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PSEUDO-PARALLEL REAL HYPERSURFACES IN COMPLEX SPACE FORMS

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ABSTRACT. Pseudo-parallel real hypersurfaces in complex space forms can be defined as an extrinsic analogues of pseudo-symmetric real hypersurfaces, that generalize the notion of semi-symmetric real hypersurface. In this paper a classification of the pseudo-parallel real hypersurfaces in a non-flat complex space forms is obtained.

1. Introduction

The class of isometric immersions in a Riemannian manifold with parallel second fundamental form is very wide, as it is shown, for instance, in the classical Ferus' paper [12]. Certain generalizations of these immersions have been studied, obtaining classification theorems in some cases. We recall the following facts found in the literature.

1. Given an isometric immersion $i : M \longrightarrow \tilde{M}$, let h be the second fundamental form and $\tilde{\nabla}$ the van der Waerden-Bortolotti connection. Then, J. Deprez defined the immersion to be *semi-parallel* if $\tilde{R}(X,Y) \cdot h = (\tilde{\nabla}_X\tilde{\nabla}_Y - \tilde{\nabla}_Y\tilde{\nabla}_X - \tilde{\nabla}_{[X,Y]})h = 0$, for any tangent vectors X, Y to M. J. Deprez mainly paid attention to the case of semi-parallel immersions in real space forms (see [7] and [8], [9]). Lumiste in [13] showed that a semi-parallel submanifold is the second orden envelope of the family of parallel submanifolds. In the case of hypersurfaces in the sphere and the hyperbolic space, F. Dillen showed that they are flat surfaces, hypersurfaces with parallel Weingarten endomorphism or rotation hypersurfaces of certain helices [11].

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2. The first author et al. ([1], [2]) generalized the works of J. Deprez and F. Dillen. They introduced the notion of pseudo-parallel immersions into space forms as an extrinsic analogue of pseudo-symmetric manifolds (in the sense of R. Deszcz, [10]) and as a direct generalization semi-parallel immersions.

   Given M a hypersurface in a real space form, let R and A be the curvature tensor and the Weingarten endomorphism of M respectively. Given X, Y in TM, let $X \wedge Y$ denote the operator of TM given by $Z \mapsto \langle Y, Z\rangle X - \langle X, Z\rangle Y$. It can be extended to act as a derivation on A as follows: $(X \wedge Y \cdot A)Z = (X \wedge Y)AZ - A(X \wedge Y)Z$, for any Z in TM. Then, a hypersurface M in a real space form is called *pseudo-parallel* if there is a smooth function f on M such that

$$R(X, Y) \cdot A = f\, X \wedge Y \cdot A,$$

   for any X, Y in TM. The classification of pseudo-parallel hypersurfaces in real space forms were obtained by the first author et al. in [2], describing them as quasi-umbilic hypersurfaces [4] or cycloids of Dupin [5].
3. The submanifolds in a complex space form $\widetilde{M}^n(c)$, $n \geq 2$, of constant holomorphic sectional curvature $4c$, with parallel second fundamental form were classified by H. Naitoh in [17]. As a result, the second fundamental form of a real hypersurface M in $\widetilde{M}^n(c)$, $n \geq 2$, $4c \neq 0$, can never be parallel. This means that the Weingarten endomorphism A of M cannot be parallel, i.e., $\nabla A \neq 0$, where ∇ is the Levi-Civita connection of M extended to act on tensors as a derivation. Thus, several authors have considered weaker conditions on real hypersurfaces in non-flat complex space forms.
4. S. Maeda [14] studied *semi-parallel* real hypersurfaces in $\widetilde{M}^n(c)$, for $c > 0$ and $n \geq 3$. He translated the Deprez's definition to the following one, by making use of the Weingarten endomorphism A and the curvature tensor R of the real hypersurface

$$R(X, Y) \cdot A = (\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})A = 0,$$

   for any tangent vectors X, Y to the real hypersurface. However, this condition is too strong since S. Maeda obtained that no real hypersurface in $\widetilde{M}^n(c)$, for $c > 0$ and $n \geq 3$, satisfies it. R. Niebergall and P. J. Ryan [18] also proved the non-existence of semi-parallel real hypersurfaces in $\widetilde{M}^2(c)$, for $4c \neq 0$. Later, the second author in [20] made a proof for $4c \neq 0$ and $n \geq 3$, showing that there are no real semi-parallel real hypersurfaces in $\widetilde{M}^n(c)$, $4c \neq 0$ and $n \geq 2$.

Thus, it is natural to find a weaker condition than the semi-parallelism one that allows to find a classification theorem. In this paper, we define and classify the pseudo-parallel real hypersurfaces in a non-flat complex space form.

DEFINITION 1.1. A real hypersurface M in $\widetilde{M}^n(c)$ is called *pseudo-parallel* if there is a real valued smooth function f on M such that

$$R(X,Y) \cdot A = f\, X \wedge Y \cdot A \tag{1.1}$$

for any $X, Y \in TM$.

Moreover, condition (1.1) can also be regarded as an extrinsic analogue of pseudo-symmetry $R(X,Y) \cdot R = fX \wedge Y \cdot R$, that for $f = 0$ is the condition of semi-symmetric -studied recently for some geometers in complex space forms, see for example [18]. The classification of the above real hypersurfaces is displayed in the next theorem.

THEOREM 1.2. *Let M be a connected pseudo-parallel real hypersurface in $\widetilde{M}^n(c)$, $n \geq 2$, $c = \pm 1$, with associated function f. Then f is constant and positive, and M is one of the following real hypersurfaces:*

- i) *If $c = +1$, then $f = \cot^2(r)$, for $0 < r < \pi/2$, and M is an open subset of a geodesic hypersphere of radius r.*
- ii) *If $c = -1$, then*
  - a) *$1 < f = \coth^2(r)$, for $r > 0$, and M is an open subset of a geodesic hypersphere of radius r;*
  - b) *$f = 1$, and M is an open subset of a horosphere;*
  - c) *$0 < f = \tanh^2(r) < 1$, for $r > 0$, and M is an open subset of a tube of radius r over a totally geodesic $\mathbb{C}H^{n-1}$.*

2. Preliminaries

Let $\widetilde{M}^n(c)$ be a non-flat complex space form endowed with the metric $\langle , \rangle$ of constant holomorphic sectional curvature $4c \neq 0$ and complex dimension $n \geq 2$. For the sake of simplicity, if $c > 0$, we will only use $c = +1$, and we will call it the complex projective space $\mathbb{C}P^n$, and if $c < 0$, we just consider $c = -1$, so that we will call it the complex hyperbolic space $\mathbb{C}H^n$. Let M be a connected C^∞ real hypersurface in $\widetilde{M}^n(c)$, $c \neq 0$, $n \geq 2$, without boundary. Let N be a local unit normal vector field to M. If J is the almost complex structure of $\widetilde{M}^n(c)$, we define $\xi = -JN$. Usually, the vector field ξ is called the structure or the Reeb vector field of M. The Levi-Civita connection of $\widetilde{M}^n(c)$ and M will be denoted by $\widetilde{\nabla}$ and ∇, respectively. The Gauss and Weingarten formulae are

$$\begin{aligned} \widetilde{\nabla}_X Y &= \nabla_X Y + \langle AX, Y\rangle N, \\ \widetilde{\nabla}_X N &= -AX, \end{aligned} \tag{2.1}$$

for any $X, Y \in TM$, where A denoted the Weingarten endomorphism of M. A local tangent vector field X is called principal if it is a eigenvector of A everywhere, and its associated eigenfunction is called principal curvature function. Given a point $p \in M$ and a principal curvature λ, we write $T_\lambda(p) = \{X \in T_pM : A_pX = \lambda(p)X\}$. This vector subspace is called the principal distribution associated with λ at p. The dimension of the principal distribution is known as the multiplicity of the principal curvature. The multiplicity of a principal distribution depends on the point, although there is a dense open subset on which it is locally constant. Given a vector field X tangent to M on a neighborhood of a point $p \in M$, we put $JX = \varphi X + \eta(X)N$, where φX and $\eta(X)N$ are the tangential and the normal component of JX respectively. Thus, φ is a skew-symmetric tensor of type (1,1) and η is a 1-form on M. Furthermore, ξ is a locally defined vector field tangent to M. The set $(\varphi, \xi, \eta, \langle , \rangle)$ is called an almost contact metric structure on M, whose elementary properties are

$$\begin{gathered} \eta(X) = \langle X, \xi\rangle, \quad \varphi^2 = -I + \eta \otimes \xi, \\ \varphi\xi = 0, \quad \eta(\xi) = 1, \quad \nabla_X \xi = \varphi AX, \\ \langle \varphi X, Y\rangle + \langle X, \varphi Y\rangle = 0, \quad \langle \varphi X, \varphi Y\rangle = \langle X, Y\rangle - \eta(X)\eta(Y), \end{gathered} \tag{2.2}$$

for any $X, Y \in TM$, where I denotes the identity transformation on TM.

Let $X \wedge Y$ denote the operator of TM given by $Z \mapsto \langle Y, Z\rangle X - \langle X, Z\rangle Y$. Then, the curvature tensor R of M is given by the equation of Gauss of M

$$\begin{aligned} R(X,Y)Z = c\{&(X \wedge Y)Z + (\varphi X \wedge \varphi Y)Z - 2\langle \varphi X, Y\rangle \varphi Z\} \\ &+ (AX \wedge AY)Z, \end{aligned} \tag{2.3}$$

for any $X, Y, Z \in TM$. The Codazzi equation of M is

$$(\nabla_X A)Y - (\nabla_Y A)X = c\{\eta(X)\varphi Y - \eta(Y)\varphi X - 2\langle \varphi X, Y\rangle \xi\}, \tag{2.4}$$

for any $X, Y \in TM$.

The operators $R(X,Y)$ and $X \wedge Y$ are extended to act on A as derivations as follows,

$$(2.5)\qquad \begin{aligned}(R(X,Y)\cdot A)Z &= ((\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]})A)Z\\ &= [R(X,Y),A]Z,\end{aligned}$$

$$(2.6)\qquad (X\wedge Y\cdot A)Z = [X\wedge Y, A]Z,$$

where $[,]$ denotes the bracket of operators.

Using the equation of Gauss (2.3), condition (1.1) may be written in the form:

$$(2.7)\qquad \begin{aligned}&(c-f)\langle AY,Z\rangle X - (c-f)\langle AX,Z\rangle Y\\ &+c\langle A\varphi Y,Z\rangle\varphi X - c\langle A\varphi X,Z\rangle\varphi Y - 2c\langle\varphi X,Y\rangle\varphi AZ\\ &+(\langle A^2Y,Z\rangle - (c-f)\langle Y,Z\rangle)AX\\ &+((c-f)\langle X,Z\rangle - \langle A^2X,Z\rangle)AY\\ &-c\langle\varphi Y,Z\rangle A\varphi X + c\langle\varphi X,Z\rangle A\varphi Y + 2c\langle\varphi X,Y\rangle A\varphi Z\\ &-\langle AY,Z\rangle A^2X + \langle AX,Z\rangle A^2Y = 0\end{aligned}$$

for any $X,Y,Z \in TM$.

We need to recall the following results in order to prove our theorem. T. E Cecil and P. J. Ryan in [6] and S. Montiel in [15] classified the real hypersurfaces in complex space forms with at most two distinct principal curvatures at each point. They needed the hypothesis $n \geq 3$ to prove that the vector field ξ is principal. If we assume that ξ is principal, from the above papers, we can obtain the following result.

PROPOSITION 2.1. *Let M be a connected orientable real hypersurface in $\widetilde{M}^n(c)$, $n \geq 2$, $c = \pm 1$. Given a normal unit vector field N on M, suppose that there exist two smooth functions a and b defined on M such that the Weingarten endomorphism A associated with N takes the form $AX = aX + b\langle X,\xi\rangle\xi$, for any $X \in TM$. Then, M is an open subset of one of the following real hypersurfaces:*

1. *in $\mathbb{C}P^n$,*
   A_1)*: a tube of radius $0 < r < \pi/2$ over a totally geodesic hyperplane $\mathbb{C}P^{n-1}$;*
2. *in $\mathbb{C}H^n$,*
   A_0)*: a horosphere;*
   A_1)*: a tube of radius $r > 0$ over a totally geodesic $\mathbb{C}H^k$, where $k = 0, n-1$.*

Usually, tubes over a totally geodesic $\widetilde{M}^k(c)$, with $k \in \{1,\dots,n-2\}$, are called real hypersurfaces of type A_2. A description of the horosphere

A_0 can be found in [16]. Real hypersurfaces of type A_0, A_1 and A_2 are simply known as real hypersurfaces of type A.

LEMMA A. [19] *Let M be a real hypersurface of $\widetilde{M}^n(c)$, $n \geq 2$, $c = \pm 1$. Suppose that ξ is principal with principal curvature μ. Then the function μ is locally constant.*

LEMMA B. [3] *Let M be a real hypersurface of $\widetilde{M}^n(c)$, $n \geq 2$, $c = \pm 1$. Suppose that ξ is principal with principal curvature μ. Let X be a principal vector orthogonal to ξ and with principal curvature λ. If $\mu^2 + 4c \neq 0$ or $\mu^2 + 4c = 0$ and $\lambda \neq \mu/2$, then φX is also principal with principal curvature ρ and is characterized by the equation*

$$(2\lambda - \mu)(2\rho - \mu) = \mu^2 + 4c. \tag{2.8}$$

REMARK C. If $\mu^2 + 4c = 0$ and $\lambda = \mu/2$, then equation (2.8) holds, but it does not give information about the value of ρ. On the other hand, if $\mu^2 + 4c = 0$ and $\lambda \neq \mu/2$, the equation (2.8) tells us that $\rho = \mu/2$.

3. Proof of Theorem 1.2

First, we show that ξ is principal. If we set $Y = Z$ and insert it in (2.7), we obtain

$$\begin{aligned} 0 = \;& (c-f)\langle AZ, Z\rangle X - (c-f)\langle AX, Z\rangle Z \\ & + c\langle A\varphi Z, Z\rangle \varphi X - c\langle A\varphi X, Z\rangle \varphi Z - 3c\langle \varphi X, Z\rangle \varphi AZ \\ & + 3c\langle \varphi X, Z\rangle A\varphi Z + (\langle A^2 Z, Z\rangle - (c-f)\langle Z, Z\rangle)AX \\ & + ((c-f)\langle X, Z\rangle - \langle A^2 X, Z\rangle)AZ \\ & - \langle AZ, Z\rangle A^2 X + \langle AX, Z\rangle A^2 Z, \end{aligned} \tag{3.1}$$

for any $X, Z \in TM$. Now, we consider $\{E_1, \ldots, E_{2n-1}\}$ a local orthonormal frame of TM. We insert $Z = E_k$ in (3.1), and taking summation over $k = 1, \ldots, 2n-1$, we get

$$\begin{aligned} 0 = \;& (c-f)\mathrm{tr}(A)X + c\,\mathrm{tr}(A\varphi)\varphi X - 4c\varphi A\varphi X + 3cA\varphi^2 X \\ & + (\mathrm{tr}(A^2) - (2n-1)(c-f))AX - \mathrm{tr}(A)A^2 X, \end{aligned}$$

for any $X \in TM$. From (2.2), it follows $\mathrm{tr}(A\varphi) = 0$ and consequently, the above equation becomes

$$\begin{aligned} 0 = \;& (c-f)\mathrm{tr}(A)X - 4c\varphi A\varphi X + 3c\langle X, \xi\rangle A\xi \\ & + (\mathrm{tr}(A^2) - (2n-1)(c-f) - 3c)AX - \mathrm{tr}(A)A^2 X, \end{aligned} \tag{3.2}$$

for any $X \in TM$. If X and $Y \in TM$ are orthogonal, from (3.2), we have

$$(3.3)\quad 0 = -4c\langle\varphi A\varphi X, Y\rangle + 3c\langle X, \xi\rangle\langle A\xi, Y\rangle + (\mathrm{tr}(A^2) - (2n-1)(c-f) - 3c)\langle AX, Y\rangle - \mathrm{tr}(A)\langle A^2X, Y\rangle.$$

Exchanging X and Y in (3.3), we obtain

$$(3.4)\quad 0 = -4c\langle\varphi A\varphi Y, X\rangle + 3c\langle Y, \xi\rangle\langle A\xi, X\rangle + (\mathrm{tr}(A^2) - (2n-1)(c-f) - 3c)\langle AY, X\rangle - \mathrm{tr}(A)\langle A^2Y, X\rangle.$$

By equation (2.2) and subtracting (3.3) to (3.4), we get $\langle X, \xi\rangle\langle A\xi, Y\rangle = \langle Y, \xi\rangle\langle A\xi, X\rangle$, for any $X, Y \in TM$. If we insert $X = \xi$ in the above equation, then we have $\langle A\xi, Y\rangle = \langle\langle A\xi, \xi\rangle\xi, Y\rangle$, for any $Y \in TM$. From here, we conclude that ξ is a principal vector of A. This means that M is a *Hopf* hypersurface in $\widetilde{M}^n(c)$. Now, by Lemma B, we have that $\mu = \langle A\xi, \xi\rangle$ is locally constant.

Next, by Lemma B, given a point of M, we can choose a unit tangent vector field U defined on a connected open neighborhood Ω of such point, satisfying $AU = \lambda U$ and $A\varphi U = \rho\varphi U$. Suppose that $\lambda \neq \rho$ on a open set $\Sigma \subset \Omega$. If we insert $X = U$, $Y = \varphi U$ and $Z = U$ in (2.7), we get

$$(R(U, \varphi U) \cdot A)U = (\lambda I - A)R(U, \varphi U)U = -(4c + \lambda\rho)(\lambda - \rho)\varphi U$$
$$= f(U \wedge \varphi U \cdot A)U = f(\lambda I - A)(U \wedge \varphi U(U)) = -f(\lambda - \rho)\varphi U,$$

and being M pseudo-parallel, it follows

$$(3.5)\qquad f = 4c + \lambda\rho \text{ on } \Sigma.$$

Now, if we insert $X = \xi$, $Y = U$ and $Z = U$ in (2.7), we obtain

$$(R(\xi, U) \cdot A)U = (\lambda I - A)R(\xi, U)U = (c + \lambda\mu)(\lambda - \mu)\xi$$
$$= f(\xi \wedge U \cdot A)U = f(\lambda I - A)(\xi \wedge U(U)) = f(\lambda - \mu)\xi,$$

and by the same argument, it follows

$$(3.6)\qquad 0 = (\lambda - \mu)(c + \lambda\mu - f) \text{ on } \Sigma.$$

Similarly, we repeat the calculations with $X = \xi$, $Y = \varphi U$ and $Z = \varphi U$, obtaining

$$(3.7)\qquad 0 = (\rho - \mu)(c + \rho\mu - f) \text{ on } \Sigma.$$

If $\rho \neq \mu$ and $\lambda \neq \mu$ at a point $p \in \Sigma$, from (3.6) and (3.7), we get $0 = (\lambda - \rho)\mu$, hence $\mu = 0$. Then from (3.6) follows $f = c$, and by Lemma B, we obtain that $\lambda\rho = c$. But from (3.5), $4c + \lambda\rho = f = c$, hence $4c = 0$, which is a contradiction. Consequently, we obtain that at each point of Σ, either $\mu = \lambda$ or $\mu = \rho$ is satisfied. However, given $p \in \Sigma$ such that $\mu(p) = \lambda(p) \neq \rho(p)$, from (3.5) and (3.7), $f(p) = 4c + \mu(p)\rho(p) = c + \mu(p)\rho(p)$, hence $c = 0$, and this is again a contradiction. A similar argument works well for the case $\mu = \rho$. Thus, Σ is the empty set, and therefore, that $\lambda = \rho$ on Ω.

If $n = 2$, we have a local orthonormal basis $\{\xi, X, \varphi X\}$ of TM, where X is a unit tangent vector field to M. From the above reasoning, $AX = \lambda X$, $A\varphi X = \lambda\varphi X$, and this real hypersurface M is one of the real hypersurfaces in Proposition 2.1. By Lemmata A and B, being M connected, μ and λ are constant.

If $n \geq 3$, let $\{\xi, E_1, \varphi E_1, \ldots, E_{n-1}, \varphi E_{n-1}\}$ be a local orthonormal basis such that $AE_i = \lambda_i E_i$, $A\varphi E_i = \lambda_i \varphi E_i$, for $i = 1, \ldots, n-1$. Now, suppose that there exist an open subset Ω of M, and $i \neq j$ such that $\lambda_i \neq \lambda_j$ on Ω. If $\mu^2 + 4c = 0$, then λ_i (or λ_j) is different of $\mu/2$ at a point of Ω. By Lema B and equation (2.8), $(2\lambda_i - \mu)(2\lambda_i - \mu) = 0$, hence $\lambda_i = \mu/2$, but this a contradiction. Therefore, $\mu^2 + 4c \neq 0$.

Next, we insert $X = E_i$, $Y = E_j$ and $Z = E_j$ in (2.7), obtaining

$$f = c + \lambda_i \lambda_j \text{ on } \Omega. \tag{3.8}$$

If we insert $X = \xi$, $Y = E_i$, $Z = E_i$ in (2.7), we obtain

$$(\lambda_i - \mu)(c + \lambda_i \mu - f) = 0 \text{ on } \Omega. \tag{3.9}$$

Similarly, inserting $X = \xi$, $Y = E_j$ and $Z = E_j$ in (2.7), we have

$$(\lambda_j - \mu)(c + \lambda_j \mu - f) = 0 \text{ on } \Omega. \tag{3.10}$$

If $\mu = \lambda_i$ (or $\mu = \lambda_j$) at a point of Ω, by equation (2.8), we have $\mu^2 = (2\lambda_i - \mu)^2 = \mu^2 + 4c$, hence $c = 0$, but this is a contradiction. Consequently, $\mu \neq \lambda_i$, $\mu \neq \lambda_j$ on the whole Ω. From (3.9) and (3.10) we obtain $\mu = 0$, therefore $f = c$, and by (3.8), $0 = \lambda_i \lambda_j$. Hence, there exists p in Ω such that $\lambda_i(p) = 0 = \mu$ or $\lambda_j(p) = 0 = \mu$, which is again a contradiction. Thus, there is a smooth function λ defined on Ω such that $AX = \lambda X$ for any tangent vector fields X to M and orthogonal to ξ. Therefore, Ω must be one of the examples in Proposition 2.1. A standard connectedness reasoning shows that the whole M is one the real hypersurfaces in Proposition 2.1.

The rest of the proof consists of checking whether these real hypersurfaces are pseudo-parallel. All of them satisfy that their Weingarten endomorphism takes the form $AX = \lambda X + (\mu - \lambda)\langle X, \xi\rangle\xi$, for any $X \in TM$, and for suitable real constants λ and μ. A straightforward computation shows

$$(R(X,Y) \cdot A)Z = (\mu - \lambda)(c + \lambda\mu)\{\langle Z, \xi\rangle (X \wedge Y)\xi + \langle (X \wedge Y)\xi, Z\rangle \xi\},$$

and

$$f(X \wedge Y \cdot A)Z = f(\mu - \lambda)\{\langle Z, \xi\rangle (X \wedge Y)\xi + \langle (X \wedge Y)\xi, Z\rangle \xi\},$$

for any $X, Y, Z \in TM$. From these equations, $f = c + \lambda\mu$. It only remains to compute the exact value of f for each case.

Case $c = +1$. See [21]. $\mu = 2\cot(2r)$, $\lambda = \cot(r)$, for $0 < r < \pi/2$. Hence, $f = 1 + 2\cot(2r)\cot(r) = \cot^2(r)$.

Case $c = -1$. See [3].

1) $\mu = 2\coth(2r)$, $\lambda = \coth(r)$, for $0 < r$. Then, $f = -1 + 2\coth(2r)\coth(r) = \coth^2(r) > 1$.
2) $\mu = 2$, $\lambda = 1$. Then, $f = 1$.
3) $\mu = 2\coth(2r)$, $\lambda = \tanh(r)$, for $0 < r$. Hence, $0 < f = \tanh^2(r) < 1$.

This concludes the proof of Theorem 1.2. □

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