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HYPERCYCLIC OPERATOR WEIGHTED SHIFTS

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ABSTRACT. We consider bilateral operator weighted shift T on $\mathbf{L}^2(\mathcal{K})$ with weight sequence $\{A_n\}_{n=-\infty}^{\infty}$ of positive invertible diagonal operators on $\mathcal{K}$. We give a characterization for T to be hypêrcyclic, and show that the conditions are far simplified in case T is invertible.

1. Introduction

A bounded linear operator T on a Hilbert space $\mathcal{H}$ is said to be *hypercyclic* if, for some $x \in \mathcal{H}$, the orbit $\{T^n x : n \in \mathcal{N}_0\}$ is dense in $\mathcal{H}$. The concept of hypercyclicity was first isolated by Rolewicz in [6]. In [5], Kitai gave a criterion for hypercyclicity of general operators that came to be known as the "hypercyclicity criterion". This criterion has been widely used to establish the hypercyclicity of different classes of operators. For instance, hypercyclic operators arise in the classes of composition operators [1], adjoints of multiplication operators [3], and weighted shifts [7]. In [4], Herrero posed the following problem : *Does T hypercyclic imply that $T \oplus T$ is also hypercyclic?* In [7], Salas proved that Herrero's question has an affirmative answer in case T is a bilateral weighted shift on $\ell^2(\mathbb{Z})$ or a unilateral backward weighted shift on $\ell^2(\mathbb{Z}_+)$. In the same paper he provided a characterization for hypercyclic bilateral weighted shifts in terms of their weight sequences. In [2], Feldman established a simpler characterization for hypercyclic bilateral weighted shifts that are invertible. In this paper, we extend these results to the case of hypercyclic bilateral operator weighted shifts on $\mathbf{L}^2(\mathcal{K})$. We also study the case when these shifts are invertible. The characterizations of Salas and Feldman turn out to be particular cases of our results when $dim\mathcal{K} = 1$.

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2. Preliminaries

Let $\mathcal{K}$ be a separable complex Hilbert space with an orthonormal basis $\{f_k\}_{k=0}^{\infty}$. Let $\mathbf{L}^2(\mathcal{K}) := \{x = (\dots, x_{-1}, [x_0], x_1, \dots) : x_i \in \mathcal{K}$ and $\sum_i \|x_i\|^2 < \infty\}$. $\mathbf{L}^2(\mathcal{K})$ is a Hilbert space with inner product $< x, y > := \sum_i < x_i, y_i >$. Let $\{A_n\}_{n=-\infty}^{\infty}$ be a uniformly bounded sequence of invertible positive diagonal operators on $\mathcal{K}$, and T be the bilateral (forward) operator weighted shift on $\mathbf{L}^2(\mathcal{K})$ with weight sequence $\{A_n\}_{n=-\infty}^{\infty}$. For $x = (x_i) \in \mathbf{L}^2(\mathcal{K})$, $T(\dots, x_{-1}, [x_0], x_1, \dots) = (\dots, A_{-2}x_{-2}, [A_{-1}x_{-1}], A_0x_0, \dots)$. Also, since $\{A_n\}_{n=-\infty}^{\infty}$ is uniformly bounded, T is bounded and $\|T\| = \sup_i \|A_i\|$. For $n > 0$,

$$T^n(\dots, x_{-1}, [x_0], x_1, \dots) = (\dots, y_{-1}, [y_0], y_1, \dots),$$

where

$$y_j = \prod_{s=0}^{n-1} A_{j+s-n}x_{j-n} \text{ or } y_{n+j} = \prod_{s=0}^{n-1} A_{j+s}x_j \quad , \tag{2.1}$$

$$\|T^n\| = \sup_j \| \prod_{s=0}^{n-1} A_{j+s}\|. \tag{2.2}$$

If $\{A_n^{-1}\}_{n=-\infty}^{\infty}$ is also uniformly bounded, then T^{-1} is the (backward) bilateral operator weighted shift on $\mathbf{L}^2(\mathcal{K})$, given by

$$T^{-1}(\dots, x_{-1}, [x_0], x_1, \dots) = (\dots, A_{-1}^{-1}x_0, [A_0^{-1}x_1], A_1^{-1}x_2, \dots).$$

For $n > 0$,

$$T^{-n}(\dots, x_{-1}, [x_0], x_1, \dots) = (\dots, z_{-1}, [z_0], z_1, \dots),$$

where

$$z_j = \prod_{s=0}^{n-1} A_{j+s}^{-1}x_{n+j} \text{ or } z_{j-n} = \prod_{s=0}^{n-1} A_{j-n+s}^{-1}x_j \quad , \tag{2.3}$$

$$\|T^{-n}\| = \sup_j \| \prod_{s=0}^{n-1} A_{j+s}^{-1}\|. \tag{2.4}$$

If T is the backward bilateral shift on $\mathbf{L}^2(\mathcal{K})$, then

$$T^n(\dots, x_{-1}, [x_0], x_1, \dots) = (\dots, y_{-1}, [y_0], y_1, \dots),$$

where

$$(2.5)\qquad y_j = \prod_{s=0}^{n-1} A_{j+s} x_{n+j} \text{ or } y_{j-n} = \prod_{s=0}^{n-1} A_{j-n+s} x_j \ ,$$

$$(2.6)\qquad \|T^n\| = \sup_j \| \prod_{s=0}^{n-1} A_{j+s} \|.$$

Further, $T^{-n}(\ldots, x_{-1}, [x_0], x_1, \ldots) = (\ldots, z_{-1}, [z_0], z_1, \ldots)$, where

$$(2.7)\qquad z_j = \prod_{s=0}^{n-1} A^{-1}_{j+s-n} x_{j-n} \text{ or } z_{n+j} = \prod_{s=0}^{n-1} A^{-1}_{j+s} x_j \ ,$$

$$(2.8)\qquad \|T^{-n}\| = \sup_j \| \prod_{s=0}^{n-1} A^{-1}_{j+s} \|$$

Also, since each A_n is an invertible diagonal operator on $\mathcal{K}$, $\|A_n\| = \sup_k \|A_n f_k\|$, $\|A_n^{-1}\| = \sup_k \|A_n^{-1} f_k\|$ and $\sup_k \|A_n f_k\| = \frac{1}{\inf_k \|A_n^{-1} f_k\|}$.

NOTATIONS. 1. For $x \in \mathcal{K}$ and $i \in \mathbb{Z}$, let $x(i) := (\ldots, y_{-1}, [y_0], y_1, \ldots)$ in $\mathbf{L}^2(\mathcal{K})$, where $y_i = x$ and $y_j = 0$ for all $j \neq i$.

2. For $n \geq 0$ and $q \in \mathbb{N}$, $H_{q,n} := \{x = (x_i) \in \mathbf{L}^2(\mathcal{K}) : x_i = 0 \ \forall \ |i+n| > q\}$. Clearly, $H_{q,n}$ is a closed subspace of $\mathbf{L}^2(\mathcal{K})$.

We include the hypercyclicity criterion, due independently to Kitai [5] and Gethner and Shapiro [3], for reference :

THEOREM 2.1. (*Hypercyclicity Criterion*) *Suppose that $T \in \mathcal{L}(X)$. If there exist two dense sets Y and Z in X and a sequence $n_k \to \infty$ such that*

1. $T^{n_k} x \to 0$ for every $x \in Y$, and

2. there exists a function $B : Z \to X$ such that $TBx = x$ for all $x \in Z$ and $B^{n_k} x \to 0$ for every $x \in Z$, then T is hypercyclic.

We also include, for reference, the characterization given by Salas in [7] :

THEOREM 2.2. *Let T be a bilateral weighted shift with positive weight sequence $\{a_n\}$. Then T is hypercyclic if and only if given $\epsilon > 0$ and $q \in \mathbb{N}$, there exists n arbitrarily large such that for all $|j| \leq q$, $\prod_{s=0}^{n-1} a_{s+j} < \epsilon$ and $\prod_{s=1}^{n} a_{j-s} > \frac{1}{\epsilon}$.*

For the invertible case, Feldman [2] gave the following simpler form of the above characterization :

THEOREM 2.3. *If $T : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$ is an invertible bilateral weighted shift with weight sequence $\{w_n\}_{n=-\infty}^{\infty}$, then T is hypercyclic if and only if there exists a sequence of integers $n_k \to \infty$ such that $\lim_{k\to\infty} \prod_{j=1}^{n_k} w_j = 0$ and $\lim_{k\to\infty} \prod_{j=1}^{n_k} \frac{1}{w_{-j}} = 0$.*

3. Hypercyclic operator weighted shift

THEOREM 3.1. *Let T be a bilateral (forward) operator weighted shift on $\mathbf{L}^2(\mathcal{K})$ with weight sequence $\{A_n\}_{n=-\infty}^{\infty}$, where $\{A_n\}$ is a uniformly bounded sequence of positive invertible diagonal operators on $\mathcal{K}$. Then T is hypercyclic if and only if given $\epsilon > 0$ and $q \in \mathbb{N}$, there exists n arbitrarily large such that for all $|j| \leq q$,*

$$\| \prod_{s=0}^{n-1} A_{s+j} \| < \epsilon \text{ and } \| \prod_{s=1}^{n} A_{j-s}^{-1} \| < \epsilon \ . \tag{3.1}$$

Proof. Suppose T is hypercyclic. Let $\epsilon > 0$, $q \in \mathbb{N}$ be given; choose $\delta > 0$. For arbitrarily fixed non negative integer i, consider the vector $\sum_{|j|\leq q} f_i(j)$ in $\mathbf{L}^2(\mathcal{K})$. Since the set of hypercyclic vectors for T is dense in $\mathbf{L}^2(\mathcal{K})$, there exists a hypercyclic vector $x = (\dots, x_{-1}, [x_0], x_1, \dots)$ such that $\|x - \sum_{|j|\leq q} f_i(j)\| < \delta$. Then, we have

$$\begin{cases} \|x_j\| < \delta & \text{for all } |j| > q, \\ \|x_j - f_i\| < \delta & \text{for all } |j| \leq q. \end{cases}$$

Since each x_j is in $\mathcal{K}$, there exist scalars $\alpha_k^{(j)}$ such that $x_j = \sum_{k=0}^{\infty} \alpha_k^{(j)} f_k$. Now, for $|j| > q$, $\|x_j\| < \delta$ implies $|\alpha_k^{(j)}| < \delta$ for all k, and for $|j| \leq q$, $\|x_j - f_i\| < \delta$ implies

$$\begin{cases} |\alpha_k^{(j)}| < \delta & \text{for } k \neq i, \\ |\alpha_k^{(j)}| > 1 - \delta & \text{for } k = i. \end{cases}$$

Also, since $orb(T, x)$ is dense in $\mathbf{L}^2(\mathcal{K})$, there exists arbitrarily large n (for convenience choose $n > 2q$) such that $\|T^n x - \sum_{|j|\leq q} f_i(j)\| < \delta$. Then,

$$\begin{cases} \|y_{j+n}\| < \delta & \text{if} |j+n| > q; \\ \|y_{j+n} - f_i\| < \delta & \text{if} |j+n| \leq q, \end{cases}$$

where $T^n x$ is denoted by $(\dots, y_{-1}, [y_0], y_1, \dots)$. As by the hypothesis $n > 2q$, we have $j + n > q$ for all $|j| \leq q$ and hence $\|y_{j+n}\| < \delta$. Furthermore, as $y_{j+n} = \prod_{s=0}^{n-1} A_{j+s} x_j = \sum_{k=0}^{\infty} \alpha_k^{(j)} \prod_{s=0}^{n-1} A_{j+s} f_k$ from

(2.1), we have $|\alpha_k^{(j)}| \| \prod_{s=0}^{n-1} A_{j+s} f_k \| < \delta$ for all k. In particular, if $k = i$ then $\| \prod_{s=0}^{n-1} A_{j+s} f_i \| < \frac{\delta}{1-\delta}$. As i is arbitrarily fixed, we have

$$(3.2) \qquad \| \prod_{s=0}^{n-1} A_{j+s} \| = \sup_i \| \prod_{s=0}^{n-1} A_{j+s} f_i \| < \frac{\delta}{1-\delta} \text{ for all } |j| \le q \,.$$

Again from (2.1), we have,

$$\begin{aligned} y_j &= \prod_{s=0}^{n-1} A_{j+s-n} x_{j-n} \\ &= \sum_{k=0}^{\infty} \alpha_k^{(j-n)} \prod_{s=0}^{n-1} A_{j+s-n} f_k. \end{aligned}$$

Therefore, for $|j| \le q$, $\|y_j - f_i\| < \delta$ gives

$$\begin{cases} \|\alpha_k^{(j-n)} \prod_{s=0}^{n-1} A_{j+s-n} f_k - f_k\| < \delta & \text{for } k = i; \\ \|\alpha_k^{(j-n)} \prod_{s=0}^{n-1} A_{j+s-n} f_k\| < \delta & \text{for } k \ne i. \end{cases}$$

The hypothesis $n > 2q$ implies that $|j - n| > q$ for all $|j| \le q$. Hence $|\alpha_k^{(j-n)}| < \delta$ for all k. In particular, $|\alpha_i^{(j-n)}| < \delta$. Also,

$$\|\alpha_i^{(j-n)} \prod_{s=0}^{n-1} A_{j+s-n} f_i - f_i\| < \delta$$

implies

$$|\alpha_i^{(j-n)}| \, \| \prod_{s=0}^{n-1} A_{j+s-n} f_i \| > 1 - \delta.$$

Therefore, $\| \prod_{s=1}^{n} A_{j-s} f_i \| = \| \prod_{s=0}^{n-1} A_{j+s-n} f_i \| > \frac{1-\delta}{\delta}$. Since this result holds for arbitrary i,

$$\inf_i \| \prod_{s=1}^{n} A_{j-s} f_i \| > \frac{1-\delta}{\delta},$$

and so, $\sup_i \| \prod_{s=1}^{n} A_{j-s}^{-1} f_i \| = \frac{1}{\inf_i \| \prod_{s=1}^{n} A_{j-s} f_i \|} < \frac{\delta}{1-\delta}$. Hence

$$(3.3) \qquad \| \prod_{s=1}^{n} A_{j+s}^{-1} \| \; < \; \frac{\delta}{1-\delta}.$$

The necessity of the conditions now follow from (3.2) and (3.3) upon choosing δ such that $\frac{\delta}{1-\delta} < \epsilon$.

For the converse we need the following lemma :

LEMMA 3.2. *Let T be a bilateral (forward) operator weighted shift on $\mathbf{L}^2(\mathcal{K})$ with weight sequence $\{A_n\}_{n=-\infty}^{\infty}$, where $\{A_n\}$ is a uniformly bounded sequence of positive invertible diagonal operators on $\mathcal{K}$. Then T is hypercyclic if the following condition holds: for $\epsilon > 0$, $q \in \mathbb{N}$ and vectors g, h in $H_{q,0}$, there exist n arbitrarily large and vector u in $H_{q,n}$ such that*

$$\|u\| < \epsilon \ , \ \|T^n(u) - g\| < \epsilon \ , \ \|T^n(h)\| < \epsilon.$$

Proof. The aim is to explicitly exhibit a hypercyclic vector f for T. For this we arbitrarily fix $x = (x_i)$ in $\mathbf{L}^2(\mathcal{K})$. For $n \in \mathbb{N}$, let P_n denote the projection of $\mathbf{L}^2(\mathcal{K})$ onto $H_{n,0}$. If $g_n := P_n x$, then $g_n = (\dots, 0, x_{-n}, x_{-n+1}, \dots, [x_0], \dots, x_{n-1}, x_n, 0, \dots)$ and $g_n \to x$ in the norm of $\mathbf{L}^2(\mathcal{K})$. We construct f in $\mathbf{L}^2(\mathcal{K})$ such that $\lim_{k\to\infty} \|T^{n_k}(f) - g_k\| = 0$ and n_k is a rapidly increasing sequence to be specified.

<u>For $k = 1$.</u> Let $n_1 = 0$, $q_1 = 1$, $\rho_1 = g_1$.

<u>For $k = 2$.</u> Let $\epsilon_2 := M^{-n_1}2^{-2}$, $q_2 := 2$, where M denotes $\|T\|$. Since g_2 , $\rho_1 \in H_{q_2,0}$, there exists n_2 arbitrarily large and $\rho_2 \in H_{q_2,n_2}$ such that $\|\rho_2\| < \epsilon_2$, $\|T^{n_2}(\rho_2) - g_{q_2}\| = \|T^{n_2}(\rho_2) - g_2\| < \epsilon_2$, $\|T^{n_2}(\rho_1)\| < \epsilon_2$. To ensure that the supports of ρ_1 and ρ_2 are disjoint, choose n_2 such that $q_2 - n_2 < -1$. Thus, $n_2 > 3$.

<u>For $k = 3$.</u> Let $\epsilon_3 := M^{-n_2}2^{-3}$, $q_3 := q_2 + n_2$. Since g_{q_3} , $\rho_1 + \rho_2 \in H_{q_3,0}$, there exists n_3 arbitrarily large and $\rho_3 \in H_{q_3,n_3}$ such that $\|\rho_3\| < \epsilon_3$, $\|T^{n_3}(\rho_3) - g_{q_3}\| < \epsilon_3$, $\|T^{n_3}(\rho_1 + \rho_2)\| < \epsilon_3$. To ensure that the supports of ρ_1 , ρ_2 and ρ_3 are disjoint, choose n_3 such that $q_3 - n_3 < -q_2 - n_2$. Thus, $n_3 > 2(n_2 + 2)$.

<u>For $k = 4$.</u> Let $\epsilon_4 := M^{-n_3}2^{-4}$, $q_4 := q_3 + n_3$. Since g_{q_4} , $\rho_1 + \rho_2 + \rho_3 \in H_{q_4,0}$, there exists n_4 arbitrarily large and $\rho_4 \in H_{q_4,n_4}$ such that $\|\rho_4\| < \epsilon_4$, $\|T^{n_4}(\rho_4) - g_{q_4}\| < \epsilon_4$, $\|T^{n_4}(\rho_1 + \rho_2 + \rho_3)\| < \epsilon_4$. To ensure that the supports of ρ_1 , ρ_2 , ρ_3 and ρ_4 are disjoint, choose n_4 such that $q_4 - n_4 < -q_3 - n_3$. Thus, $n_4 > 2(n_2 + n_3 + 2)$. Continuing this process, we get, $\|\rho_{k+1}\| < M^{-n_k}2^{-(k+1)}$, $\|T^{n_{k+1}}(\rho_{k+1}) - g_{q_{k+1}}\| < M^{-n_k}2^{-(k+1)}$, and $\|T^{n_{k+1}}(\rho_1 + \dots + \rho_k)\| < M^{-n_k}2^{-(k+1)}$. Let $f := \sum_{j=1}^{\infty} \rho_j$. Since the supports of ρ_j are pairwise disjoint, it follows that $\sum_{j=1}^{\infty} \|\rho_j\|^2 < \infty$ and $f \in \mathbf{L}^2(\mathcal{K})$. Also, $\|T^{n_k}(f) - g_{q_k}\| \le \|T^{n_k}(\sum_{j=1}^{k-1} \rho_j)\| + \|T^{n_k}(\rho_k) - g_{q_k}\| + \sum_{j=k+1}^{\infty} \|T^{n_k}(\rho_j)\| \le M^{-n_{k-1}}2^{-k} + M^{-n_{k-1}}2^{-k} + \sum_{j=k+1}^{\infty} M^{n_k}\|\rho_j\| \le 2^{-k+2}$. Therefore, $\|T^{n_k}(f) - x\| \le \|T^{n_k}(f) - g_{q_k}\| + \|g_{q_k} - x\| \to 0$ as $k \to \infty$. Hence, $orb(T, f)$ is dense in $\mathbf{L}^2(\mathcal{K})$, and T is hypercyclic. □

CONVERSE OF THEOREM 3.1. Let $\epsilon > 0$, $q \in \mathbb{N}$ and $g, h \in H_{q,0}$. Then, $T^n h \in H_{q,-n}$ and $T^{-n} g \in H_{q,n}$. Using (2.1) and (2.3), we get

$$\|T^n h\| \leq \sup_{|j|\leq q} \|\prod_{s=0}^{n-1} A_{s+j}\| \, \|h\| \tag{3.4}$$

$$\|T^{-n} g\| \leq \sup_{|j|\leq q} \|\prod_{s=1}^{n} A_{j-s}^{-1}\| \, \|g\|. \tag{3.5}$$

Choose $\delta < \min(\frac{\epsilon}{\|h\|}, \frac{\epsilon}{\|g\|})$. From the hypothesis, there exists $n > 2q$ such that for all $|j| \leq q$, $\|\prod_{s=0}^{n-1} A_{s+j}\| < \delta$ and $\|\prod_{s=1}^{n} A_{j-s}^{-1}\| < \delta$. Using this in (3.4) and (3.5), we get $\|T^{-n}g\| \leq \sup_{|j|\leq q} \|\prod_{s=1}^{n} A_{j-s}^{-1}\| \, \|g\| \leq \delta\|g\| < \epsilon$ and $\|T^n h\| \leq \delta\|h\| < \epsilon$. Thus, for $u := T^{-n}g$, we have $u \in H_{q,n}$ and $\|u\| = \|T^{-n}g\| < \epsilon$, $\|T^n u - g\| = 0$, $\|T^n h\| < \epsilon$. The result now immediately follows from Lemma 3.2. □

In Theorem 3.1, T is a forward bilateral operator weighted shift on $\mathbf{L}^2(\mathcal{K})$. For a backward bilateral operator weighted shift on $\mathbf{L}^2(\mathcal{K})$, a similar result can be stated as :

THEOREM 3.3. *Let T be a backward bilateral operator weighted shift on $\mathbf{L}^2(\mathcal{K})$ with weight sequence $\{A_n\}_{n=-\infty}^{\infty}$, where $\{A_n\}$ is a uniformly bounded sequence of positive invertible diagonal operators on $\mathcal{K}$. Then T is hypercyclic if and only if given $\epsilon > 0$ and $q \in \mathbb{N}$, there exists n arbitrarily large such that for all $|j| \leq q$,*

$$\|\prod_{s=0}^{n-1} A_{s+j}^{-1}\| < \epsilon \text{ and } \|\prod_{s=1}^{n} A_{j-s}\| < \epsilon. \tag{3.6}$$

REMARK. Theorem 2.1 and Theorem 2.5 of [7] follows by taking $dim\mathcal{K} = 1$ and $dim\mathcal{K} = m < \infty$ respectively in Theorem 3.1.

4. Invertibly hypercyclic operator weighted shifts

In this section we assume that $\{A_n\}_{n=-\infty}^{\infty}$ is a uniformly bounded sequence of positive invertible diagonal operators on $\mathcal{K}$, and the sequence $\{A_n^{-1}\}_{n=-\infty}^{\infty}$ is also uniformly bounded. Thus T is invertible on $\mathbf{L}^2(\mathcal{K})$ and T^{-1} is the backward bilateral operator weighted shift on $\mathbf{L}^2(\mathcal{K})$ defined as $T^{-1}(\ldots, x_{-1}, [x_0], x_1, \ldots) = (\ldots, A_{-1}^{-1}x_0, [A_0^{-1}x_1], A_1^{-1}x_2, \ldots)$.

LEMMA 4.1. *Let T be an invertible bilateral operator weighted shift on $\mathbf{L}^2(\mathcal{K})$ with weight sequence $\{A_n\}_{n=-\infty}^{\infty}$ of positive invertible diagonal operators on $\mathcal{K}$, and let $\{n_k\}$ be a sequence of positive integers such that $n_k \to \infty$. If there exists $n \in \mathbb{Z}$ such that for any $x \in \mathcal{K}$, $T^{n_k}x(n) \to 0$, then $T^{n_k}x(j) \to 0$ for all $j \in \mathbb{Z}$ as $k \to \infty$.*

Proof. For any $j \in \mathbb{Z}$, there exists $p \in \mathbb{Z}$ and a positive invertible operator A on $\mathcal{K}$ such that $AT^p x(n) = x(j)$. Since T is invertible, T^p is continuous (even if p is negative). So assuming $T^{n_k}x(n) \to 0$, we get $T^{n_k}x(j) = AT^pT^{n_k}x(n) \to 0$ as $k \to \infty$, for all $j \in \mathbb{Z}$. □

THEOREM 4.2. *Let T be an invertible bilateral operator weighted shift on $\mathbf{L}^2(\mathcal{K})$ with weight sequence $\{A_n\}_{n=-\infty}^{\infty}$ of positive invertible diagonal operators on $\mathcal{K}$. Then T is hypercyclic if and only if there exists a sequence of integers $n_k \to \infty$ such that $\lim_{k\to\infty} \|\prod_{j=1}^{n_k} A_j\| = 0$ and $\lim_{k\to\infty} \|\prod_{j=1}^{n_k} A_{-j}^{-1}\| = 0$.*

Proof. If T is hypercyclic, the result follows immediately from Theorem 3.1. For the converse, let $Z := \{x = (x_i) \in \mathbf{L}^2(\mathcal{K})$: all but finitely many x_i's are zero$\}$. Then Z is dense in $\mathbf{L}^2(\mathcal{K})$. Let B be the restriction of T^{-1} to Z. Choosing a non zero element x in Z, we can assume, without loss of generality, that the n^{th} entry of x, denoted by y, is non zero. As $\|T^{n_k}y(1)\| = \|\prod_{j=1}^{n_k} A_j y\| \le \|\prod_{j=1}^{n_k} A_j\|\,\|x\|$ and $\|B^{n_k}y(0)\| = \|\prod_{j=1}^{n_k} A_{-j}^{-1}y\| \le \|\prod_{j=1}^{n_k} A_{-j}^{-1}\|\,\|x\|$, the hypothesis implies that $T^{n_k}y(1) \to 0$ and $B^{n_k}y(0) \to 0$. This implies $T^{n_k}y(n) \to 0$ and $B^{n_k}y(n) \to 0$ as $k \to \infty$, by Lemma 4.1. Since for $x = (x_i) \in Z$, only finitely many entries x_i are non zero, it follows that $T^{n_k}x \to 0$ and $B^{n_k}x \to 0$ for every $x \in Z$. The result now follows from the hypercyclicity criterion in Theorem 2.1. □

5. Relaxing invertibility

If T is a bilateral operator weighted shift on $\mathbf{L}^2(\mathcal{K})$ with weight sequence $\{A_n\}_{n=-\infty}^{\infty}$, then T is invertible if and only if there exists $m > 0$ such that $\|A_n^{-1}\| \le m$ for all n. In this section, we show that the assumption $\|A_n^{-1}\| \le m$ for all $n < 0$ is sufficient to establish Theorem 4.2.

THEOREM 5.1. *Let T be the bilateral operator weighted shift on $\mathbf{L}^2(\mathcal{K})$ with weight sequence $\{A_n\}_{n=-\infty}^{\infty}$, where $\{A_n\}_{n=-\infty}^{\infty}$ is a uniformly bounded sequence of positive invertible diagonal operators on $\mathcal{K}$, and there exists $m > 0$ such that $\|A_n^{-1}\| \le m$ for all $n < 0$. Then T is* *hypercyclic if and only if there exists a sequence of integers $n_k \to \infty$ such that* $\lim_{k\to\infty} \| \prod_{j=1}^{n_k} A_j\| = 0$ *and* $\lim_{k\to\infty} \| \prod_{j=1}^{n_k} A_{-j}^{-1}\| = 0$.

Proof. If T is hypercyclic the result follows from Theorem 3.1. For the converse, let $\epsilon > 0$, $\delta > 0$ and $q \in \mathbb{N}$. Choose n_k arbitrarily large such that $\| \prod_{s=1}^{n_k} A_s\| < \delta$ and $\| \prod_{s=1}^{n_k} A_{-s}^{-1}\| < \delta$. For $j \in \mathbb{Z}$ and $|j| \le q$, let

$$c_j := \begin{cases} \| \prod_{s=1}^{j-1} A_s^{-1}\| & \text{if } 1 < j \le q, \\ 1 & \text{if } j = 1, \\ \| \prod_{s=j}^{0} A_s\| & \text{if } -q \le j < 1. \end{cases}$$

Let $n := n_k + q + 1$. For all $|j| \le q$, $n_k \le n + j - 1$ and

(5.1)

$$\| \prod_{s=0}^{n-1} A_{s+j}\| = \| \prod_{s=j}^{n+j-1} A_s\| \le c_j \| \prod_{s=1}^{n+j-1} A_s\| \le c_j \| \prod_{s=1}^{n_k} A_s\| \, \| \prod_{n_k+1}^{n+j-1} A_s\|.$$

By (2.2), $\|T^r\| = \sup_k \| \prod_{s=0}^{r-1} A_{k+s}\| = \sup_k \| \prod_{s=k}^{r+k-1} A_s\|$ for all $r > 0$. Hence, for all $|j| \le q$, $\|T^{q+j}\| = \sup_k \| \prod_{s=k}^{q+j+k-1} A_s\|$, which implies that $\| \prod_{s=n_k+1}^{n+j-1} A_s\| = \| \prod_{s=n_k+1}^{n_k+q+j} A_s\| \le \|T^{q+j}\|$. Using this in (5.1) we get, $\| \prod_{s=0}^{n-1} A_{s+j}\| \le c_j \| \prod_{s=1}^{n_k} A_s\| \, \|T^{q+j}\| < c_j \delta \|T^{q+j}\|$ for all $|j| \le q$. If $c := \max\{c_j : |j| \le q\}$ and $m_1 := \max\{\|T^{q+j}\| : |j| \le q\}$, then

$$\| \prod_{s=0}^{n-1} A_{s+j}\| < c m_1 \delta \text{ for all } |j| \le q. \tag{5.2}$$

Again, let

$$c_j' := \begin{cases} \| \prod_{s=1-j}^{0} A_{-s}^{-1}\| & \text{if } 0 < j \le q, \\ 1 & \text{if } j = 0, \\ \| \prod_{s=1}^{-j} A_{-s}\| & \text{if } -q \le j < 0. \end{cases}$$

For $|j| \le q$, $0 < n_k + 1 \le n - j$ and

$$\begin{aligned} \| \prod_{s=1}^{n} A_{j-s}^{-1}\| &= \| \prod_{s=1-j}^{n-j} A_{-s}^{-1}\| \\ &\le c_j' \| \prod_{s=1}^{n_k} A_{-s}^{-1}\| \, \| \prod_{s=n_k+1}^{n-j} A_{-s}^{-1}\| < c_j' \delta \prod_{n_k+1}^{n-j} \|A_{-s}^{-1}\| \\ &\le c_j' \delta \, m^{(n-j)-(n_k+1)+1} = c_j' \delta \, m^{q-j+1}. \end{aligned}$$

If $c' := \max\{c'_j : |j| \leq q\}$ and $m_2 := \max\{m^{q-j+1} : |j| \leq q\}$, then

$$(5.3) \qquad \|\prod_{s=1}^{n} A_{j-s}^{-1}\| < c' m_2 \delta \text{ for all } |j| \leq q.$$

Choosing $\delta < \min\{\frac{\epsilon}{c' m_2}, \frac{\epsilon}{c m_1}\}$, we get

$$\|\prod_{s=0}^{n-1} A_{s+j}\| < \epsilon$$

and

$$\|\prod_{s=1}^{n} A_{j-s}^{-1}\| < \epsilon$$

for all $|j| \leq q$. The result now follows from Theorem 3.1. □

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