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A TRIAL SOLUTION APPROACH TO THE $GI/M/1$ QUEUE WITH N-POLICY AND EXPONENTIAL VACATIONS†

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Abstract

We present a trial solution approach to $GI/M/1$ queues with generalized vacations. Specific types of generalized vacations we consider are N-policy and a combination of N-policy and exponential multiple vacations. Discussions about how to find trial solutions are given.

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1. Introduction

The subject of $M/G/1$ queues with generalized vacations has been studied extensively in the literature by a number of researchers. (For a survey see Doshi, 1986.) Even books are written on this subject (Takagi, 1991 and 1993). On the contrary, the subject of $GI/M/1$ queues with generalized vacations has been studied by quite fewer researchers.

Karaesmen and Gupta (1996) stated in their paper that $GI/M/1$ queues with generalized vacations are "considerably more difficult" to analyze than corresponding $M/G/1$ queues. The main objective of this paper is to present a simple method for analyzing $GI/M/1$ queues with generalized vacations.

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Our method is a trial solution method similar to what Ross (1997) used for the $GI/M/1$ queue and what Gross and Harris (1985) used for the $GI/M/c$ queue. Once trial solutions of proper forms are found for balance equations of arrival-epoch embedded Markov chains (AEMCs), the rest are straightforward. Thus the issue is how to find (or guess) proper trial solutions. In this paper, we find trial solutions by an approach based on the regenerative processes.

The list of specific types of generalized vacations is long. It includes multiple vacations, single vacation, setup time, N-policy, D-policy, *etc.* $M/G/1$ queues with each of these types, and some combinations of these, are well investigated. Under the $GI/M/1$ setting, however, only some of these types are investigated so far.

An almost exclusive listing is as follows. The $GI/M/1$ queue with exponential multiple vacations (EMV) is investigated by three teams of researchers at about the same time (Chatterjee and Mukherjee, 1990; Choi and Park, 1991; Tian *et al.*, 1989). Three extensions of this $GI/M/1$ queue with EMV are the finite capacity version (Karaesmen and Gupta, 1996), a batch service version (Choi and Han, 1994), and the discrete-time version (Tian and Zhang, 2002). Analyses of the $GI/M/1$ queue with exponential single vacation, and a variant of it, can be found in Choi and Park (1991) and Daniel and Krishnamoorthy (1986). And only recently the N-policy $GI/M/1/K$ queue (Ke and Wang, 2002), the N-policy $GI/M/1$ queue (Zhang and Tian, 2004) and the N-policy $GI/M/1/K$ queue with EMV (Ke, 2003) are investigated, where K stands for the finite capacity.

It should be noted that the N-policy $GI/M/1$ queue (Zhang and Tian, 2004) is a special case of the N-policy $GI/M/1/K$ queue (Ke and Wang, 2002) such that $K = \infty$. However, the ten-step algorithm presented by Ke and Wang (2002) is intended for the finite capacity K. Likewise, the fifteen-step algorithm presented by Ke (2003) for the N-policy $GI/M/1/K$ queue with EMV is intended for the finite capacity K.

We analyze the N-policy $GI/M/1$ queue with EMV by using the trial solution approach in Section 4. In Section 3, we demonstrate that the result of Zhang and Tian (2004) could have been obtained easily by using the trial solution approach. In Section 2, we introduce the trial solution approach.

2. PRELIMINARIES AND ANALYSIS OF THE $GI/M/1$ QUEUE

In this section, we define queueing systems and the notation that will be used in this paper. Also, we examine the method of Ross (1997) and introduce the trial solution approach.

2.1. The N-policy GI/M/1 queue

Customers' interarrival times A_n are *iid* (independent and identically distributed) random variables having general distribution $A(t)$ with a mean λ^{-1}. Customers are served one at a time by a single server. The service times S_n are *iid* exponential with a mean μ^{-1}. A_n and S_n are mutually independent.

An N-policy operates as follows. The server is turned off each time the system empties. When the queue length reaches a predetermined constant N, the server is turned on and begins to serve the customers exhaustively.

2.2. The N-policy GI/M/1 queue with EMV

As soon as the system empties, the sever has a vacation of random length. When a vacation is over, if the queue length is below N the server has another vacation; otherwise he returns from the vacation and begins to serve the customers exhaustively. The vacation times V_n are *iid* exponential with a mean ν^{-1}. V_n, A_n, S_n are mutually independent.

2.3. Notation

$\mathbf{P}$: matrix of state transition probabilities of an AEMC,

π_{n0} : steady-state probabilities of an AEMC that an arriving customer finds the server idling (either turned off or on vacation) and sees n customers in the queue, $n = 0, 1, 2, \Lambda$,

π_{n1} : steady-state probabilities of an AEMC that an arriving customer finds the server busy and sees n customers in the system, $n = 1, 2, 3, \Lambda$,

p_{n0} (p_{n1}) : long-run proportions of time during which the server is idling (busy) and there are n customers in the system, $n = 0, 1, 2, \Lambda$ ($n = 1, 2, 3, \Lambda$),

$$b_k = \int_0^\infty \left\{ \frac{(\mu t)^k}{k!} \exp(-\mu t) \right\} dA(t), \quad k = 0, 1, 2, \Lambda,$$

$$A^*(\theta) = \int_0^\infty \exp(-\theta t) dA(t)$$: LST (Laplace-Stieltjes transform) of A_n.

2.4. AEMC based analysis of the GI/M/1 queue

We consider the AEMC of the standard $GI/M/1$ queue where $\lambda < \mu$. Let $\mathbf{\Pi}$ denote the vector $(\pi_{00}, \pi_{11}, \pi_{21}, \pi_{31}, \Lambda)$, then $\mathbf{\Pi}$ is the unique solution of

$$\mathbf{\Pi} = \mathbf{\Pi P}, \tag{2.1}$$

$$1 = \pi_{00} + \sum_{n=1}^{\infty} \pi_{n1}, \tag{2.2}$$

where $\mathbf{P}$, with deleted first column, is

$$\begin{bmatrix} b_0 & & & \\ b_1 & b_0 & & \\ b_2 & b_1 & b_0 & \\ M & M & M & O \end{bmatrix}. \tag{2.3}$$

The trial solution used by Ross (1997) is of the form

$$\pi_{n1} = \pi_{00}\, r^n, \quad n \geq 1. \tag{2.4}$$

The conditions that the trial solution (2.4) should satisfy are obtained by substituting it into (2.1) and (2.2). Since one of the so-called balance equations belonging to (2.1) is redundant, we choose to ignore the first equation which corresponds to the deleted first column of $\mathbf{P}$.

Substituting (2.4) into the rest of the balance equations, we have

$$\pi_{00}\, r^n = \sum_{i=n-1}^{\infty} \pi_{00}\, r^i\, b_{i-n+1}, \quad n \geq 1. \tag{2.5}$$

Throughout the paper we will ignore trivial cases. The only non-trivial root of (2.5) is

$$r = \sum_{i=0}^{\infty} b_i\, r^i = A^*(\mu - \mu\, r), \quad 0 < r < 1. \tag{2.6}$$

Finally, we have $\pi_{00} = 1 - r$ from (2.2).

2.5. A regenerative process approach to the GI/M/1 queue

Let π_{n2}, $n = 0, 1, 2, \Lambda$, denote the long-run proportions of customers who leave behind n customers in the system when they depart. Note that π_{n0}, $n = 0, 1, 2, \Lambda$ (π_{n1}, $n = 1, 2, 3, \Lambda$) can also be interpreted as the long-run proportions of customers who find the server idling (busy) and see n customers in the system when they arrive.

For the $GI/M/1$ queue, it is well known that (see Wolff, 1989, p. 387)

$$\pi_{02} = \pi_{00}, \tag{2.7.a}$$

$$\pi_{n2} = \pi_{n1}, \quad n \geq 1. \tag{2.7.b}$$

Another well known relation based on PASTA (Wolff, 1982) property is that (see Wolff, 1989, p. 397)

$$\mu\, p_{n1} = \lambda\, \pi_{n-1,2}, \quad n \geq 1. \tag{2.8}$$

Our approach based on the regenerative process is as follows. Let R_n and $R_n^*(\theta)$ respectively denote the remaining interarrival time and its LST at an instant a customer departs the system leaving behind n customers, $n = 0, 1, 2, \Lambda$. Then, based on the standard renewal reward arguments, we claim that

$$p_{00} = \lambda_2\, \pi_{02}\, E(R_0), \tag{2.9.a}$$

$$p_{11} = \lambda_2\, \pi_{12}\, E[\min(R_1, S)] + \lambda\, \pi_{00}\, E[\min(A, S)], \tag{2.9.b}$$

$$p_{n1} = \lambda_2\, \pi_{n2}\, E[\min(R_n, S)] + \lambda\, \pi_{n-1,1}\, E[\min(A, S)], \quad n \geq 2, \tag{2.9.c}$$

where

$$\lambda_2 = \lambda. \tag{2.10}$$

We interpret (2.9) as follows. (Readers interested in a rigorous approach are referred to Theorem 3.1 of El-Taha and Stidham Jr., 1999.) λ_2 (λ) is the expected number of departing (arriving) customers per unit time. Thus $\lambda_2\pi_{n2}$ is the expected number of departing customers who leave behind n customers per unit time, $n \geq 0$. Likewise, $\lambda\pi_{00}$ ($\lambda\pi_{n1}, n \geq 1$) is the expected number of arriving customers who find the server idling (busy) and see 0 (n) customers per unit time. Furthermore, each time a customer departs the system leaving behind 0 customers, the expected time duration until the next arrival is $E(R_0)$. On the other hand, each time a customer departs the system leaving behind n customers, $n \geq 1$, the expected time duration until either the next arrival or the next departure (whichever occurs first) is $E[\min(R_n, S)]$. Similarly, each time a customer arrives at the system seeing n customers, $n \geq 0$, the expected time duration until either the next arrival or the next departure (whichever occurs first) is $E[\min(A, S)]$, due to the memoryless property of the exponential service time. $E[\min(R_n, S)]$ and $E[\min(A, S)]$ are as follows (see Appendix).

$$E[\min(R_n, S)] = \{1 - R_n^*(\mu)\}\mu^{-1}, \quad n \geq 1, \tag{2.11.a}$$

$$E[\min(A,S)] = (1-b_0)\mu^{-1}. \tag{2.11.b}$$

Substituting (2.7), (2.8), (2.10), (2.11) into (2.9.b) and (2.9.c) and solving for π_{n1}, we have

$$\pi_{n1} = \pi_{00}\prod_{i=1}^{n}\frac{b_0}{\{1-R_i^*(\mu)\}}, \quad n \geq 1. \tag{2.12}$$

Comparing (2.12) with (2.4), we obtain following relations.

$$R_n^*(\mu) = R_1^*(\mu), \quad n \geq 2, \tag{2.13}$$

$$r = \frac{b_0}{\{1-R_1^*(\mu)\}}. \tag{2.14}$$

3. THE N-POLICY $GI/M/1$ QUEUE

In this section, we solve balance equations of AEMC using a trial solution. Then we explain how the trial solution is found by the approach introduced in Section 2.5.

3.1. *Solving* $\mathbf{\Pi} = \mathbf{\Pi P}$

Let $\mathbf{\Pi}$ denote the vector $(\pi_{00}, \pi_{10}, \pi_{20}, \Lambda, \pi_{N-1,0}, \pi_{11}, \pi_{21}, \pi_{31}, \Lambda)$. As we did in Section 2, we delete the first column of $\mathbf{P}$ of the N-policy $GI/M/1$ queue. Then we can express the rest of $\mathbf{P}$ in terms of submatrices as

$$\left[\begin{array}{c|c|c} \mathbf{I} & \mathbf{O}_2 & \\ \cline{1-2} & & \mathbf{O}_3 \\ \mathbf{O}_1 & \mathbf{P}_1 & \\ \cline{3-3} & & \mathbf{P}_2 \end{array}\right], \tag{3.1}$$

where $\mathbf{I}$ is an identity matrix of size $N-1$, whose columns correspond to states $\{(n,0),\ 1 \leq n \leq N-1\}$; $\mathbf{O}_1, \mathbf{O}_2, \mathbf{O}_3$ are matrices with all entries zero; $\mathbf{P}_2$ is the same as (2.3) except that corresponding column states are now $\{(n,1),\ n \geq N+1\}$

and corresponding row states are $\{(n,1),\ n \geq N\}$; and

$$\mathbf{P}_1 = \begin{array}{c} \\ N-1 \\ 11 \\ 21 \\ M \\ N-1,1 \\ N,1 \\ M \end{array} \begin{array}{c} \begin{array}{cccccc} 11 & 21 & 31 & \Lambda & N-1,1 & N,1 \end{array} \\ \begin{pmatrix} b_{N-1} & b_{N-2} & b_{N-3} & \Lambda & b_1 & b_0 \\ b_1 & b_0 & & & & \\ b_2 & b_1 & b_0 & & & \\ M & M & M & & O & \\ b_{N-1} & b_{N-2} & b_{N-3} & \Lambda & b_1 & b_0 \\ b_N & b_{N-1} & b_{N-2} & \Lambda & b_2 & b_1 \\ M & M & M & & M & M \end{pmatrix} \end{array}. \tag{3.2}$$

$\mathbf{\Pi}$ is the unique solution of

$$\mathbf{\Pi} = \mathbf{\Pi P}, \tag{3.3}$$

$$1 = \sum_{n=0}^{N-1} \pi_{n0} + \sum_{n=1}^{\infty} \pi_{n1}. \tag{3.4}$$

As we did in Section 2, we ignore the first equation of (3.3) which corresponds to the deleted first column of $\mathbf{P}$. Solving the next $N-1$ equations, which correspond to $\mathbf{I}$ in (3.1), we obtain

$$\pi_{N-1,0} = \pi_{N-2,0} = \Lambda = \pi_{10} = \pi_{00}. \tag{3.5}$$

The rest of the balance equations are as follows.

$$\pi_{n1} = \begin{cases} \pi_{N-1,0} b_{N-1} + \sum_{i=1}^{\infty} \pi_{i1} b_i, & n = 1, \qquad (3.6.a) \\ \pi_{N-1,0} b_{N-n} + \sum_{i=n-1}^{\infty} \pi_{i1} b_{i-n+1}, & 2 \leq n \leq N, \qquad (3.6.b) \\ \sum_{i=n-1}^{\infty} \pi_{i1} b_{i-n+1}, & n \geq N+1. \qquad (3.7) \end{cases}$$

Note that (3.6) and (3.7) correspond to $\mathbf{P}_1$ and $\mathbf{P}_2$ in (3.1), respectively.

The trial solution we use is of the form

$$\pi_{n1} = \pi_{N1} r^{n-N}, \quad n \geq N. \tag{3.8}$$

Substituting (3.8) into (3.7) and dividing both sides by $\pi_{N1} r^{n-N}$, we obtain (2.6). That is, r in (3.8) should satisfy the equation (2.6).

Next, π_{n1}, $1 \leq n \leq N$, must be determined by (3.6). We substitute (3.8) into (3.6.b), one at a time, in the reversed order of $n = N, N-1, \Lambda, 3, 2$. Substituting (3.5) and (3.8) into the case $n = N$ of (3.6.b), we have

$$\pi_{N1} = \pi_{00} b_0 + \pi_{N-1,1} b_0 + \pi_{N1} r^{-1} \sum_{i=1}^{\infty} b_i r^i. \tag{3.9}$$

From (2.6) we have "$\sum_{i=1}^{\infty} b_i r^i = r - b_0$". Thus from (3.9), we obtain

$$\pi_{N-1,1} = \pi_{N1} r^{-1} - \pi_{00}. \tag{3.10}$$

The rest of π_{n1}, $1 \leq n \leq N-2$, are obtained in a recursive fashion from the cases $2 \leq n \leq N-1$ of (3.6.b). For example, substituting (3.10) together with (3.8), (3.5), (2.6) into the case $n = N-1$ of (3.6.b) and solving for $\pi_{N-2,1}$, we have

$$\pi_{N-2,1} = \pi_{N1} r^{-2} - \pi_{00} C_2, \tag{3.11}$$

where

$$C_2 = b_0^{-1}. \tag{3.12}$$

For convenience, let us define C_n as follows:

$$b_0 C_n = C_{n-1} - \sum_{i=2}^{n-1} C_i b_{n-i}, \quad n \geq 3. \tag{3.13}$$

Then, from the cases $n = N-2, N-3, \Lambda, 3, 2$ of (3.6.b), we obtain

$$\pi_{N1} = \pi_{N1} r^{n-N} - \pi_{00} C_{N-n}, \quad 1 \leq n \leq N-3. \tag{3.14}$$

Note that (3.14) is valid for $1 \leq n \leq N-1$ since (3.11) and (3.10) are also of the same form if we let C_1 be 1.

Finally, substituting (3.14), (3.8), (3.5), (2.6) into (3.6.a), we obtain "$\pi_{N1} = \pi_{00} C_N r^N$". Thus, putting all these together, we have

$$\pi_{n1} = \begin{cases} \pi_{00}(C_N r^n - C_{N-n}), & 1 \leq n \leq N-1, \\ \pi_{00} C_N r^n, & n \geq N, \end{cases} \tag{3.15}$$

where π_{00}, obtained by substituting (3.5) and (3.15) into (3.4), is as follows:

$$\pi_{00} = (1-r)\left\{(1-r)(N - \sum_{n=1}^{N-1} C_n) + C_N r\right\}^{-1}. \tag{3.16}$$

It can be confirmed that (3.15) coincide with the result of Zhang and Tian (2004). (Details are omitted due to space consideration.)

3.2. A regenerative process approach to the N-policy $GI/M/1$ queue

For the N-policy $GI/M/1$ queue where $N \geq 2$, (2.7.a), (2.8), (2.10), (2.11) are still valid but (2.7.b) should be modified as

$$\pi_{n2} = \begin{cases} \pi_{n0} + \pi_{n1}, & 1 \leq n \leq N-1, \qquad (3.17.a) \\ \pi_{n1}, & n \geq N. \qquad (3.17.b) \end{cases}$$

(2.9) is extended from the case $N = 1$ to the case $N \geq 2$ as follows. (2.9.a) is still valid but now we additionally have

$$p_{n0} = \lambda \pi_{n-1,0} E(A) = \pi_{n-1,0}, \quad 1 \leq n \leq N-1. \tag{3.18}$$

Furthermore, instead of (2.9.b) and (2.9.c), we now have

$$p_{11} = \lambda_2 \pi_{12} E[\min(R_1, S)], \tag{3.19.a}$$

$$p_{n1} = \lambda_2 \pi_{n2} E[\min(R_n, S)] + \lambda \pi_{n-1,1} E[\min(A, S)], \quad 2 \leq n \leq N-1, \tag{3.19.b}$$

$$\begin{aligned} p_{N1} = {} & \lambda_2 \pi_{N2} E[\min(R_N, S)] + \lambda \pi_{N-1,0} E[\min(A, S)] \\ & + \lambda \pi_{N-1,1} E[\min(A, S)], \end{aligned} \tag{3.19.c}$$

$$p_{n1} = \lambda_2 \pi_{n2} E[\min(R_n, S)] + \lambda \pi_{n-1,1} E[\min(A, S)], \quad n \geq N+1. \tag{3.19.d}$$

Interpretation of (3.19) is omitted since it is (almost) the same as that of (2.9). Substituting (2.8), (2.10), (2.11), (3.17.b) into (3.19.d) and solving for π_{n1}, we have

$$\pi_{n1} = \pi_{N1} \prod_{i=N+1}^{n} \frac{b_0}{\{1 - R_i^*(\mu)\}}, \quad n \geq N+1. \tag{3.20}$$

Since (3.20) resembles (2.12), there may exist N-policy counterparts of (2.13) and (2.14). The trial solution, (3.8), is based on the following conjectures.

$$R_n^*(\mu) = R_N^*(\mu), \quad n \geq N+1, \tag{3.21}$$

$$r = \frac{b_0}{\{1 - R_N^*(\mu)\}}, \tag{3.22}$$

which include (2.13) and (2.14) as a special case that $N = 1$. Substituting (3.21) and (3.22) into (3.20), we have (3.8). Note that (3.8) resembles the solution used by Gross and Harris (1985) for the $GI/M/c$ queue if N in (3.8) is replaced by c.

4. THE N-POLICY $GI/M/1$ QUEUE WITH EMV

As a further demonstration that our trial solution method is simple and straightforward, the $GI/M/1$ queue with both N-policy and EMV is analyzed in this section. Note that the case $N = 1$ corresponds to the $GI/M/1$ queue with EMV investigated by Chatterjee and Mukherjee (1990), Choi and Park (1991), and Tian *et al.* (1989).

4.1. **P** *for the N-policy GI/M/1 queue with EMV*

Tian *et al.* (1989) define ν_n, $n \geq 0$, and β as follows.

$$\nu_n = \int_0^\infty \int_0^t \frac{(\mu t - \mu x)^n}{n!} e^{-\mu t - \mu x} \nu e^{-\nu x} dx dA(t), \quad n \geq 0, \tag{4.1}$$

$$\beta = \sum_{i=0}^{\infty} \nu_i A^*(\nu)^i \Big\{ \sum_{j=0}^{\infty} b_j A^*(\nu)^j - A^*(\nu) \Big\}^{-1}. \tag{4.2}$$

Note that $A^*(\nu)$ in (4.2) is the probability that an exponential vacation (with a mean ν^{-1}) does not end during an interarrival time. ν_n, on the other hand, is the joint probability of two events. The first event is that a vacation ends sometime during an interarrival time and the second is that n customers are served during the remaining interarrival time. Substituting (4.1) into (4.2) and carrying out integrations, Tian *et al.* (1989) show that

$$\beta = \frac{\nu}{[\nu - \mu\{1 - A^*(\nu)\}]}.$$

As before, we delete the first column of **P**. Then the rest can be expressed in terms of submatrices as

$$\begin{bmatrix} \mathbf{I} & \mathbf{O}_3 & \mathbf{O}_5 \\ \mathbf{O}_1 & \mathbf{P}_1 & \mathbf{P}_2 \\ \mathbf{O}_2 & \mathbf{O}_4 & \mathbf{P}_3 \end{bmatrix},$$

where **I** is an identity matrix of size $N - 1$ whose columns correspond to states $\{(n, 0), 1 \leq n \leq N - 1\}$; $\mathbf{O}_1$ through $\mathbf{O}_5$ are matrices with all entries zero; $\mathbf{P}_1$ is a diagonal matrix whose diagonal entries are all $A^*(\nu)$ and whose columns correspond to states $\{(n, 0), n \geq N\}$; and $\mathbf{P}_2$ and $\mathbf{P}_3$ are as follows:

$$\begin{array}{cc} & \begin{array}{ccccccc} 11 & 21 & \Lambda & N,1 & N+1,1 & N+2,1 & \Lambda \end{array} \\ \mathbf{P}_2 = \begin{array}{c} N-1,0 \\ N,0 \\ N+1,0 \\ M \end{array} & \begin{pmatrix} \nu_{N-1} & \nu_{N-2} & \Lambda & \nu_0 & & & \\ \nu_N & \nu_{N-1} & \Lambda & \nu_1 & \nu_0 & & \\ \nu_{N+1} & \nu_N & \Lambda & \nu_2 & \nu_1 & \nu_0 & \\ M & M & & M & M & M & O \end{pmatrix} \end{array}$$

$$
\mathbf{P}_3 = \begin{array}{c} \\ 11 \\ 11 \\ 31 \\ M \end{array} \begin{array}{c} \begin{array}{ccccc} 11 & 21 & 31 & 41 & \Lambda \end{array} \\ \left(\begin{array}{ccccc} b_1 & b_0 & & & \\ b_2 & b_1 & b_0 & & \\ b_3 & b_2 & b_1 & b_0 & \\ M & M & M & M & O \end{array} \right) \end{array}. \tag{4.3}
$$

4.2. Solving $\mathbf{\Pi} = \mathbf{\Pi P}$

Let $\mathbf{\Pi}$ denote the vector $(\pi_{00}, \pi_{10}, \pi_{20}, \Lambda, \pi_{11}, \pi_{21}, \pi_{31}, \Lambda)$. Then solving the balance equations corresponding to $\mathbf{I}$ and $\mathbf{P}_1$, we have

$$
\pi_{n0} = \begin{cases} \pi_{00}, & 1 \le n \le N-1, \quad (4.4.a) \\ \pi_{00} A^*(\nu)^{n-N+1}, & n \ge N. \quad (4.4.b) \end{cases}
$$

The rest of the equations, corresponding to $\mathbf{P}_2$ and $\mathbf{P}_3$, are as follows:

$$
\pi_{11} = \sum_{i=N-1}^{\infty} \pi_{i0}\nu_i + \sum_{j=1}^{\infty} \pi_{j1} b_j, \tag{4.5}
$$

$$
\pi_{n1} = \begin{cases} \displaystyle\sum_{i=N-1}^{\infty} \pi_{i0}\nu_{i-n+1} + \sum_{j=n-1}^{\infty} \pi_{j1} b_{j-n+1}, & 2 \le n \le N, \quad (4.6) \\ \displaystyle\sum_{i=n-1}^{\infty} \pi_{i0}\nu_{i-n+1} + \sum_{j=n-1}^{\infty} \pi_{j1} b_{j-n+1}, & n \ge N+1. \quad (4.7) \end{cases}
$$

The trial solution we use is of the form

$$
\pi_{n1} = \pi_{00}\{D_N r^{n-N+1} - B_N A^*(\nu)^{n-N+1}\}, \quad n \ge N. \tag{4.8}
$$

The conditions that (4.8) should satisfy are obtained as follows. First, substituting (4.8) into (4.7) together with (2.6) and (4.2), we obtain

$$
B_N = \beta. \tag{4.9}
$$

Next, we substitute (4.8) into (4.6), together with (2.6) and (4.2), in the reversed order of $n = N, N-1, \Lambda, 3, 2$. From the case $n = N$, we obtain

$$
\pi_{N-1,1} = \pi_{00}\{D_N - B_N\}. \tag{4.10}
$$

Then combining (4.8) and (4.10), we have

$$
\pi_{n1} = \pi_{00}\{D_N r^{n-N+1} - B_N A^*(\nu)^{n-N+1}\}, \quad n \ge N-1. \tag{4.11}
$$

Now, for convenience, we define K_n, $n \geq 2$, as follows:

$$b_0 K_2 = \frac{\nu_0}{A^*(\nu)}, \tag{4.12}$$

$$b_0 K_n = \sum_{i=0}^{n-2} \nu_i A^*(\nu)^{i-n+1} + K_{n-1} - \sum_{j=2}^{n-1} K_j b_{n-j}, \quad n \geq 3. \tag{4.13}$$

Then from the cases $n = N-1, N-2, \Lambda, 3, 2$, we obtain the following results in a recursive fashion.

$$\pi_{n1} = \pi_{00}\{D_N r^{n-N+1} - B_N A^*(\nu)^{n-N+1} + K_{N-n}\}, \quad 1 \leq n \leq N-2. \tag{4.14}$$

Finally, substituting (2.6), (4.9), (4.11), (4.14) into (4.5), we obtain

$$D_N = r^{N-1}\{\beta A^*(\nu)^{1-N} - K_N\}. \tag{4.15}$$

Thus, putting all these together, we have

$$\frac{\pi_{n1}}{\pi_{00}} = \begin{cases} \{\beta A^*(\nu)^{1-N} - K_N\}r^n - \beta A^*(\nu)^{n-N+1} + K_{N-n}, & 1 \leq n \leq N-2, \quad (4.16.a) \\ \{\beta A^*(\nu)^{1-N} - K_N\}r^n - \beta A^*(\nu)^{n-N+1}, & n \geq N-1, \quad (4.16.b) \end{cases}$$

where r, β, K_n, $2 \leq n \leq N$, are as given in (2.6), (4.2), (4.12) and (4.13).

We make some remarks on (4.16) and (4.4). If we separate the case $N = 2$ when deriving (4.16), we will end up with (4.16.b) but without (4.16.a). Moreover, (4.16.b) is valid for the case $N = 1$ if we let K_1 be zero. In addition, (4.4.a) no longer exists when $N = 1$. Finally, π_{00} in (4.4) and (4.16) can be obtained by substituting (4.4) and (4.16) into the normalization condition, $\sum_{n=0}^{\infty} \pi_{n0} + \sum_{n=1}^{\infty} \pi_{n1} = 1$.

4.3. A regenerative process approach to the N-policy GI/M/1 queue with EMV

For the N-policy $GI/M/1$ queue with EMV where $N \geq 2$, (2.7.a), (2.8), (2.10), (2.11) are still valid but either (2.7.b) or (3.17) should be modified as

$$\pi_{n2} = \pi_{n0} + \pi_{n1}, \quad n \geq 1. \tag{4.17}$$

Concerning p_{n0}, (2.9.a) and (3.18) are still valid. But we additionally have

$$p_{n0} = \lambda \pi_{n-1,0} E[\min(A, V)] = \lambda \pi_{n-1,0}\{1 - A^*(\nu)\}\nu^{-1}, \quad n \geq N. \tag{4.18}$$

Note that $E[\min(A, V)]$ is of the same form as $E[\min(R_n, S)]$ and $E[\min(A, S)]$ in (2.11) (see Appendix).

Concerning p_{n1}, (3.19.a) and (3.19.b) are still valid, but instead of (3.19.c) and (3.19.d) we now have

$$\begin{aligned} p_{n1} &= \lambda_2 \pi_{n2} E[\min(R_n, S)] + \lambda \pi_{n-1,1} E[\min(A, S)] \\ &\quad + \lambda \pi_{n-1,0} \{1 - A^*(\nu) - \nu_0\} \mu^{-1}, \quad n \geq N, \end{aligned} \tag{4.19}$$

where ν_0 is as defined in (4.1). Among the three terms at the right hand side of (4.19), the first two can be interpreted the same way as before and the last will be interpreted in Appendix.

A remark on the case $N = 1$ is as follows. For p_{00}, (2.9.a) is valid; for p_{n0}, $n \geq 1$, (4.18) is valid; for p_{n1}, $n \geq 2$, (4.19) is valid; and for p_{11}, (4.19) is valid if we delete the middle term at the right hand side.

Now we make conjectures that the relations (3.21) and (3.22) hold for the N-policy $GI/M/1$ queue with EMV. Then, substituting (2.8), (2.10), (2.11), (3.21), (3.22), (4.4), (4.17) into (4.19) and solving for π_{n1}, we have

$$\begin{aligned} \pi_{n1} &= \pi_{N-1,1} r^{n-N+1} + \pi_{N-1,0} \Big[\frac{r}{b_0} \{A^*(\nu) - \nu_0\} - A^*(\nu) \Big] \\ &\quad \times \frac{\{r^{n-N+1} - A^*(\nu)^{n-N+1}\}}{r - A^*(\nu)} \\ &= \pi_{00} \Big\{ \Big(\frac{\pi_{N-1,1}}{\pi_{00}} + B_N \Big) r^{n-N+1} - B_N A^*(\nu)^{n-N+1} \Big\}, \quad n \geq N, \end{aligned} \tag{4.20}$$

where

$$B_N = \Big[\frac{r\{A^*(\nu) - \nu_0\}}{b_0} - A^*(\nu) \Big] \{r - A^*(\nu)\}^{-1}. \tag{4.21}$$

The trial solution, (4.8), is based on (4.20). Note that D_N in (4.8) corresponds to $(\pi_{N-1,1}/\pi_{00}) + B_N$.

4.4. FIFO sojourn time

FIFO sojourn time LST for the N-policy $GI/M/1$ queue with EMV, denoted by $W^*(\theta)$, can be obtained as follows:

$$\begin{aligned} W^*(\theta) &= \sum_{n=0}^{N-2} \pi_{n0} A^*(\theta)^{N-1-n} V^*(\theta) S^*(\theta)^{n+1} \\ &\quad + \sum_{n=N-1}^{\infty} \pi_{n0} V^*(\theta) S^*(\theta)^{n+1} + \sum_{n=1}^{\infty} \pi_{n1} S^*(\theta)^{n+1} \end{aligned} \tag{4.22}$$

where $V^*(\theta)$ is the LST of V_n. Note that $V^*(\theta) = \nu/(\theta + \nu)$.

5. Concluding Remarks

As demonstrated with N-policy $GI/M/1$ queues, with and without EMV, the trial solution method we presented in this paper is simple and straightforward once we have a proper trial solution. Moreover, the regenerative process approach we presented helps us not only to find a trial solution but to better understand the underlying process.

Originally we intended to present a simple tool just for analyzing $GI/M/1$ queues with generalized vacations. But now we suspect that the tool could be used for some other types of $GI/M/1$ queues and even for some types of $GI/M/c$ queues as well. Interested readers may try our method for the $GI/M/c$ queue with EMV and for the $GI/M/c$ queue with phase-type multiple vacations, which are recently investigated by Chao and Zhao (1998) and Tian and Zhang (2003), respectively.

Appendix : Derivation of (2.11), (4.18), (4.19)

Suppose X and Y are non-negative valued independent random variables having general distribution X_t, $t \geq 0$, and exponential distribution $Y(s) = 1 - e^{-\gamma s}$, $s \geq 0$, respectively. Then we have

$$\begin{aligned} E[\min(X,Y)] &= \int_0^\infty \Big(\int_0^t s\gamma e^{-\gamma s} ds + \int_t^\infty t\gamma e^{-\gamma s} ds \Big) dX(t) \\ &= \int_0^\infty \Big[\Big\{ \frac{1}{\gamma}(1 - e^{-\gamma t}) - te^{-\gamma t} \Big\} + te^{-\gamma t} \Big] dX(t) \\ &= \frac{1}{\gamma} \Big\{ \int_0^\infty dX(t) - \int_0^\infty e^{-\gamma t} dX(t) \Big\} \\ &= \frac{1}{\gamma}\{1 - X^*(\gamma)\}. \end{aligned} \tag{A.1}$$

Note that $E[\min(R_n, S)]$ and $E[\min(A, S)]$ in (2.11) and $E[\min(A, V)]$ in (4.18) are all of the same from as (A.1). Thus, we only need to show that two random variables in each pair are independent and that one of each pair has an exponential distribution.

In the pair (R_n, S), R_n is partly determined by the past service times but it is independent of a future service time S. And this S has an exponential distribution.

In the pair (A, S) ((A, V)), S (V) originally implies the remaining service (vacation) time at an epoch a customer arrives. Due to the memoryless property of the exponential service (vacation) time, however, the remaining service (vacation) time can be replaced by a new service (vacation) time S (V). And this S (V) is independent of an interarrival time A.

In the last term of (4.19), $\lambda\pi_{n-1,0}$ is interpreted as the expected number, per unit time, of arriving customers who find the server taking a vacation and see $n-1$ customers already waiting in the system when they arrive.

Each time a customer arrives at the system seeing $n-1$ customers who are waiting for the vacationing server, the expected time duration until either the next arrival or the end of the ongoing vacation (whichever occurs first) is $E[\min(A,V)]$, due to the memoryless property of the exponential vacation time. Note that this $E[\min(A,V)]$ is what we have in (4.18).

Suppose the ongoing vacation ends before the next arrival, then a busy period begins in the presence of n customers. We now show that $\{1-A^*(\nu)-\nu_0\}/\mu$ in (4.19) is the joint quantity of (i) and (ii), where (i) is the probability that the ongoing vacation ends before the next arrival and (ii) is the expected time duration from the instant a busy period begins in the presence of n customers to either the instant the next customer arrives or the instant the first service in the busy period ends (whichever comes first) as follows:

$$\begin{aligned}
&\int_0^\infty \Big[0\cdot Pr(V>t)+\int_0^t\Big\{\int_0^{t-x} s\mu e^{-\mu s}ds \\
&\qquad\qquad +\int_{t-x}^\infty (t-x)\mu e^{-\mu s}ds\Big\}\nu e^{-\nu x}dx\Big]dA(t) \\
&=\int_0^\infty\int_0^t\Big[\Big\{\frac{1}{\mu}(1-e^{-\mu(t-x)})-(t-x)e^{-\mu(t-x)}\Big\} \\
&\qquad\qquad +(t-x)e^{-\mu(t-x)}\Big]\nu e^{-\nu x}dxdA(t) \\
&=\frac{1}{\mu}\Big\{\int_0^\infty\int_0^t \nu e^{-\nu x}dxdA(t)-\int_0^\infty\int_0^t e^{-\mu(t-x)}\nu e^{-\nu x}dxdA(t)\Big\} \\
&=\frac{1}{\mu}\Big\{\int_0^\infty(1-e^{-\nu t})dA(t)-\nu_0\Big\} \\
&=\frac{1}{\mu}\{1-A^*(\nu)-\nu_0\}.
\end{aligned}$$

Note that we made use of the definition of ν_0 by substituting $n=0$ into (4.1).

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