Properties of a Generalized Impulse Response Gramian with Application to Model Reduction

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Abstract: In this paper we investigate the properties of a generalized impulse response Gramian. The recursive relationship satisfied by the family of Gramians is established. It is shown that the generalized impulse response Gramian contains information on the characteristic polynomial of a linear time-invariant continuous system. The results are applied to model reduction problem.

Keywords: Generalized impulse response Gramian, Lyapunov equation, Markov parameter, model reduction, time-moment.

1. INTRODUCTION

In identification or model reduction problems, an important task is the computation of the characteristic polynomial of the original or reduced-order system. Several literature have shown that the characteristic polynomial of a system can be extracted from the information generated by the impulse response data. For continuous systems, the Gram matrix [1,2] and the impulse response Gramian [3] are good examples of information from which the characteristic polynomial of a system can be obtained. For discrete systems, the Hankel matrix [4] and the impulse response Gramian [5] possess the same properties. Recently a new impulse response Gramian was introduced in [6] that can also be utilized to compute the characteristic polynomial of a discrete system. In addition to computing the characteristic polynomial, those impulse response data are useful for the order reduction of linear time-invariant systems [6-10].

In this paper we investigate the properties of a generalized impulse response Gramian [11], which includes the Gram matrix of [1,2] and the impulse response Gramian in [3] as special cases. The recursive relationship satisfied by the family of Gramians is established. It is also shown that the generalized impulse response Gramian contains information on the characteristic polynomial of a linear time-invariant continuous system. The results are applied to model reduction problem.

This paper is organized as follows. In Section 2, some preliminaries are presented. The properties of a generalized impulse response Gramian are studied in Section 3. An application to a model reduction problem is considered in Section 4 and the paper is concluded in Section 5.

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2. PRELIMINARIES

2.1. Canonical realizations

Consider a stable nth-order linear time-invariant system described by the transfer function

$$H(s)=\frac{b_1 s^{n-1}+b_2 s^{n-2}+\cdots+b_{n-1}s+b_n}{s^n+a_1 s^{n-1}+\cdots+a_{n-1}s+a_n} \tag{1}$$

or by the minimal state-space realization

$$\dot{\mathbf{x}}(t)=A\mathbf{x}(t)+br(t), \tag{2}$$

$$y(t)=c\mathbf{x}(t), \tag{3}$$

where $\mathbf{x}(t)\in\Re^n$. The transfer function $H(s)$ can be expanded into the following two forms

$$H(s)=-t_1-t_2 s-t_3 s^2-t_4 s^3-\cdots, \tag{4}$$

$$H(s)=\frac{m_1}{s}+\frac{m_2}{s^2}+\frac{m_3}{s^3}+\cdots, \tag{5}$$

where t_i s and m_i s respectively denote the time-moments and Markov parameters of the system, and are computed from the coefficients of $H(s)$ or from the system matrices as follows:

$$t_i = -\frac{1}{a_n}(b_{n-i+1} + \sum_{j=1}^{i-1} a_{n-i+j} t_j), \quad (6)$$

$$m_i = b_i - \sum_{j=1}^{i-1} a_{i-j} m_j, \quad (7)$$

with $a_0 = 1$, $a_i = 0$ for $i < 0$, $b_i = 0$ for $i \le 0$ in (6), and $a_i = b_i = 0$ for $i > n$ in (7), or

$$t_i = cA^{-i}b, \quad (8)$$

$$m_i = cA^{i-1}b. \quad (9)$$

Let C be the standard controllability matrix for the realization (A,b,c) given in (2) and (3). As is well known, the system provided in (2) and (3) can be transformed to the following controllability canonical form by the similarity transform [12]

$$\dot{\mathbf{x}}(t) = \hat{A}\mathbf{x}(t) + \hat{b}r(t), \quad (10)$$

$$y(t) = \hat{c}\mathbf{x}(t), \quad (11)$$

where

$$\hat{A} = C^{-1}AC = \begin{bmatrix} 0 & \cdots & 0 & -a_n \\ 1 & \cdots & 0 & -a_{n-1} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & -a_1 \end{bmatrix}, \quad (12)$$

$$\hat{b} = C^{-1}b = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}^T, \quad (13)$$

$$\hat{c} = cC = \begin{bmatrix} m_1 & m_2 & \cdots & m_n \end{bmatrix}. \quad (14)$$

For each $u \in \mathrm{N}$, let $b_u = \hat{A}^{-u}\hat{b}$ and $c_u = \hat{c}\hat{A}^u$. Then we have a class of canonical realizations $\{ (\hat{A}, b_u, c_u), u \in \mathrm{N} \}$ of $H(s)$. It is easily seen that, for $0 \le u \le n-1$, b_{-u} takes the form such that all elements are zero except for the $(u+1)$th element which is equal to one, and $b_{-n} = \begin{bmatrix} -a_n & \cdots & -a_1 \end{bmatrix}^T$.

Conversely, from the Cayley-Hamilton theorem, we have

$$cA^i b = -a_n cA^{i-n}b - a_{n-1}cA^{i-n+1}b - \cdots - a_1 cA^{i-1}b. \quad (15)$$

Then, using (8), (9) and (15), it can be shown that c_u s take the following form:

(i) For $u \ge 0$: $c_u = \begin{bmatrix} m_{u+1} & m_{u+2} & \cdots & m_{u+n} \end{bmatrix}$.

(ii) For $-(n-1) \le u \le -1$:

$$c_u = \begin{bmatrix} t_{-u} & \cdots & t_1 & m_1 & \cdots & m_{n+u} \end{bmatrix}.$$

(iii) For $u \le -n$: $c_u = \begin{bmatrix} t_{-u} & t_{-u-1} & \cdots & t_{-u-n+1} \end{bmatrix}$.

2.2. Generalized impulse response Gramian

For the system described in (2) and (3), the impulse response is given by

$$h(t) = ce^{At}b. \quad (16)$$

For $i \ge 0$, recursively define [1-3]

$$h_{-(i+1)}(t) = \int_{\infty}^{t} h_{-i}(\alpha)d\alpha, \quad (17)$$

$$h_{i+1}(t) = \frac{d}{dt}h_i(t), \quad (18)$$

with $h_0(t) = h(t)$.

Definition: For each $u \in \mathrm{N}$, the nth-order generalized impulse response Gramian $G_{u,n}$ is defined by

$$G_{u,n} = \int_0^\infty \begin{bmatrix} h_u^2(t) & h_u(t)h_{u+1}(t) & \cdots & h_u(t)h_{u+n-1}(t) \\ h_u(t)h_{u+1}(t) & h_{u+1}^2(t) & \cdots & h_{u+1}(t)h_{u+n-1}(t) \\ \vdots & \vdots & \vdots & \vdots \\ h_u(t)h_{u+n-1}(t) & h_{u+1}(t)h_{u+n-1}(t) & \cdots & h_{u+n-1}^2(t) \end{bmatrix} dt. \quad (19)$$

The $(n+1)$th-order generalized impulse response Gramian $G_{u,n+1}$ is defined similarly. The Gramian of the form given in (19) was first introduced in [11] in relation to the model reduction problem. However, detailed analysis was not performed on the properties of the Gramian. Note that $G_{-(n-1),n}$ corresponds to the Gram matrix of [1,2] and $G_{0,n}$ is the impulse response Gramian defined in [7]. In particular, $G_{-(n-1),n+1}$ and $G_{0,n+1}$ respectively denote the characteristic Gram matrix [1,2] and the characteristic impulse response Gramian [3] of the system.

3. MAIN RESULTS

In this section some important properties of the generalized impulse response Gramian are studied.

Lemma 1 [11]: For each $u \in \mathrm{N}$ and the realization $(\hat{A}, b_u, c_u)$, the nth-order generalized impulse response Gramian $G_{u,n}$ is a unique positive definite solution to the Lyapunov equation

$$\hat{A}^T G_{u,n} + G_{u,n}\hat{A} = -c_u^T c_u. \quad (20)$$

In some applications such as model reduction, it is necessary to compute the generalized impulse response Gramian $G_{u,n}$ for different values of u, which requires the repeated solving of (20). However, it is sufficient to solve (20) only once for some u as indicated in the following lemma.

Lemma 2: For each n, we have

$$G_{u+1,n} = \hat{A}^T G_{u,n}\hat{A}, \quad u \in \mathrm{N}. \quad (21)$$

Proof: Premultiplying $\hat{A}^T$ and postmultiplying $\hat{A}$ on both sides of (20), we have

$$\begin{aligned}\hat{A}^T(\hat{A}^T G_{u,n}\hat{A}) + (\hat{A}^T G_{u,n}\hat{A})\hat{A} &= -\hat{A}^T c_u^T c_u \hat{A} \\ &= -c_{u+1}^T c_{u+1}.\end{aligned} \quad (22)$$

Since each $G_{u,n}$ is a unique solution to (20), the result follows. □

In [1-3], it was indicated that the Gram matrix and the impulse response Gramian contain information on the characteristic polynomial of linear time-invariant continuous systems. It can be demonstrated that the generalized impulse response Gramian possesses the same property.

Theorem 1: For each $u \in \mathrm{N}$, partition the $(n+1)$th-order generalized impulse response Gramian $G_{u,n+1}$ as

$$G_{u,n+1} = \begin{bmatrix} G_{u,n} & \mathbf{g}_{u,n} \\ \mathbf{g}_{u,n}^T & g_{u,n+1} \end{bmatrix}, \quad (23)$$

where

$$\mathbf{g}_{u,n} = \int_0^\infty \begin{bmatrix} h_u(t)h_{u+n}(t) \\ h_{u+1}(t)h_{u+n}(t) \\ \vdots \\ h_{u+n-1}(t)h_{u+n}(t) \end{bmatrix} dt. \quad (24)$$

Then the coefficients a_is in (1) or (12) are given by

$$\mathbf{a} = \begin{bmatrix} a_n & a_{n-1} & \cdots & a_1 \end{bmatrix}^T = -G_{u,n}^{-1}\mathbf{g}_{u,n}. \quad (25)$$

Proof: Since each $h_u(t)$ satisfies the characteristic equation, we have

$$\begin{aligned} h_{u+n}(t) &= -a_n h_u(t) - a_{n-1}h_{u+1}(t) - \cdots - a_1 h_{u+n-1}(t) \\ &= -\begin{bmatrix} h_u(t) & h_{u+1}(t) & \cdots & h_{u+n-1}(t) \end{bmatrix}\mathbf{a}. \end{aligned} \quad (26)$$

Then

$$\begin{aligned}\mathbf{g}_{u,n} &= \int_0^\infty \begin{bmatrix} h_u(t) \\ h_{u+1}(t) \\ \vdots \\ h_{u+n-1}(t) \end{bmatrix} h_{u+n}(t)dt \\ &= -G_{u,n}\mathbf{a}\end{aligned} \quad (27)$$

and we have (25). □

A different formula can be derived from the recursive relationship given in (21). For the ease of presentation, let g_{ij} denote the (i,j)th element of the $(n+1)$th-order generalized impulse response Gramian $G_{u,n+1}$, and let

$$G_{u,n+1}[k,l] = \begin{bmatrix} g_{kk} & g_{k,k+1} & \cdots & g_{kl} \\ g_{k,k+1} & g_{k+1,k+1} & \cdots & g_{k+1,l} \\ \vdots & \vdots & \ddots & \\ g_{kl} & g_{k+1,l} & \cdots & g_{ll} \end{bmatrix}. \quad (28)$$

Theorem 2: The a_is in (1) and (12) are computed by

$$a_n = \sqrt{\frac{|G_{u,n+1}[2,n+1]|}{|G_{u,n+1}[1,n]|}}, \quad (29)$$

$$\mathbf{a}_{n-1} = -G_{u,n+1}^{-1}[2,n](\mathbf{g}_{1,n-1}a_n + \mathbf{g}_{2,n-1}), \quad (30)$$

where

$$\mathbf{a}_{n-1} = \begin{bmatrix} a_{n-1} & a_{n-2} & \cdots & a_1 \end{bmatrix}^T,$$

$$\mathbf{g}_{1,n-1} = \begin{bmatrix} g_{12} & g_{13} & \cdots & g_{1n} \end{bmatrix}^T,$$

$$\mathbf{g}_{2,n-1} = \begin{bmatrix} g_{2,n+1} & g_{3,n+1} & \cdots & g_{n,n+1} \end{bmatrix}^T.$$

Proof: From (21), we obtain n equations as follows:

$$\mathbf{g}_{1,n-1}a_n + G_{u,n+1}[2,n]\mathbf{a}_{n-1} + \mathbf{g}_{2,n-1} = \mathbf{0}, \quad (31)$$

$$g_{11}a_n^2 + \mathbf{g}_{1,n-1}^T\mathbf{a}_{n-1}a_n - \mathbf{g}_{2,n-1}^T\mathbf{a}_{n-1} - g_{n+1,n+1} = 0. \quad (32)$$

Since $G_{u,n+1}[2,n+1]$ is positive definite, $G_{u,n+1}^{-1}[2,n]$ exists and (30) follows from (31). Substituting (30) into (32), we obtain the quadratic equation of the form

$$d_1 a_n^2 + d_2 = 0, \quad (33)$$

where

$$\begin{aligned} d_1 &= g_{11} - \mathbf{g}_{1,n-1}^T G_{u,n+1}^{-1}[2,n]\mathbf{g}_{1,n-1} \\ &= \frac{|G_{u,n+1}[1,n]|}{|G_{u,n+1}[2,n]|}, \\ d_2 &= \mathbf{g}_{2,n-1}^T G_{u,n+1}^{-1}[2,n]\mathbf{g}_{2,n-1} - g_{n+1,n+1} \\ &= -\frac{|G_{u,n+1}[2,n+1]|}{|G_{u,n+1}[2,n]|}. \end{aligned}$$

Then (29) follows from the fact that a_n is positive since the given system is stable. □

For the system matrix $\hat{A}$ given in (12), $\hat{A}^{-1}$ is given by

$$\hat{A}^{-1} = \begin{bmatrix} -\frac{a_{n-1}}{a_n} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{a_1}{a_n} & 0 & \cdots & 1 \\ -\frac{1}{a_n} & 0 & \cdots & 0 \end{bmatrix}, \tag{34}$$

which determines the characteristic polynomial of the reciprocal system [2].

Theorem 3: For each $u \in \mathrm{N}$, partition $G_{u,n+1}$ as

$$G_{u,n+1} = \begin{bmatrix} g_{u,1} & \bar{\mathbf{g}}_{u,n}^T \\ \bar{\mathbf{g}}_{u,n} & \bar{G}_{u,n} \end{bmatrix}, \tag{35}$$

where

$$\bar{\mathbf{g}}_{u,n} = \int_0^\infty \begin{bmatrix} h_{u+1}(t) \\ h_{u+2}(t) \\ \vdots \\ h_{u+n}(t) \end{bmatrix} h_u(t)dt. \tag{36}$$

Then

$$\hat{\mathbf{a}} = \begin{bmatrix} \frac{a_{n-1}}{a_n} & \frac{a_{n-2}}{a_n} & \cdots & \frac{1}{a_n} \end{bmatrix}^T = -\bar{G}_{u,n}^{-1}\bar{\mathbf{g}}_{u,n}. \tag{37}$$

Proof: From (26), we have

$$\begin{aligned} h_u(t) &= -\frac{a_{n-1}}{a_n}h_{u+1}(t) - \cdots - \frac{1}{a_n}h_{u+n}(t) \\ &= -\begin{bmatrix} h_{u+1}(t) & h_{u+2}(t) & \cdots & h_{u+n}(t) \end{bmatrix}\hat{\mathbf{a}}. \end{aligned} \tag{38}$$

Then,

$$\begin{aligned} \bar{\mathbf{g}}_{u,n} &= \int_0^\infty \begin{bmatrix} h_{u+1}(t) \\ h_{u+2}(t) \\ \vdots \\ h_{u+n}(t) \end{bmatrix} h_u(t)dt \\ &= -\bar{G}_{u,n}\hat{\mathbf{a}} \end{aligned} \tag{39}$$

and the proof is completed. □

4. APPLICATION TO MODEL REDUCTION

In this section the results of Section 3 are applied to the model reduction problem considered in [7,9,10]. For an nth-order stable system described by (1) or (10) and (11), the objective is to find a reduced model that preserves impulse energies as well as some time-moments and Markov parameters of the original system. The Gramian technique of [7] yields the kth-order reduced model that preserves the first $k \times k$ elements of the impulse response Gramian and the first k Markov parameters of an original system. In [9] it was revealed that the reduced model retaining the first $k \times k$ elements of the Gram matrix and the first k time-moments can be derived by applying the same technique to the reciprocal system of the original system. In [10], more general reduced models were obtained based on the methods of [7] and [9]. Assuming two different forms of the reduced model, and applying the techniques of [7] and [9] separately to (17) or (18), multiple reduced models were derived so that some time-moments and/or Markov parameters as well as diagonal elements of impulse response Gramian and/or Gram matrix are preserved in the reduced models. It will be shown that the reduced models of [10] can be obtained more efficiently by applying the results of the previous section.

From lemma 1, $G_{u,n}$ and $G_{u+1,n}$ are respectively unique solutions to the following two Lyapunov equations for the realization $(\hat{A}, b_u, c_u)$

$$\hat{A}^T G_{u,n} + G_{u,n}\hat{A} = -c_u^T c_u, \tag{40}$$

$$\hat{A}^T G_{u+1,n} + G_{u+1,n}\hat{A} = -c_{u+1}^T c_{u+1} = -\hat{A}^T c_u^T c_u \hat{A}. \tag{41}$$

Premultiplying $\hat{A}^{-T}$ and postmultiplying $\hat{A}^{-1}$ on both sides of (41), we have

$$G_{u+1,n}\hat{A}^{-1} + \hat{A}^{-T}G_{u+1,n} = -c_u^T c_u. \tag{42}$$

The Lyapunov equations derived in [7] and [9] are special cases of (40) and (42) respectively with $u = 0$ and $u = -n$.

Now let $(\hat{A}_{u,k}, b_{u,k}, c_{u,k})$ be the realization in canonical form for the kth-order reduced model and let $G_{u,k}$ denote the $k \times k$ principal leading submatrix of $G_{u,n}$. If

$$\hat{A}_{u,k}^T G_{u,k} + G_{u,k}\hat{A}_{u,k} = -c_{u,k}^T c_{u,k}, \qquad (43)$$

then the reduced model clearly preserves the first $k \times k$ elements of the original nth-order impulse response Gramian $G_{u,n}$. Similarly if

$$G_{u+1,k}\hat{A}_{u,k}^{-1} + \hat{A}_{u,k}^{-T} G_{u+1,k} = -c_{u,k}^T c_{u,k}, \qquad (44)$$

then the first $k \times k$ elements of $G_{u+1,n}$ are retained in the reduced model. If $b_{u,k}$ and $c_{u,k}$ are chosen appropriately, then some time-moments and/or Markov parameters of the original system also can be preserved in the reduced model. Consequently two kth-order reduced models, denoted by $H_{1,k}^u$ and $H_{2,k}^u$ respectively, can be obtained for each u.

Let

$$\hat{A}_{u,k} = \begin{bmatrix} 0 & \cdots & 0 & -\bar{a}_k \\ 1 & \cdots & 0 & -\bar{a}_{k-1} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & -\bar{a}_1 \end{bmatrix}. \qquad (45)$$

For $H_{1,k}^u$, partition the original (k+1)th-order generalized impulse response Gramian $G_{u,k+1}$ as

$$G_{u,k+1} = \begin{bmatrix} G_{u,k} & \mathbf{g}_{u,k} \\ \mathbf{g}_{u,k}^T & g_{u,k+1} \end{bmatrix} \qquad (46)$$

and compute $\bar{a}_i$ s in (45) as in Theorem 1, i.e.,

$$\begin{bmatrix} \bar{a}_k & \bar{a}_{k-1} & \cdots & \bar{a}_1 \end{bmatrix}^T = -G_{u,k}^{-1}\mathbf{g}_{u,k}. \qquad (47)$$

Let $c_{u,k}$ be the k-dimensional row vector that consists of the first k elements of c_u and let $b_{u,k} = \hat{A}_{u,k}^{-u} b_{0,k}$, where $b_{0,k}$ denotes the k-dimensional column vector that consists of the first k elements of $\hat{b}$. Then the first reduced model $H_{1,k}^u$ is completed. It is easily seen that $H_{1,k}^u$ preserves the first $(-u)$ time-moments and the first $(k+u)$ Markov parameters of the original system. Alternatively, it can be shown that the Lyapunov (43) holds. Hence the first $k \times k$ elements of the original nth-order impulse response Gramian $G_{u,n}$ are also retained in the reduced model.

For $H_{2,k}^u$, we compute $\hat{A}_{u,k}^{-1}$ instead of $\hat{A}_{u,k}$ using Theorem 3. Let

$$\tilde{A}_{u,k} = \hat{A}_{u,k}^{-1} = \begin{bmatrix} -\tilde{a}_k & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\tilde{a}_2 & 0 & \cdots & 1 \\ -\tilde{a}_1 & 0 & \cdots & 0 \end{bmatrix} \qquad (48)$$

and partition $G_{u,k+1}$ as

$$G_{u,k+1} = \begin{bmatrix} g_{u,1} & \bar{\mathbf{g}}_{u,k}^T \\ \bar{\mathbf{g}}_{u,k} & \bar{G}_{u,k} \end{bmatrix}. \qquad (49)$$

Then

$$\begin{bmatrix} \tilde{a}_k & \tilde{a}_{k-1} & \cdots & \tilde{a}_1 \end{bmatrix}^T = -\bar{G}_{u,k}^{-1}\bar{\mathbf{g}}_{u,k} \qquad (50)$$

and the second reduced model $H_{2,k}^u$ is completed by computing $\hat{A}_{u,k} = \tilde{A}_{u,k}^{-1}$ and choosing the vectors $b_{u,k}$ and $c_{u,k}$ as in the first model. It is again easily seen that $H_{2,k}^u$ preserves the first $(-u)$ time-moments and the first $(k+u)$ Markov parameters of the original system. However the first $k \times k$ elements of $G_{u+1,n}$ are maintained in the reduced model since the Lyapunov equation (44) holds in this case.

Applying the above method for different values of u, we can obtain a family of reduced models. Note that the kth-order reduced models derived by the methods of [7] and [9] respectively correspond to $H_{1,k}^0$ and $H_{2,k}^{-k}$. On the other hand, $2k$ reduced models $H_{1,k}^u$, $H_{2,k}^{u-1}$, $-(k-1) \le u \le 0$, are the same as those obtained by the technique of [10].

Example: Consider a fourth-order system with the transfer function [10]

$$H(s) = \frac{0.01s^2 + 0.1s + 1}{0.0121s^4 + 0.03s^3 + 1.1s^2 + 2.3s + 1}. \qquad (51)$$

The first two time-moments and the first four Markov parameters of (51) are respectively computed by

$t_1 = -1, \quad t_2 = 2.2,$
$m_1 = 0, \quad m_2 = 0.826, \quad m_3 = 6.216, \quad m_4 = -7.893.$

Then the state-space realization $(\hat{A}, \hat{b}, \hat{c})$ of (51) in controllability canonical form is given by

$$\hat{A} = \begin{bmatrix} 0 & 0 & 0 & -82.645 \\ 1 & 0 & 0 & -190.083 \\ 0 & 1 & 0 & -90.909 \\ 0 & 0 & 0 & -2.479 \end{bmatrix}, \quad \hat{b} = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}^T,$$

$$\hat{c} = \begin{bmatrix} 0 & 0.826 & 6.216 & -7.893 \end{bmatrix}.$$

Solving the Lyapunov (20) for $u = 0$, we obtain the fourth-order generalized impulse response Gramian $G_{0,4}$ by

$$G_{0,4} = \begin{bmatrix} 0.232 & 0 & -1.258 & 0.342 \\ 0 & 1.258 & -0.342 & -94.375 \\ -1.258 & -0.342 & 89.238 & -19.316 \\ 0.342 & -94.375 & -19.316 & 7944.849 \end{bmatrix}. \tag{52}$$

From the recursive relationship given in (21), we have

$$G_{-1,4} = \begin{bmatrix} 1.29 & -0.5 & -0.232 & 0.826 \\ -0.5 & 0.232 & 0 & -1.258 \\ -0.232 & 0 & 1.258 & -0.342 \\ 0.826 & -1.258 & -0.342 & 89.238 \end{bmatrix}, \tag{53}$$

$$G_{-2,4} = \begin{bmatrix} 4.569 & -2.42 & 0.91 & 0.5 \\ -2.42 & 1.29 & -0.5 & -0.232 \\ 0.91 & -0.5 & 0.232 & 0 \\ 0.5 & -0.232 & 0 & 1.258 \end{bmatrix}. \tag{54}$$

Now we derive four second-order reduced models $H_{1,2}^{u}$, $H_{2,2}^{u-1}$, $-1 \le u \le 0$. For $H_{1,2}^{0}$, we have from (46), (47) and (52)

$$\begin{bmatrix} \bar{a}_2 \\ \bar{a}_1 \end{bmatrix} = -\begin{bmatrix} 0.232 & 0 \\ 0 & 1.258 \end{bmatrix}^{-1} \begin{bmatrix} -1.258 \\ -0.342 \end{bmatrix} = \begin{bmatrix} 5.422 \\ 0.272 \end{bmatrix}. \tag{55}$$

Hence $H_{1,2}^{0}$ is given by

$$\hat{A}_{0,2} = \begin{bmatrix} 0 & -5.422 \\ 1 & -0.272 \end{bmatrix}, \quad b_{0,2} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$c_{0,2} = \begin{bmatrix} m_1 & m_2 \end{bmatrix} = \begin{bmatrix} 0 & 0.826 \end{bmatrix}.$$

Similarly $H_{1,2}^{-1}$ is obtained from (46), (47) and (53) as

$$\hat{A}_{-1,2} = \begin{bmatrix} 0 & -1.092 \\ 1 & -2.354 \end{bmatrix}, \quad b_{-1,2} = \hat{A} b_{0,2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$c_{-1,2} = \begin{bmatrix} t_1 & m_1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \end{bmatrix}.$$

For $H_{2,2}^{-1}$, we obtain from (49), (50) and (53)

$$\begin{bmatrix} \tilde{a}_2 \\ \tilde{a}_1 \end{bmatrix} = -\begin{bmatrix} 0.232 & 0 \\ 0 & 1.258 \end{bmatrix}^{-1} \begin{bmatrix} -0.5 \\ -0.232 \end{bmatrix} = \begin{bmatrix} 2.155 \\ 0.184 \end{bmatrix}.$$

Then

$$\hat{A}_{-1,2} = \begin{bmatrix} -2.155 & 1 \\ -0.184 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & -5.423 \\ 1 & -11.688 \end{bmatrix}.$$

$b_{-1,2}$ and $c_{-1,2}$ are given as in $H_{1,2}^{-1}$. Using the same procedure, $H_{2,2}^{-2}$ is obtained from (49), (50) and (54) as

$$\hat{A}_{-2,2} = \begin{bmatrix} 0 & -1.365 \\ 1 & -2.948 \end{bmatrix},$$

$$b_{-2,2} = \hat{A}^2 b_{0,2} = \begin{bmatrix} -1.365 \\ -2.948 \end{bmatrix},$$

$$c_{-1,2} = \begin{bmatrix} t_2 & t_1 \end{bmatrix} = \begin{bmatrix} 2.2 & -1 \end{bmatrix}.$$

Note that four second-order reduced models acquired above are the same as those derived in [10].

5. CONCLUSIONS

In this paper we studied the properties of a generalized impulse response Gramian. The recursive relationship satisfied by the family of Gramians was established. It was shown that the characteristic polynomial of the linear time-invariant continuous system can be determined from the generalized impulse response Gramian. Usefulness of the generalized impulse response Gramian for model reduction was also discussed.

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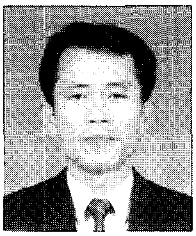

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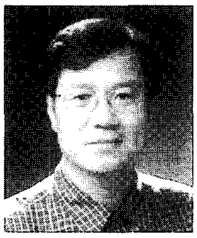

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