int_a^b |p(t)|\,dt\right]\frac{1}{n!}\int_a^b |x-t|^n\,dt. \end{aligned} \tag{3.9}$$

Observe that

$$\begin{aligned} \frac{1}{n!}\int_a^b |x-t|^n\,dt &= \frac{1}{n!}\left[\int_a^x (x-t)^n\,dt + \int_x^b (t-x)^n\,dt\right] \\ &= \frac{1}{(n+1)!}\left[(x-a)^{n+1} + (b-x)^{n+1}\right]. \end{aligned} \tag{3.10}$$

Using (3.10) in (3.9) we get the desired inequality in (2.7).

REMARK 2. We note that the idea employed to establish Theorems 1 and 2 can be used to deal with the various special versions of identities

(1.1) and (1.3) as discussed in [1] and [2]. We also note that one can very easily obtain the bounds on the right hand sides in (2.5) and (2.7) when $f^{(n)}, g^{(n)}, p^{(n)}, q^{(n)}$ all belongs to $L_p[a,b]$ for $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ or $L_1[a,b]$. The precise formulations of such results are very close to those given in Theorems 1 and 2 and in view of the results given in [1] and [2] with suitable modifications. We do not discuss it here.

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57 Shri Niketan Colony, Near Abhinay Talkies, Aurangabad 431 001 (Maharashtra) India
E-mail: bgpachpatte@hotmail.com