Journal of the Korean Statistical Society (2004), **33**: *3, pp 271–282*

ON COMPLETE CONVERGENCE OF WEIGHTED SUMS OF ϕ-MIXING RANDOM VARIABLES WITH APPLICATION TO MOVING AVERAGE PROCESSES[†]

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ABSTRACT

We discuss complete convergence of weighted sums for arrays of ϕ-mixing random variables. As application, we obtain the complete convergence of moving average processes for ϕ-mixing random variables. The result of Baum and Katz (1965) as well as the result of Li *et al.* (1992) on *iid* case are extended to ϕ-mixing setting.

AMS 2000 subject classifications. Primary 60G50; Secondary 60F15.
Keywords. Complete convergence, ϕ-mixing sequence, moving average processes.

1. INTRODUCTION

Let $\{X_n|\ n = 1, 2, \dots\}$ be a sequence of random variables. Hsu and Robbins (1947) introduced the concept of complete convergence of $\{X_n\}$. A sequence $\{X_n\}$ of random variables is said to converge completely to a constant c if

$$\sum_{n=1}^{\infty} P(|X_n - c| > \varepsilon) < \infty \quad \text{for all} \quad \varepsilon > 0.$$

Moreover, it was proved that the sequence of arithmetic means of independent identically distributed (*iid*) random variables converges completely to the expected value if the variance of the summands is finite.

Received October 2003; accepted March 2004.
[†]This paper was supported by grant NO. 2001-042-D00008 from Korea Research Foundation.
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This result has been generalized and extended in several directions and carefully studied by many authors (see Pruitt, 1966; Rohatgi, 1971; Hu *et al.*, 1989; Gut, 1992; Wang *et al.*, 1993; Kuczmaszewska and Szynal, 1994; Hu *et al.*, 1999; Ahmed *et al.*, 2002). Li *et al.* (1992) also had obtained the result on complete convergence in *iid* random variables.

The concept of ϕ-mixing was introduced by Ibragimov (1962). It is well known that ϕ-mixing variables include *iid*, m-dependent, ψ-mixing variables, and ϕ-mixing variables are used in many practical questions. So, the notion of ϕ-mixing has been received considerable attention recently. We refer to Ibragimov (1962) for fundamental properties, Ibragimov and Linnik (1971) for the central limit theorem, Roussas (1988) for a nonparametric estimation, Peligrad (1985, 1993) for an invariance principle and asymptotic results, Hu (1991) for a law of the iterated logarithm, Liu *et al.* (1998) for the law of large numbers, Chen *et al.* (2003) for the law of iterated logarithm, Prakasa (2003) for the moment inequality for supremum of empirical processes. The purpose of this paper is to discuss a general conclusion for complete convergence of weighted sums for arrays of ϕ-mixing random variables. As application, we obtain that the famous result of Baum and Katz (1965) from *iid* case to ϕ-mixing array case. Also we discuss the complete convergence of moving average processes for ϕ-mixing random variables, which extends the result of Li *et al.* (1992) in *iid* setting.

This paper is organized as follows. In Section 2, we provide some lemmas, which will be used in the proof of our main theorem and in Section 3, we derive the complete convergence for arrays of $\phi-$mixing random variables under some suitable conditions. In Section 4, we obtain a result on complete convergence of moving average processes.

Finally, $a \ll b$ means $a = O(b)$ and C will represent positive constants whose value may change from one place to another, for $x \geq 0$ the symbol $[x]$ denotes the greatest integer in x, and for a finite set A the symbol $\sharp A$ denotes the number of elements in the set A.

2. PRELIMINARIES

This section will be contain some background materials which will be used in obtaining the main results in the next section.

DEFINITION 2.1 (Ibragimov, 1962). *Let* $\{X_n |\ n \geq 1\}$ *be a sequence of random variables on a probability space* $(\Omega, \mathcal{F}, P)$. *Let* $F_n^m = \sigma(X_i,\ n \leq i \leq m)$, $1 \leq$

$n \le m < \infty$. *We say that* $\{X_n|\ n \ge 1\}$ *is* ϕ*-mixing if*

$$\phi(n) = \sup_{m \in N} \sup_{\{A \in F_1^m, P(A) \neq 0, B \in F_{n+m}^\infty\}} |P(B|A) - P(B)| \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

LEMMA 2.1 (Ibragimov, 1962). *Let* $p,\ q > 1$ *with* $p^{-1} + q^{-1} = 1$. *Suppose that* ξ *and* η *are measurable with* F_1^k *and* F_{k+m}^∞. *Moreover, suppose that* $E|\xi|^p < \infty$ *and* $E|\eta|^p < \infty$. *Then*

$$|E\xi\eta - E\xi E\eta| \le 2(\phi(n))^{1/2} E^{1/p}|\xi|^p E^{1/q}|\eta|^q.$$

LEMMA 2.2 (Shao, 1993). *Let* $\{X_n|n \ge 1\}$ *be a* ϕ*-mixing sequence and* $S_n = \sum_{k=1}^n X_k,\ n \ge 1$. *If there exists a sequence* $\{C_n\}$ *of positive numbers such that*

$$\max_{1 \le i \le n} ES_i^2 \le C_n,$$

then for any $q \ge 2$, *there exists* $C = C(q, \phi(\cdot))$ *such that*

$$E \max_{1 \le i \le n} |S_i|^q \le C(C_n^{q/2} + E \max_{1 \le i \le n} |X_i|^q).$$

LEMMA 2.3 (Burton and Dehling, 1990). *Let* $\sum_{i=-\infty}^\infty a_i$ *be an absolutely convergent series of real numbers with* $a = \sum_{i=-\infty}^\infty a_i,\ b = \sum_{i=-\infty}^\infty |a_i|$. *Suppose* $\Phi : [-b, b] \to \mathbb{R}$ *is a function satisfying the following conditions:*

(i) Φ *is bounded and continuous at* a.

(ii) *There exist* $\delta > 0$ *and* $C > 0$ *such that for all* $|x| \le \delta, |\Phi(x)| \le C|x|$.

Then,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=-\infty}^{\infty} \Phi\Big(\sum_{j=i+1}^{i+n} a_j\Big) = \Phi(a).$$

REMARK 2.1. Taking $\Phi(x) = |x|^q,\ q \ge 1$, from Lemma 2.3 we have

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=-\infty}^{\infty} \Big|\sum_{j=i+1}^{i+n} a_j\Big|^q = |a|^q.$$

3. MAIN RESULT

THEOREM 3.1. *Let* $\{X_{ni}|\ i \ge 1, n \ge 1\}$ *be an array of* ϕ*-mixing random variables with* $EX_{ni} = 0$ *and* $P(|X_{ni}| > x) = O(1)P(|X| > x)$ *for all* $i \ge 1,\ n \ge 1$ *and* $x \ge 0$. *Let* $\{a_{ni}|\ i \ge 1, n \ge 1\}$ *be an array of constants such that*

$$\sup_{i>1} |a_{ni}| = O(n^{-r}) \quad \text{for some } \ r > 0 \tag{3.1}$$

and

$$\sum_{i=1}^{\infty} |a_{ni}| = O(n^{\alpha}) \quad \textit{for some } \alpha \in [0, r). \tag{3.2}$$

Let β be such that $\alpha + \beta \neq -1$ and fix $\delta > 0$ such that $(\alpha/r) + 1 < \delta \leq 2$, and $s = \max(1 + (1 + \alpha + \beta)/r, \delta)$. If

$$E|X|^s < \infty \textit{ and } \sum_{m=1}^{\infty} \phi^{1/2}(m) < \infty, \tag{3.3}$$

then

$$\sum_{n=1}^{\infty} n^{\beta} P\left(\left|\sum_{i=1}^{\infty} a_{ni} X_{ni}\right| > \varepsilon\right) < \infty \quad \textit{for all } \varepsilon > 0.$$

PROOF. Note that the result is of interest only for $\beta \geq -1$. Moreover, since $\delta > \alpha/r + 1$, we have that $\alpha - r(\delta - 1) < 0$.

We can assume, without loss of generality, that

$$\sup_{i \geq 1} |a_{ni}| = n^{-r}, \ \sum_{i=1}^{\infty} |a_{ni}| = n^{\alpha}, \ \frac{1}{|a_{ni}|} := b_{ni}, \ |a_{ni}| \leq 1,$$

and

$$I_{nk} = \{i | (nk)^r \leq |a_{ni}|^{-1} = |b_{ni}| < (n(k+1))^r\}, \ k \geq 1, \ n \geq 1. \tag{3.4}$$

Note that by (3.4) $\bigcup_{j=1} I_{nj} = N$ for all $n \geq 1$, where N is the set of positive integers. Note also that for all $j \geq 1, \ n \geq 1$,

$$\sum_{j=1}^{k} \sharp I_{nj} = \sharp\{i \in N | \ b_{ni} < (n(k+1))^r\} := \sharp A_{nk},$$

and for all $k \geq 1, \ n \geq 1$, by (3.4), we have

$$n^{\alpha} \geq \sum_{i \in A_{nk}} |a_{ni}| = \sum_{i \in A_{nk}} \frac{1}{|b_{ni}|} \geq \frac{\sharp A_{nk}}{(n(k+1))^r}$$

and hence

$$\sharp A_{nk} \leq n^{\alpha + r}(k+1)^r \text{ and } \sum_{j=1}^{k} \sharp I_{nj} = \sharp A_{nk} \leq n^{\alpha + r}(k+1)^r. \tag{3.5}$$

Using $EX_{ni} = 0$, we get that

$$\begin{aligned}\sum_{i=1}^{\infty} |Ea_{ni}X_{ni}I(|a_{ni}X_{ni}| \le 1)| &= \sum_{i=1}^{\infty} E|a_{ni}X_{ni}|^{\delta}|a_{ni}X_{ni}|^{1-\delta}I(|a_{ni}X_{ni}| > 1) \\ &\le \sum_{i=1}^{\infty} E|a_{ni}X_{ni}|^{\delta}I(|a_{ni}X_{ni}| > 1) \\ &= \sum_{i=1}^{\infty} |a_{ni}|^{\delta-1}|a_{ni}|E|X_{ni}|^{\delta}I(|a_{ni}X_{ni}| > 1) \\ &\ll \sup_{i\ge1} |a_{ni}|^{\delta-1}\sum_{i=1}^{\infty} |a_{ni}|E|X|^{\delta}I(|X| > 1) \\ &\ll n^{\alpha-r(\delta-1)}E|X|^{\delta} = o(1) \quad as\ n \to \infty. \qquad (3.6)\end{aligned}$$

Note that

$$\begin{aligned}&\sum_{n=1}^{\infty} n^{\beta}P\left(\left|\sum_{n=1}^{\infty} a_{ni}X_{ni}\right| > \varepsilon\right) \\ &\le \sum_{n=1}^{\infty} n^{\beta}P\left(\sup_{i\ge1} |a_{ni}X_{ni}| > 1\right) + \sum_{n=1}^{\infty} n^{\beta}P\left(|\sum_{i=1}^{\infty} a_{ni}X_{ni}I(|a_{ni}X_{ni}| \le 1)| \ge \frac{\varepsilon}{2}\right) \\ &:= I_1 + I_2.\end{aligned}$$

Hence, it suffices to show that $I_1 < \infty$ and $I_2 < \infty$.

To prove $I_1 < \infty$, using (3.5), we get

$$\begin{aligned}I_1 &\le \sum_{n=1}^{\infty} n^{\beta}\sum_{i=1}^{\infty} P(|a_{ni}X_{ni}| > 1) \\ &= \sum_{n=1}^{\infty} n^{\beta}\sum_{i=1}^{\infty} P(|X_{ni}| > b_{ni}) \\ &\ll \sum_{n=1}^{\infty} n^{\beta}\sum_{i=1}^{\infty} P(|X| \ge b_{ni}) \\ &\le \sum_{n=1}^{\infty} n^{\beta}\sum_{j=1}^{\infty} (\sharp I_{nj})P(|X| \ge (nj)^r) \\ &\le \sum_{n=1}^{\infty} n^{\beta}\sum_{j=1}^{\infty} (\sharp I_{nj}) \sum_{k=nj}^{\infty} P(k \le |X|^{1/r} < k+1) \\ &\le \sum_{n=1}^{\infty} n^{\beta}\sum_{k=n}^{\infty} P(k \le |X|^{1/r} < k+1)n^{\alpha+r}\left(\left[\frac{k}{n}\right] + 1\right)^r\end{aligned}$$

$$\begin{aligned}
&\ll \sum_{n=1}^{\infty} n^{\beta} n^{\alpha+r} n^{-r} \sum_{k=n}^{\infty} k^{r} P(k \le |X|^{1/r} < k+1) \\
&\le \sum_{k=1}^{\infty} k^{r} P(k \le |X|^{1/r} < k+1) \sum_{n=1}^{k} n^{\alpha+\beta} \\
&\ll \sum_{k=1}^{\infty} k^{\alpha+\beta+r+1} P(k \le |X|^{1/r} < k+1) \\
&\ll E|X|^{1+(\alpha+\beta+1)/r} < \infty \quad \text{by (3.3).}
\end{aligned}$$

To prove $I_2 < \infty$, we need only to prove from (3.6) that for all $\varepsilon > 0$,

$$I_2^* := \sum_{n=1}^{\infty} n^{\beta} P\left(\left| \sum_{i=1}^{\infty} a_{ni} X_{ni} I(|a_{ni} X_{ni}| \le 1) - E a_{ni} X_{ni} I(|a_{ni} X_{ni}| \le 1) \right| \ge \varepsilon \right) < \infty.$$

Note that, by writing $\xi_{ni} = a_{ni} X_{ni} I(|a_{ni} X_{ni}| \le 1)$, we obtain from Lemma 2.1 and $\sum_{m=1}^{\infty} \phi^{1/2}(m) < \infty$ that

$$\begin{aligned}
&\sup_{1 \le m \le \infty} E \left\{ \sum_{i=1}^{m} (a_{ni} X_{ni} I(|a_{ni} X_{ni}| \le 1)) - E \sum_{i=1}^{m} (a_{ni} X_{ni} I(|a_{ni} X_{ni}| \le 1)) \right\}^2 \\
&= \sup_{1 \le k \le \infty} \left[\sum_{i=1}^{m} E(\xi_{ni} - E\xi_{ni})^2 + 2 \sum_{s=1}^{m-1} \sum_{t=s+1}^{m} E\{(\xi_{ns} - E\xi_{ns})(\xi_{nt} - E\xi_{nt})\} \right] \\
&\le \sup_{1 \le k \le \infty} \left\{ \sum_{i=1}^{m} E(\xi_{ni} - E\xi_{ni})^2 \right. \\
&\qquad \left. + 2 \sum_{s=1}^{m-1} \sum_{t=s+1}^{m} 2\phi^{1/2}(t-s) E^{1/2}(\xi_{ns} - E\xi_{ns})^2 E^{1/2}(\xi_{nt} - E\xi_{nt})^2 \right\} \\
&= \sup_{1 \le k \le \infty} \left\{ \sum_{i=1}^{m} E(\xi_{ni} - E\xi_{ni})^2 \right. \\
&\qquad \left. + 4 \sum_{l=1}^{m} \phi^{1/2}(l) \sum_{s=1}^{m-l} E^{1/2}(\xi_{ns} - E\xi_{ns})^2 E^{1/2}(\xi_{n(l+s)} - E\xi_{n(l+s)})^2 \right\} \\
&\le \sup_{1 \le k \le \infty} \left[\sum_{i=1}^{m} E(\xi_{ni} - E\xi_{ni})^2 \right. \\
&\qquad \left. + 4 \sum_{l=1}^{m} \phi^{1/2}(l) \left\{ \sum_{s=1}^{m-l} E(\xi_{ns} - E\xi_{ns})^2 \right\}^{1/2} \left\{ \sum_{s=1}^{m-l} E(\xi_{n(l+s)} - E\xi_{n(l+s)})^2 \right\}^{1/2} \right]
\end{aligned}$$

$$
\begin{aligned}
&\leq \sup_{1\leq k\leq\infty}\left\{\sum_{i=1}^{m}E(\xi_{ni}-E\xi_{ni})^2+4\sum_{l=1}^{\infty}\phi^{1/2}(l)\cdot\sum_{s=1}^{m}E(\xi_{ns}-E\xi_{ns})^2\right\}\\
&\leq C\sup_{1\leq k\leq\infty}\left\{\sum_{i=1}^{m}E(\xi_{ni}-E\xi_{ni})^2\right\}\\
&\leq C\sum_{i=1}^{\infty}E(\xi_{ni})^2=C\sum_{i=1}^{\infty}E(a_{ni}X_{ni}I(|a_{ni}X_{ni}|\leq 1))^2.
\end{aligned}
$$

So, according to Lemma 2.2, we have for $M \geq 2$,

$$
\begin{aligned}
I_2^* &\ll \sum_{n=1}^{\infty}n^{\beta}E\left|\sum_{i=1}^{\infty}(a_{ni}X_{ni}I(|a_{ni}X_{ni}|\leq 1)-E(a_{ni}X_{ni}I(|a_{ni}X_{ni}|\leq 1))\right|^{M}\\
&\ll \sum_{n=1}^{\infty}n^{\beta}\left\{\left(\sum_{i=1}^{\infty}E|a_{ni}^2X_{ni}|^2I(|a_{ni}X_{ni}|\leq 1)\right)^{M/2}\right.\\
&\qquad\left.+\sum_{i=1}^{\infty}E|a_{ni}X_{ni}|^{M}I(|a_{ni}X_{ni}|\leq 1)\right\}\\
&:= I_3+I_4.
\end{aligned}
$$

In order to prove $I_3 < \infty$ and $I_4 < \infty$, we need to verify that

$$
E|a_{ni}X_{ni}I(|a_{ni}X_{ni}|\leq 1)| \ll E|a_{ni}XI(|a_{ni}X|\leq 1)|+P(|a_{ni}X|\geq 1). \tag{3.7}
$$

In fact,

$$
\begin{aligned}
&E|a_{ni}X_{ni}|I(|a_{ni}X_{ni}|\leq 1)\\
&=\int_0^1 x dP(|a_{ni}X_{ni}|\leq x)\\
&=-\int_0^1 x dP(|a_{ni}X_{ni}|\geq x)\\
&=-\left\{xP(|a_{ni}X_{ni}|\geq x)\Big|_0^1-\int_0^1 P(|a_{ni}X_{ni}|\geq x)dx\right\}\\
&=-P(|a_{ni}X_{ni}|\geq 1)+\int_0^1 P(|a_{ni}X_{ni}|\geq x)dx.
\end{aligned}\tag{3.8}
$$

Similarly,

$$
E|a_{ni}X|I(|a_{ni}X|\leq 1)=-P(|a_{ni}X|\geq 1)+\int_0^1 P(|a_{ni}X|\geq x)dx,
$$

which, together with (3.8), follows (3.7).

Next, we will use the fact: for any $\eta \geq 1$,

$$\sum_{n=1}^{\infty} n^{\beta}\Big\{\sum_{i=1}^{\infty} P(|a_{ni}X|>1)\Big\}^{\eta} \ll \sum_{n=1}^{\infty} n^{\beta}\sum_{i=1}^{\infty} P(|a_{ni}X|>1)<\infty.$$

We now verify $I_3<\infty$. Taking M large enough such that $M>(\beta+1)/(r(\delta$ $1)-\alpha)$, we have

$$\begin{aligned}
I_3 &\ll \sum_{n=1}^{\infty} n^{\beta}\Big(\sum_{i=1}^{\infty}(a_{ni}^{\delta}E|X|^{\delta}I(|a_{ni}X|\leq 1))\Big)^{M/2} \\
&\quad+\sum_{n=1}^{\infty} n^{\beta}\Big(\sum_{i=1}^{\infty} P(|a_{ni}X_{ni}|>1)\Big)^{M/2} \\
&\ll \sum_{n=1}^{\infty} n^{\beta}\Big(\sum_{i=1}^{\infty}|a_{ni}|^{\delta}E|X|^{\delta}\Big)^{M/2}+C \\
&\leq \sum_{n=1}^{\infty} n^{\beta}\Big(\sup_{i\geq 1}|a_{ni}|^{\delta-1}\sum_{i=1}^{\infty}|a_{ni}|E|X|^{\delta}\Big)^{M/2}+C \\
&\ll \sum_{n=1}^{\infty} n^{\beta}(n^{-r(\delta-1)}n^{\alpha}E|X|^{\delta})^{M/2}+C \\
&\leq \sum_{n=1}^{\infty} n^{\beta+M(\alpha-r(\delta-1))}+C<\infty.
\end{aligned}$$

As to I_4, we have

$$\begin{aligned}
I_4 &\leq \sum_{n=1}^{\infty} n^{\beta}\sum_{j=1}^{\infty}\sum_{i\in A_{nj}} E|a_{ni}X|^{M}I(|a_{ni}X|\leq 1)+\sum_{n=1}^{\infty} n^{\beta}\sum_{i=1}^{\infty} P(|a_{ni}X|>1) \\
&\leq \sum_{n=1}^{\infty} n^{\beta}\sum_{j=1}^{\infty}(\sharp I_{nj})(nj)^{-rM}E|X|^{M}I(|X|\leq (n(j+1))^{r})+C \\
&\leq \sum_{n=1}^{\infty} n^{\beta}\sum_{j=1}^{\infty}(\sharp I_{nj})(nj)^{-rM}\sum_{0\leq k\leq (n(j+1))} E|X|^{M}I(k\leq |X|^{1/r}<k+1)+C \\
&\leq \sum_{n=1}^{\infty} n^{\beta}\sum_{j=1}^{\infty}(\sharp I_{nj})(nj)^{-rM}\sum_{k=0}^{2n} E|X|^{M}I(k\leq |X|^{1/r}<k+1)+C \\
&\quad+\sum_{n=1}^{\infty} n^{\beta}\sum_{j=1}^{\infty}(\sharp I_{nj})(nj)^{-rM}\sum_{k=2n+1}^{n(j+1)} E|X|^{M}I(k\leq |X|^{1/r}<k+1)+C \\
&:= I_5+I_6+C.
\end{aligned}$$

Note that for all $m \geq 1$, we have

$$n^\alpha = \sum_{i=1}^{\infty} |a_{ni}^+| = \sum_{j=1}^{\infty} \sum_{i \in I_{nj}} |a_{ni}| \geq \sum_{j=1}^{\infty} \sharp I_{nj} (n(j+1))^{-r}$$
$$\geq \sum_{j=m}^{\infty} (\sharp I_{nj})(n(j+1))^{-r} \geq \sum_{j=m}^{\infty} (\sharp I_{nj})(n(j+1))^{-rM} (n(j+1))^{r(M-1)}.$$

So,

$$\sum_{j=m}^{\infty} (\sharp I_{nj})(nj)^{-rM} \ll n^{\alpha - r(M-1)} m^{-r(M-1)}.$$

Hence, by choosing $M > 1 + (\alpha + \beta + 1)/r$, we have

$$I_5 \ll \sum_{n=1}^{\infty} n^\beta n^{\alpha - r(M-1)} \sum_{k=0}^{2n} E|X|^M I(k \leq |X|^{1/r} < k+1)$$
$$\ll \sum_{k=1}^{\infty} \sum_{n=[k/2]}^{\infty} n^{\beta + \alpha - r(M-1)} E|X|^M I(k \leq |X|^{1/r} < k+1)$$
$$\ll \sum_{k=1}^{\infty} k^{\beta + \alpha + 1 - r(M-1)} E|X|^M I(k \leq |X|^{1/r} < k+1)$$
$$\ll E|X|^{1 + (\alpha + \beta + 1)/r} < \infty.$$

$$I_6 \ll \sum_{n=1}^{\infty} n^\beta \sum_{k=2n+1}^{\infty} \sum_{j \geq k/n - 1} (\sharp I_{nj})(nj)^{-rM} E|X|^M I(k \leq |X|^{1/r} < k+1)$$
$$\ll \sum_{n=1}^{\infty} n^\beta \sum_{k=2n+1}^{\infty} n^{\alpha - r(M-1)} \left(\frac{k}{n}\right)^{-r(M-1)} E|X|^M I(k \leq |X|^{1/r} < k+1)$$
$$= \sum_{n=1}^{\infty} n^{\beta + \alpha} \sum_{k=2n+1}^{\infty} k^{-r(M-1)} E|X|^M I(k \leq |X|^{1/r} < k+1)$$
$$\ll \sum_{k=2}^{\infty} \sum_{n=1}^{[k/2]} n^{\beta + \alpha} k^{-r(M-1)} E|X|^M I(k \leq |X|^{1/r} < k+1)$$
$$\ll \sum_{k=2}^{\infty} k^{\beta + \alpha + 1 - r(M-1)} E|X|^M I(k \leq |X|^{1/r} < k+1)$$
$$\ll E|X|^{1 + (\beta + \alpha + 1)/r} < \infty.$$

□

Let $1 \leq t < 2$, $p > 1$. As application, in Theorem 3.1, taking $\beta = p - 2$, $a_{ni} = n^{1/t}$ for $1 \leq i \leq n$ and $a_{ni} = 0$ for $i > n$, then we obtain the following corollary.

COROLLARY 3.1. *Let $1 \leq t < 2$, $p > 1$, and let $\{X_{ni} \,|\, 1 \leq i \leq n,\ n \geq 1\}$ be an array of ϕ-mixing random variables with $EX_{ni} = 0$ and $P(|X_{ni}| > x) = O(1)P(|X| > x)$ for all $i \geq 1$, $n \geq 1$ and $x \geq 0$. If $E|X|^{pt} < \infty$ and $\sum_{m=1}^{\infty} \phi^{1/2}(m) < \infty$, then*

$$\sum_{n=1}^{\infty} n^{p-2} P\left(\left|\sum_{i=1}^{n} X_{ni}\right| > \varepsilon n^{1/t}\right) < \infty \quad \textit{for all } \varepsilon > 0.$$

REMARK 3.1. When $X_{ni} = X_i$ for $1 \leq i \leq n$ and $\{X_i \,|\, i \geq 1\}$ is a sequence of *iid* random variables, Corollary 3.1 becomes into the famous result of Baum and Katz (1965).

4. COMPLETE CONVERGENCE OF MOVING AVERAGE PROCESSES

In this section, we present one result about the convergence of moving average processes which follows from Theorem 3.1. The result is the extension on the ϕ-mixing setting of the result of Li *et al.* (1992) in *iid* case.

THEOREM 4.1. *Assume that $\{Y_i | -\infty < i < \infty\}$ is a sequence of ϕ-mixing random variables with $EY_i = 0$ and $P(|Y_i| > x) = O(1)P(|Y| > x)$ for all $i \geq 1$ and $x \geq 0$. Let $\{a_i| -\infty < i < \infty\}$ be an absolutely summable sequence of real numbers and $X_k = \sum_{i=-\infty}^{\infty} a_{i+k} Y_i$, $k \geq 1$. If $E|X|^{(\beta+2)t} < \infty$, where $1 \leq t < 2$, $(\beta+2)t \neq 1$, $\beta > -1$, and $\sum_{m=1}^{\infty} \phi^{1/2}(m) < \infty$, then*

$$\sum_{n=1}^{\infty} n^{\beta} P(|\sum_{k=1}^{n} X_k| \geq \varepsilon n^{1/t}) < \infty \quad \textit{for all } \varepsilon \geq 0.$$

PROOF. Let $X_{ni} = Y_i$ and $a_{ni} = (1/n^{1/t}) \sum_{k=1}^{n} a_{i+k}$ for all $n \geq 1$, $i \geq 1$. Then $(1/n^{1/t}) \sum_{k=1}^{n} X_k = \sum_{i=-\infty}^{\infty} a_{ni} X_{ni}$. By Lemma 2.3, since $a = \sum_{i=-\infty}^{\infty} a_i$ and $b = \sum_{i=-\infty}^{\infty} |a_i|$, we have that $\sup_{i \geq 1} |a_{ni}| \leq bn^{-1/t}$ and $\sum_{i=-\infty}^{\infty} |a_{ni}| \leq C|n|^{1-1/t}$. The result follows by Theorem 3.1 with $\alpha = 1 - 1/t$, $r = 1/t$ and $t < \delta < 2$. □

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