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OSCILLATION OF PARABOLIC NEUTRAL DELAY DIFFERENCE EQUATIONS

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ABSTRACT. Some new oscillation criteria for parabolic neutral delay difference equations corresponding to two sets of boundary conditions are obtained. Our results improve the well known results in the literature.

1. Introduction

Qualitative theory for discrete dynamic systems with one dimension, i.e., ordinary difference equations which parallel the qualitative theory of differential equations, has been investigated by several authors (see [1, 2, 10]) and the references cited therein. On the other, few papers have been devoted to the qualitative theory of the nonlinear discrete dynamic systems involving functions of two or more independent variables, i.e., partial difference equations (PDEs) (see [17, 21]) and the references cited therein. In fact, partial difference equations arise in the approximation of solutions of partial differential equations by finite difference methods, random walk problems, the study on molecular orbits, mathematical physics problems and other problems in population dynamics, we refer the reader to [5, 6, 10, 15]. In this paper, to develop the qualitative theory of partial difference equations, we shall consider the following parabolic nonlinear neutral delay difference equation

$$
\begin{aligned}
(1.1)\qquad & \Delta_2(y_{m,n} - p_n y_{m,n-\tau}) + \sum_{i\in I} q^{(i)}_{m,n} f(y_{m,n-\sigma_i}) \\
&= r_n \nabla^2 y_{m-1,n+1} + \sum_{j\in J} R_{j,n} \nabla^2 y_{m-1,n+1-\gamma_j},
\end{aligned}
$$

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where $I := \{1, 2, \ldots, I_0\}$, $J := \{1, 2, \ldots, J_0\}$, $\{y_{m,n}\} = \{y_{m_1,m_2,\ldots,m_l,n}\}$ which is defined in $\Omega \times \mathbb{N}_{n_0}$, $\mathbb{N}_{n_0} = \{n_0, n_0+1, \ldots\}$ and $\Omega = \{p_1^{(1)}, \ldots, p_{M_1}^{(1)}\} \times \ldots \{p_1^{(l)}, \ldots, p_{M_l}^{(l)}\}$ is a convex connected solid net, and every $p_i^{(j)} \in \mathbb{Z}$ (for the definition of the convex connected solid net, we refer to [17]).

We assume throughout this paper that:

(h1) $r_n \in \mathbb{N}_{n_0} \to \mathbb{R}^+$ and $R_{j,n} \in J \times \mathbb{N}_{n_0} \to \mathbb{R}^+$;

(h2) $q_{m,n}^{(i)} \in I \times \Omega \times \mathbb{N}_{n_0} \to \mathbb{R}^+$, $q_{i,n} = \min_{m\in\Omega}\{q_{m,n}^{(i)}\}$, for $i \in I$ and $n \in \mathbb{N}_{n_0}$;

(h3) $\tau \geq 1$, $\sigma_i \in I \to \mathbb{N}_1$, $\gamma_j \in J \to \mathbb{N}_1$;

(h4) $f_i \in C(\mathbb{R}, \mathbb{R})$ is convex, $uf_i(u) > 0$ for $u \neq 0$, $\frac{f_i(u)}{u} \geq k > 0$ for $i \in I$,

(h5) $p_n \in \mathbb{N}_{n_0} \to \mathbb{R}^+$, and there exists a positive integer $K > 0$ such that

$$p_{K+i\tau} \leq 1, \quad \text{for } i = 0, 1, 2, \ldots.$$

We write ∇^2 is the discrete Laplacian operator, which is defined by $\nabla^2 y_{m-1,n+1} = \sum_{i=1}^{l} \Delta_i^2 y_{m_1,m_2,\ldots,m_{i-1},m_i-1,\ldots,m_l,n+1}$, where Δ_i^2 is the a partial difference operator of order two, i.e., $\Delta_i^2 y_{m,n} = \Delta_i(\Delta_i y_{m,n})$, $\Delta_1 y_{m,n} = y_{m+1,n} - y_{m,n}$ $\Delta_2 y_{m,n} = y_{m,n+1} - y_{m,n}$.

Consider the initial boundary value problem (IBVP) (1.1) with two kinds of the boundary conditions

$$\text{(B1)} \qquad \Delta_N y_{m-1,n} = 0, \quad \text{on} \quad \partial\Omega \times \mathbb{N}_{n_0},$$

$$\text{(B2)} \qquad \Delta_N y_{m-1,n} + g_{m,n} y_{m,n} = 0, \quad \text{on} \quad \partial\Omega \times \mathbb{N}_{n_0},$$

and the initial condition (IC)

$$\text{(1.2)} \qquad y_{m,s} = \mu_{m,s}, \quad \text{for } n_0 - M \leq s \leq n_0,$$

where $\Delta_N y_{m-1,n}$ is the normal difference at $(m, n) \in \partial\Omega \times \mathbb{N}_{n_0}$ which is defined by

$$\Delta_N y_{m-1,n} = \sum_{all\ \mathrm{m}\pm 1 \notin \Omega} (\Delta_1 y_{m,n} - \Delta_1 y_{m-1,n}) = \sum_{all\ \mathrm{m}\pm 1 \notin \Omega} (\Delta_1^2 y_{m,n})$$

and $M = \max\{\tau, \sigma_i, \gamma_j\colon\ i \in I \text{ and } j \in J)$ and $g_{m,n} \in \partial\Omega \times \mathbb{N}_{n_0} \to \mathbb{R}^+$.

By a solution of initial boundary value problem (1.1), (B1), (1.2) (for short IBVPB1) we mean a sequence $\{y_{m,n}\}$ which satisfies Eq.(1.1) for $(m, n) \in \Omega \times \mathbb{N}_{n_0}$, satisfies (B1) for $(m, n) \in \partial\Omega \times \mathbb{N}_{n_0}$ and satisfies IC (1.2) for $(m, n) \in \Omega \times \{n_0 - M, \ldots, n_0\}$. The definition of the solution of the initial boundary value problem (1.1), (B2), (1.2) (IBVPB2) is defined similarly.

Our objective in this paper is to present sufficient conditions which imply that every solution $\{y_{m,n}\}$ of IBVPB1 and IBVPB2 are oscillatory in $\Omega \times \mathbb{N}_{n_0}$, in the sense that there does not exist an $n_1 \in \mathbb{N}_{n_0}$ such that $y_{m,n} > 0$ or $y_{m,n} < 0$ for $n \in \mathbb{N}_{n_1}$. In Section 2, we shall consider IBVPB1 and IBVPB2 will be considered in Section 3. Our results in Section 3 improve the results obtained in [17].

For the oscillation of ordinary neutral delay difference equations we refer the reader to [4, 8, 11-14, 16, 19, 22] and the references therein.

2. Oscillation of IBVPB1

In this section we will establish some oscillation criteria for IBVPB1. Before stating our main results we need the following lemma.

LEMMA 2.1 [17]. (Discrete Gaussian formula). *Let Ω be a convex connected solid net. Then we have*

$$\sum_{m\in\Omega} \nabla^2 y_{m-1,n+1} = \sum_{m\in\partial\Omega} \Delta_N y_{m-1,n+1}.$$

Throughout this paper, we will assume that

$$\sigma = \min_{i\in I}\{\sigma_i\} \quad \text{and} \quad Q_n = \sum_{i\in I} kq_{i,n}.$$

THEOREM 2.1. *Assume that (h1)-(h5) hold, and every solution of the delay difference equation*

$$\Delta x_n + Q_n x_{n-\sigma} = 0, \quad n \in \mathbb{N}_{n_0}, \tag{2.1}$$

oscillates. Then, every solution of IBVPB1 is oscillatory in $\Omega \times \mathbb{N}_{n_0}$.

Proof. Suppose to the contrary that $\{y_{m,n}\}$ is a nonoscillatory solution of IBVPB1. Without loss of generality, we may assume that there exists $n_1 \in \mathbb{N}_{n_0}$ such that $y_{m,n-M} > 0$ for all $n \in \mathbb{N}_{n_1}$. Summing the equation (1.1) over Ω, we have

$$\begin{aligned} &\Delta_2\Big(\sum_{m\in\Omega} y_{m,n} - p_n \sum_{m\in\Omega} y_{m,n-\tau}\Big) + \sum_{i\in I}\sum_{m\in\Omega} q^{(i)}_{m,n} f(y_{m,n-\sigma_i}) \\ &= r_n \sum_{m\in\Omega} \nabla^2 y_{m-1,n+1} + \sum_{j\in J} R_{j,n} \sum_{m\in\Omega} \nabla^2 y_{m-1,n+1-\gamma_j}, \end{aligned} \tag{2.2}$$

for $(m,n) \in \Omega \times \mathbb{N}_{n_1}$.

From Lemma 2.1 and (B1) we obtain

$$\sum_{m\in\Omega} \nabla^2 y_{m-1,n+1} = \sum_{m\in\partial\Omega} \Delta_N y_{m-1,n+1} = 0, \quad \text{for } n \in \mathbb{N}_{n_1}, \tag{2.3}$$

$$\begin{aligned} &\sum_{m\in\Omega} \nabla^2 y_{m-1,n+1-\gamma_j} \\ &= \sum_{m\in\partial\Omega} \Delta_N y_{m-1,n+1-\gamma_j} = 0, \quad \text{for } j \in J \text{ and } n \in \mathbb{N}_{n_1}. \end{aligned} \tag{2.4}$$

From (h2) and using the Jensens's inequality, we have

$$\begin{aligned} &\sum_{m\in\Omega} q^{(i)}_{m,n} f(y_{m,n-\sigma_i}) \geq q_{i,n} \sum_{m\in\Omega} f(y_{m,n-\sigma_i}) \\ &\geq q_{i,n} \sum_{m\in\Omega} f\left(\frac{1}{|\Omega|}\sum_{m\in\Omega} y_{m,n-\sigma_i}\right)|\Omega|, \quad \text{for } i \in I \text{ and } n \in \mathbb{N}_{n_1}. \end{aligned} \tag{2.5}$$

Set

$$z_n = \frac{1}{|\Omega|}\sum_{m\in\Omega} y_{m,n}. \tag{2.6}$$

Thus, we obtain by (h4), (2.2)-(2.6) that

$$\Delta(z_n - p_n z_{n-\tau}) + \sum_{i\in I} k q_{i,n} z_{n-\sigma_i} \leq 0, \tag{2.7}$$

where Δ is the ordinary difference operator. Set

$$x_n = z_n - p_n z_{n-\tau}, \tag{2.8}$$

then by (h2) and (h4) $\Delta x_n \leq 0$ and Lemma 1 in [8, 14] yields that $x_n \geq z_n > 0$, for $n \in \mathbb{N}_{n_1}$. Now, since $\{x_n\}$ is positive and nonincreasing sequence and being $\sigma \leq \sigma_i$ for all i$\in I$, then, (2.7) and (2.8) imply that x_n is a positive solution of the delay difference inequality

$$\Delta x_n + \sum_{i\in I} k q_{i,n} x_{n-\sigma} \leq 0.$$

But, then by Lemma 1 in [23] the delay difference equation (2.1) has an eventually positive solution also, which contradicts the assumption that every solution of Eq.(2.1) oscillates. Then every solution of IBVPB1 is oscillatory in $\Omega \times \mathbb{N}_{n_0}$. □

Theorem 2.1 shows that the oscillation of IBVPB1 is equivalent to the oscillation of the delay difference equation (2.1). Thus, we can use the results of [7, 9, 18, 20, 3] to obtain some sufficient conditions for oscillation of all solutions of IBVPB1 in $\Omega \times \mathbb{N}_{n_0}$. Now, by applying

Theorem 2.1 and using the results in [7, 9, 18, 20, 3] respectively we have the following results.

COROLLARY 2.1. *Assume that (h1)-(h5) hold. If*

$$\limsup_{n\to\infty}\sum_{i=0}^{\sigma} Q_{n-i} > 1, \tag{2.9}$$

or

$$\liminf_{n\to\infty} Q_n > \frac{\sigma^{\sigma}}{(\sigma+1)^{\sigma+1}}, \tag{2.10}$$

or

$$\liminf_{n\to\infty}\frac{1}{\sigma}\sum_{i=1}^{\sigma} Q_{n-i} > \frac{\sigma^{\sigma}}{(\sigma+1)^{\sigma+1}}. \tag{2.11}$$

Then every solution of IBVPB1 is oscillatory in $\Omega \times \mathbb{N}_{n_0}$.

COROLLARY 2.2. *Assume that (h1)-(h5) hold. If*

$$\liminf_{n\to\infty}\sum_{i=1}^{\sigma} Q_{n-i} > L > 0,$$

and

$$\limsup_{n\to\infty} Q_n > 1 - \frac{L^2}{4}. \tag{2.12}$$

Then every solution of IBVPB1 is oscillatory in $\Omega \times \mathbb{N}_{n_0}$.

COROLLARY 2.3. *Assume that (h1)-(h5) hold. If*

$$0 \le \alpha = \liminf_{n\to\infty}\sum_{i=1}^{\sigma} Q_{n-i} \le \frac{\sigma^{\sigma+1}}{(\sigma+1)^{\sigma+1}},$$

and

$$\limsup_{n\to\infty}\sum_{i=0}^{\sigma} Q_{n-i} > 1 - \frac{\alpha^2}{4} \tag{2.13}$$

or

$$\limsup_{n\to\infty}\sum_{i=0}^{\sigma} Q_{n-i} > 1 - \frac{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}{2}. \tag{2.14}$$

Then every solution of IBVPB1 is oscillatory in $\Omega \times \mathbb{N}_{n_0}$.

In the following theorems, we give new oscillation criteria for IBVPB1 when none of the conditions (2.9)-(2.14) are satisfied.

THEOREM 2.2. *Assume that (h1)-(h5) hold. If* $\sum_{i=n+1}^{n+\sigma} Q_i > 0$ *and*

$$\sum_{n=n_0}^{\infty} Q_n \left[\left(\sum_{i=n+1}^{n+\sigma} Q_i \right)^{\frac{1}{1+\sigma}} (\sigma+1) - \sigma \right] = \infty. \tag{2.15}$$

Then every solution of IBVPB1 is oscillatory in $\Omega \times \mathbb{N}_{n_0}$.

Proof. We proceed as in the proof of Theorem 2.1, we may assume that IBVPB1 has a nonoscillatory solution $\{y_{m.n}\}$. Without loss of generality, we may assume that there exists $n_1 \in \mathbb{N}_{n_0}$ such that $y_{m,n-M} > 0$ for all $n \in \mathbb{N}_{n_1}$. Then by Theorem 2.1 the delay difference equation (2.1) has a positive solution for all $n \geq n_1$. Define the sequence $\{\lambda_n\}$ by

$$\lambda_n = -\frac{\Delta x_n}{x_n}. \tag{2.16}$$

Since $\{x_n\}$ is a nonincreasing sequence, we have $0 \leq \lambda_n < 1$ for $n \geq n_1$. From (2.16) we have $\frac{x_{n+1}}{x_n} = 1 - \lambda_n$ and $\frac{x_{n-\sigma}}{x_n} = \prod_{i=n-\sigma}^{n-1} (1-\lambda_i)^{-1}$. Then by (2.1)

$$\lambda_n = Q_n \prod_{i=n-\sigma}^{n-1} (1-\lambda_i)^{-1}.$$

By employing the arithmetic mean-geometric inequality, we have

$$\lambda_n \geq Q_n \left(1 - \frac{1}{\sigma} \sum_{i=n-\sigma}^{n-1} \lambda_i \right)^{-\sigma}. \tag{2.17}$$

Let $b_n = \sum_{i=n+1}^{n+\sigma} Q_i$. Then (2.17) can be rewritten as

$$\lambda_n \geq Q_n \left(1 - \frac{1}{\sigma b_n} b_n \sum_{i=n-\sigma}^{n-1} \lambda_i \right)^{-\sigma}. \tag{2.18}$$

By using the inequality

$$\left[1 - \frac{1}{\sigma} r x\right]^{-\sigma} \geq x + \frac{\left[r^{\frac{1}{\sigma+1}}(\sigma+1) - \sigma\right]}{r} \quad \text{for } r > 0 \text{ and } x < \frac{\sigma}{r}, \tag{2.19}$$

in the right hand side of (2.18), we obtain

$$\lambda_n \geq Q_n \left[\frac{1}{b_n} \sum_{i=n-\sigma}^{n-1} \lambda_i + \frac{1}{b_n} \left((b_n)^{\frac{1}{\sigma+1}} (\sigma+1) - \sigma \right) \right].$$

It follows that

$$\lambda_n b_n - Q_n \sum_{i=n-\sigma}^{n-1} \lambda_i \geq Q_n \left(\left(\sum_{i=n+1}^{n+\sigma} Q_i \right)^{\frac{1}{\sigma+1}} (\sigma+1) - \sigma \right).$$

Then, for $N > n_1$,

(2.20)

$$\sum_{n=n_1}^{N} \lambda_n b_n - \sum_{n=n_1}^{N} Q_n \sum_{i=n-\sigma}^{n-1} \lambda_i \geq \sum_{n=n_1}^{N} Q_n \left(\left(\sum_{i=n+1}^{n+\sigma} Q_i \right)^{\frac{1}{\sigma+1}} (\sigma+1) - \sigma \right).$$

Interchanging the bound of summation, we find

$$(2.21) \qquad \begin{aligned} \sum_{n=n_1}^{N} Q_n \sum_{i=n-\sigma}^{n-1} \lambda_i &\geq \sum_{n=n_1}^{N-\sigma-1} \sum_{n=i+1}^{i+\sigma} \lambda_i Q_n \\ &= \sum_{i=n_1}^{N-\sigma-1} \lambda_i \sum_{n=i+1}^{i+\sigma} Q_n = \sum_{n=n_1}^{N-\sigma-1} \lambda_n \sum_{i=n+1}^{n+\sigma} Q_i. \end{aligned}$$

Combining (2.20) and (2.21) yields that

$$(2.22) \qquad \sum_{n=N-\sigma}^{N} \lambda_n \sum_{i=n+1}^{n+\sigma} Q_i \geq \sum_{n=n_1}^{N} Q_n \left(\left(\sum_{i=n+1}^{n+\sigma} Q_i \right)^{\frac{1}{\sigma+1}} (\sigma+1) - \sigma \right).$$

Summing (2.1) from $n+1$ to $n+\sigma$, we get

$$x_{n+1+\sigma} - x_{n+1} + \sum_{i=n+1}^{n+\sigma} Q_i x_{i-\sigma} = 0.$$

Using the fact that $\{x_n\}$ is positive and nonincreasing, we have

$$x_{n+1} > x_n \sum_{i=n+1}^{n+\sigma} Q_i,$$

and so

$$(2.23) \qquad \sum_{i=n+1}^{n+\sigma} Q_i < 1,$$

eventually. Then, form (2.22) and (2.23) we have

(2.24)

$$\sum_{n=N-\sigma}^{N} \lambda_n \geq \sum_{n=n_1}^{N} Q_n \left(\left(\sum_{i=n+1}^{n+\sigma} Q_i \right)^{\frac{1}{\sigma+1}} (\sigma+1) - \sigma \right) \to \infty \text{ as } N \to \infty$$

by (2.15). But, from the definition of λ_n we have

$$\lambda_n = \left(1 - \frac{x_{n+1}}{x_n}\right).$$

Hence,

$$\sum_{n=N-\sigma}^{N} \lambda_n = \sum_{n=N-\sigma}^{N} \left(1 - \frac{x_{n+1}}{x_n}\right) < \sigma + 1,$$

and this contradicts (2.24). Then every solution of IBVPB1 is oscillatory in $\Omega \times \mathbb{N}_{n_0}$. □

From the above results it is clear that the oscillation criteria depending only on the sequences $\{q_{i,n}\}$. In the following theorems we present some infinite integral conditions on the combined growth of the sequences $\{p_n\}$, and $\{q_{i,n}\}$.

THEOREM 2.3. *Assume that (h1)-(h5) hold. If* $\sum_{i=n+1}^{n+\sigma} A_i > 0$ *and*

$$\sum_{n=n_0}^{\infty} A_n \left[\left(\sum_{i=n+1}^{n+\sigma} A_i\right)^{\frac{1}{1+\sigma}} (\sigma+1) - \sigma\right] = \infty, \tag{2.25}$$

where $A_n = \sum_{i\in I} kq_{i,n}(1 + p_{n-\sigma_i})$. *Then every solution of IBVPB1 is oscillatory in* $\Omega \times \mathbb{N}_{n_0}$.

Proof. We proceed as in the proof of Theorem 2.2, to obtain (2.7). Defining again the sequence $\{x_n\}$ by (2.8), then we have $\{x_n\}$ is positive and nonincreasing sequence. So that (2.7) implies that

$$\Delta x_n + \sum_{i\in I} kq_{i,n} z_{n-\sigma_i} \leq 0. \tag{2.26}$$

Hence from (2.8) and (2.26) we have

$$\begin{aligned} \Delta x_n \leq & -\sum_{i\in I} kq_{i,n}[x_{n-\sigma_i} + p_{n-\sigma_i} z_{n-\tau-\sigma_i}] \\ \leq & -\sum_{i\in I} kq_{i,n} x_{n-\sigma_i} - \sum_{i\in I} kq_{i,n} p_{n-\sigma_i} z_{n-\tau-\sigma_i}. \end{aligned}$$

From (2.8) again and the last inequality, we have

$$\Delta x_n \leq -\sum_{i\in I} kq_{i,n} x_{n-\sigma_i} - \sum_{i\in I} kq_{i,n} p_{n-\sigma_i} x_{n-\sigma_i-\tau}. \tag{2.27}$$

Since $\{x_n\}$ is nonincreasing sequence

$$\Delta x_n + \sum_{i\in I} kq_{i,n}[1 + p_{n-\sigma_i}]x_{n-\sigma} \le 0. \tag{2.28}$$

Defining again $\{\lambda_n\}$ as before, then (2.28) implies that

$$\lambda_n \ge A_n \frac{x_{n-\sigma}}{x_n}. \tag{2.29}$$

Then as in Theorem 2.2 we obtain

$$\lambda_n \ge A_n \prod_{i=n-\sigma}^{n-1} (1-\lambda_i)^{-1} \ge A_n \left(1 - \frac{1}{\sigma} \sum_{i=n-\sigma}^{n-1} \lambda_i\right)^{-\sigma}. \tag{2.30}$$

The remainder of the proof is similar to that of Theorem 2.2 and hence is omitted. □

THEOREM 2.4. *Assume that (h1)-(h5) hold. If* $\sum_{i=n+1}^{n+\sigma} B_i > 0$ *and*

$$\sum_{n=n_0}^{\infty} B_n \left[\left(\sum_{i=n+1}^{n+\sigma} B_i\right)^{\frac{1}{1+\sigma}} (\sigma+1) - \sigma\right] = \infty, \tag{2.31}$$

where $B_n = \sum_{i\in I} kq_{i,n}(1 + p_{n-\sigma_i} p_{n-\tau-\sigma_i})$. *Then every solution of IBVPB1 is oscillatory in* $\Omega \times \mathbb{N}_{n_0}$.

Proof. Following the proof of Theorem 2.3 we obtain (2.28), which by using (2.8) implies that

$$\Delta x_n + \sum_{i\in I} kq_{i,n}(1 + p_{n-\sigma_i} p_{n-\tau-\sigma_i})x_{n-\sigma} \le 0, \tag{2.32}$$

$$\lambda_n \ge B_n \frac{x_{n-\sigma}}{x_n}. \tag{2.33}$$

Then as in Theorem 2.2 we have

$$\lambda_n \ge B_n \prod_{i=n-\sigma}^{n-1} (1-\lambda_i)^{-1} \ge B_n \left(1 - \frac{1}{\sigma} \sum_{i=n-\sigma}^{n-1} \lambda_i\right)^{-\sigma}.$$

The remainder of the proof is similar to that of Theorem 2.2 and hence is omitted. □

In view of the results established in [3, 7, 9, 18, 20] and the fact that every solution of IBVPB1 is oscillatory in $\Omega \times \mathbb{N}_{n_0}$ when each one of (2.28) and (2.32) has no eventually positive solution, we can present some other oscillation criteria for of all solutions of the IBVPB1 *in* $\Omega \times \mathbb{N}_{n_0}$. The details are left to the reader.

THEOREM 2.5. *Assume that (h1)-(h5) hold. If* $\sum_{i=n+1}^{n+\sigma} G_i > 0$ *and*

$$\sum_{n=n_0}^{\infty} G_n \left[\left(\sum_{i=n+1}^{n+\sigma} G_i \right)^{\frac{1}{1+\sigma}} (\sigma + 1) - \sigma \right] = \infty, \tag{2.34}$$

where $G_n = \sum_{l=0}^{N} \prod_{j=1}^{l} (\sum_{i \in I} kq_{i,n} p_{n-\sigma_i-(j-1)\tau})$. *Then every solution of IBVPB1 is oscillatory in* $\Omega \times \mathbb{N}_{n_0}$.

Proof. Following the proof of Theorem 2.3 we obtain

$$\begin{aligned} \Delta x_n &\leq -\sum_{i \in I} kq_{i,n} z_{n-\sigma_i} \leq -\sum_{i \in I} kq_{i,n} [x_{n-\sigma_i} + p_{n-\sigma_i} z_{n-\tau-\sigma_i}] \\ &\leq -\sum_{i \in I} kq_{i,n} x_{n-\sigma_i} - \sum_{i \in I} kq_{i,n} p_{n-\sigma_i} z_{n-\tau-\sigma_i}. \end{aligned}$$

Hence

$$\Delta x_n + \sum_{i \in I} kq_{i,n} x_{n-\sigma_i} + \sum_{i \in I} kq_{i,n} p_{n-\sigma_i} z_{n-\tau-\sigma_i} \leq 0, \tag{2.35}$$

and then by induction we see that

$$\begin{aligned} &\Delta x_n + \sum_{l=0}^{N} \sum_{i \in I} kq_{i,n} \prod_{j=1}^{l} p_{n-\sigma_i-(j-1)\tau} x_{n-\sigma_i-i\tau} \\ &+ \sum_{i \in I} kq_{i,n} \prod_{j=1}^{N+1} p_{n-\sigma_i-(j-1)\tau} z_{n-\sigma_i-(N+1)\tau} \leq 0. \end{aligned}$$

Hence for n sufficiently large, and $\tau \leq N$

$$\Delta x_n + \left(\sum_{l=m}^{N} \sum_{i \in I} kq_{i,n} \prod_{j=1}^{l} p_{n-\sigma_i-(j-1)\tau} \right) x_{n-\sigma_i-l\tau} \leq 0. \tag{2.36}$$

Since $\Delta x_n \leq 0$ and n-$\sigma \geq n - \sigma - i\tau$ for all i, we have

$$\Delta x_n + \left(\sum_{l=m}^{N} \sum_{i \in I} kq_{i,n} \prod_{j=1}^{l} p_{n-\sigma_i-(j-1)\tau} \right) x_{n-\sigma} \leq 0. \tag{2.37}$$

The remainder of the proof is similar to that of Theorem 2.2 and the details are left to the reader. □

3. Oscillation of IBVPB2

In this section we will establish some new oscillation criteria for IBVPB2.

THEOREM 3.1. *Assume that (h1)-(h5) hold, and every solution of the delay difference equation* (2.1) *oscillates. Then every solution of IBVPB2 is oscillatory in* $\Omega \times \mathbb{N}_{n_0}$.

Proof. Suppose to the contrary that $\{y_{m,n}\}$ is a nonoscillatory solution of IBVPB2. Without loss of generality, we may assume that there exists $n_1 \in \mathbb{N}_{n_0}$ such that $y_{m,n-M} > 0$ for all $n \in \mathbb{N}_{n_1}$. Summing the equation (1.1) over Ω, we have

$$\begin{aligned} &\Delta_2(\sum_{m\in\Omega} y_{m,n} - p_n \sum_{m\in\Omega} y_{m,n-\tau}) + \sum_{i\in I}\sum_{m\in\Omega} q^{(i)}_{m,n} f(y_{m,n-\sigma_i}) \\ &= r_n \sum_{m\in\Omega} \nabla^2 y_{m-1,n+1} + \sum_{i\in J} R_{j,n} \sum_{m\in\Omega} \nabla^2 y_{m-1,n+1-\gamma_j}, \\ &\text{for } (m,n) \in \Omega \times \mathbb{N}_{n_1}. \end{aligned} \tag{3.1}$$

From Lemma 2.1 and (B2) we find that

$$\begin{aligned} &\sum_{m\in\Omega} \nabla^2 y_{m-1,n+1} \\ &= \sum_{m\in\partial\Omega} \Delta_N y_{m-1,n+1} = -\sum_{m\in\partial\Omega} g_{m,n+1} y_{m,n+1} \le 0, \\ &\text{for } n \in \mathbb{N}_{n_1}, \end{aligned} \tag{3.2}$$

$$\begin{aligned} &\sum_{m\in\Omega} \nabla^2 y_{m-1,n+1-\gamma_j} \\ &= \sum_{m\in\partial\Omega} \Delta_N y_{m-1,n+1-\gamma_j} = -\sum_{m\in\partial\Omega} g_{m,n+1} y_{m,n+1-\gamma_j} \le 0, \\ &\text{for } j \in J \text{ and } n \in \mathbb{N}_{n_1}. \end{aligned} \tag{3.3}$$

From (h2) and using the Jensens's inequality, we have

$$\begin{aligned} &\sum_{m\in\Omega} q^{(i)}_{m,n} f(y_{m,n-\sigma_i}) \\ &\ge q_{i,n} \sum_{m\in\Omega} f(y_{m,n-\sigma_i}) \ge q_{i,n} \sum_{m\in\Omega} f\left(\frac{1}{|\Omega|}\sum_{m\in\Omega} y_{m,n-\sigma_i}\right)|\Omega|, \\ &\text{for all } i \in I \text{ and } n \in N_{n_1}. \end{aligned} \tag{3.4}$$

Set

$$z_n = \frac{1}{|\Omega|} \sum_{m \in \Omega} y_{m,n}. \tag{3.5}$$

Thus, we obtain by (h4), (3.1)-(3.5) that

$$\Delta(z_n - p_n z_{n-\tau}) + \sum_{i \in I} k q_{i,n} z_{n-\sigma_i} \leq 0 \tag{3.6}$$

where Δ is the ordinary difference operator. Define $\{x_n\}$ as in (2.8), then as in the proof of Theorem 2.1 we have $\Delta x_n \leq 0$ and $x_n \geq z_n > 0$, for $n \in \mathbb{N}_{n_1}$, and x_n is a positive solution of the delay difference equation (2.1), which contradicts the assumption that every solution of Eq.(2.1) oscillates. Then every solution of IBVPB2 is oscillatory in $\Omega \times \mathbb{N}_{n_0}$. □

Theorem 3.1 shows that the oscillation of the IBVPB2 is equivalent to the oscillation of the delay differential equation (2.1). Thus, we can use the results of [7, 9, 18, 20, 3] to obtain several oscillation criteria for IBVPB2. Now, by applying Theorem 3.1 and using the results in [7, 9, 18, 20, 3] respectively we have the following results.

COROLLARY 3.1. *Assume that all the assumptions of Corollary 2.1 hold. Then every solution of IBVPB2 is oscillatory in* $\Omega \times \mathbb{N}_{n_0}$.

COROLLARY 3.2. *Assume that all the assumptions of Corollary 2.2 hold, except that the condition (B1) is replaced by (B2). Then every solution of IBVPB2 is oscillatory in* $\Omega \times \mathbb{N}_{n_0}$.

COROLLARY 3.3. *Assume that all the assumptions of Corollary 2.3 hold, except that the condition (B1) is replaced by (B2). Then every solution of IBVPB2 is oscillatory in* $\Omega \times \mathbb{N}_{n_0}$.

We note that Theorem 3.1 and Corollaries 3.1-3.3 improve Theorems 3.1 and 3.2 in [17].

In the following theorems, we give new oscillation criteria for IBVPB2 when none of the conditions (2.9)-(2.14) are satisfied. The details are left to the reader.

THEOREM 3.2. *Assume that all the assumption of Theorem 2.2 hold, except that the condition (B1) is replaced by (B2). Then every solution of IBVPB2 is oscillatory in* $\Omega \times \mathbb{N}_{n_0}$.

THEOREM 3.3. *Assume that all the assumption of Theorem 2.3 hold, except that the condition (B1) is replaced by (B2). Then every solution of IBVPB2 is oscillatory in* $\Omega \times \mathbb{N}_{n_0}$.

THEOREM 3.4. *Assume that all the assumption of Theorem 2.4 hold, except that the condition (B1) is replaced by (B2). Then every solution of IBVPB2 is oscillatory in* $\Omega \times \mathbb{N}_{n_0}$.

THEOREM 3.5. *Assume that all the assumption of Theorem 2.5 hold, except that the condition (B1) is replaced by (B2). Then every solution of IBVPB2 is oscillatory in* $\Omega \times \mathbb{N}_{n_0}$.

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