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Kinematics of the Nonsteady Axi-symmetric Ideal Plastic Flow Process

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Abstract: A nonsteady axi-symmetric ideal flow solution is obtained here. It is based on the rigid perfect-plastic constitutive law with the Tresca yield condition and its associated flow rule. The process is to deform a circular solid disk into a spherical shell of prescribed geometry. It is assumed that there are no rigid zones and friction stresses. The solution obtained provides the distribution of kinematic variables and involves one undetermined function of the time. This function can be in general found by superimposing an optimality criterion.

Keywords: Rigid perfect plasticity, Nonsteady axi-symmetric ideal flow, Tresca yield condition

Introduction

The theory of ideal plastic flows has been originated by Richmond and Devenpeck [1] and developed by Hill [2], Chung and Richmond [3], Richmond and Alexandrov [4] among others. The theory provides a tool for preliminary design of forming processes which mainly involves plasticity. In the case of bulk forming, several design solutions have been obtained for steady processes (Richmond and Devenpeck [1], Richmond and Morrison [5], Richmond [6], Sortais and Kobayashi [7]). Non-trivial nonsteady design solutions have been found very recently (Chung *et al.* [8-10], Alexandrov *et al.* [11]). These solutions have been based on the plane strain condition. In the present paper, kinematics of a nonsteady axi-symmetric ideal flow process is described. As shown in the general theory (Richmond and Alexandrov [4]), a stress distribution compatible with this solution as well as the equilibrium equation and the Tresca yield condition exists.

The process considered is to deform a circular solid disk into a spherical shell of prescribed geometry. It is assumed that there are no rigid zones and friction stresses. The solution obtained provides the distribution of kinematic variables and involves one undetermined function of the time. This function can be in general found by superimposing an optimality criterion. The solution is an axi-symmetric analogy to the plane strain solution reported by Alexandrov *et al.* [11], employing similar ideas. Several coordinate systems, including Eulerian and Lagrangian coordinates, are introduced and, then, corresponding transformation equations constitute a basis for the proof that an ideal flow has been found.

Statement of the Problem

Consider a solid disk of initial radius R_0 and thickness H_0 and assume that it should be deformed into a spherical shell of a constant thickness H_f such that the ideal flow conditions are satisfied as shown in Figure 1. There are no shear stresses at the surfaces, but normal surface tractions should be, in general, found from the solution determining the system of loading that produces the ideal flow. Intermediate shapes should be also found from the solution.

The material of the disk is rigid-perfect plastic obeying the Tresca yield criterion, which can be written in the form

$$|\sigma_1 - \sigma_2| \le 2k,\ |\sigma_2 - \sigma_3| \le 2k,\ |\sigma_3 - \sigma_1| \le 2k \tag{1}$$

where σ_1, σ_2 and σ_3 are the principal stresses and k is the shear yield stress, a material constant. The following corner regimes will be used in the present paper: (*i*) $\sigma_1 = \sigma_2$ and $\sigma_1 > \sigma_3$, and (*ii*) $\sigma_1 = \sigma_2$ and $\sigma_1 < \sigma_3$. In these regimes, the associated flow rule takes the form

$$\xi_1 = \pm\lambda_1,\ \ \xi_2 = \pm\lambda_2,\ \ \xi_3 = \mp(\lambda_2 + \lambda_1) \tag{2}$$

Here the upper sign corresponds to regime (*i*) and the lower sign to regime (*ii*). Also ξ_1, ξ_2, and ξ_3 are the principal strain rates and λ_1 and λ_2 are non-negative multipliers. equations (1) and (2) should be supplemented with the equilibrium equations.

Solution

With no loss of generality, it is possible to assume that the material point located at the intersection of the axis of symmetry and the upper surface of the deformed body is motionless (point E in Figure 1). It is convenient to adopt four coordinate systems as shown in Figure 2. An Eulerian cylindrical system $r\varphi z$ is chosen such that its origin coincides with the motionless point of the body. Another cylindrical coordinate system $r'\varphi z'$ moves along the z-axis such that the z- and z'-axes coincide. The speed of this motion is determined by the solution. A spherical coordinate system $\rho\theta\varphi$ moves

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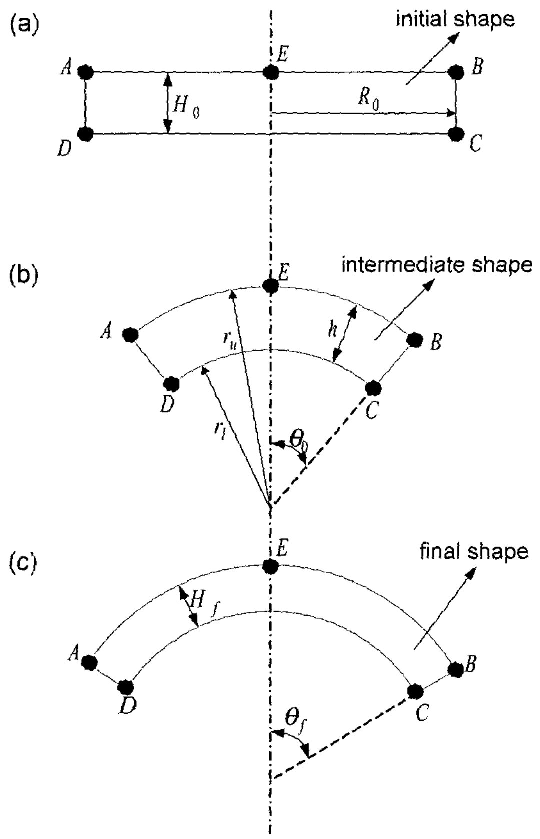

Figure 1. Notations for (a) initial, (b) intermediate and (c) final shapes.

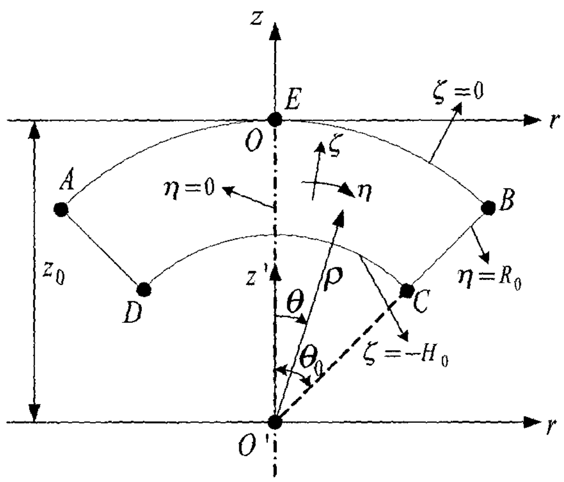

Figure 2. Coordinate systems for an intermediate shape.

along with the system $r'\varphi z'$ and is defined by the usual transformation equations

$$r' = \rho\sin\theta \quad \text{and} \quad z' = \rho\cos\theta \tag{3}$$

Finally, in an arbitrary cross-section $\varphi = const$, it is possible to introduce a Lagrangian system of coordinate $\zeta\eta$ defined by the condition that

$$\zeta = z \quad \text{and} \quad \eta = r \tag{4}$$

at the initial instant. Therefore, the boundary of the body is determined by the equations

$$\zeta = 0, \ \eta = 0, \ \zeta = -H_0, \ \text{and} \ \eta = R_0 \tag{5}$$

It follows from the definition of the coordinate systems that

$$r' = r \quad \text{and} \quad z' = z + z_0 \tag{6}$$

where z_0 is an arbitrary function of the time, t.

The main assumption for an intermediate shape, which will be verified *a pasteriori*, is that

$$\rho = P(\zeta, t) \quad \text{and} \quad \theta = \Theta(\eta, t) \tag{7}$$

where $P(\zeta, t)$ is an arbitrary function of ζ and t, $\Theta(\eta, t)$ is an arbitrary function of η and t. It follows from equation (7) that the curves AB and DC are circular arcs and AD and CB are straight (Figure 1) in the planes $\varphi = const$ after any amount of deformation. Combining equations (3), (6) and (7) leads to

$$r = P\sin\Theta \quad \text{and} \quad z = P\cos\Theta - z_0 \tag{8}$$

Using these equations the non-zero components of the metric tensor of the Lagrangian coordinate system can be found as

$$g_{\eta\eta} = P^2\Theta_{,\eta}^2, \ g_{\varphi\varphi} = P^2\sin^2\Theta, \ \text{and} \ g_{\zeta\zeta} = P_{,\zeta}^2 \tag{9}$$

It follows from the initial conditions equation (4) that $g_{\zeta\zeta}g_{\varphi\varphi}g_{\eta\eta} = \eta^2$ at the initial instant. Substituting this equation and equation (9) into the incompressibility equation gives

$$\Theta_{,\eta}P_{,\zeta}P^2\sin\Theta = \eta \tag{10}$$

This equation has solutions if and only if

$$\Theta_{,\eta}\sin\Theta = b\eta/R_0^2 \quad \text{and} \quad P_{,\zeta}P^2 = R_0^2/b \tag{11}$$

where b is an arbitrary function of the time. Equation (11) can be immediately integrated, with the use of equation (5), to give

$$\cos\Theta = 1 - \frac{b\eta^2}{2R_0^2} \quad \text{and} \quad P^3 = \frac{3R_0^2}{b}\zeta + z_0^3 \tag{12}$$

Here and from now on, z_0 can be considered as a function of b rather than t.

It is now necessary to show that the solution equation (12) satisfies equation (4). Substituting equation (12) into equation

(8) gives

$$r = \frac{\eta z_0\sqrt{b}}{R_0}\sqrt[3]{\frac{3R_0^2\zeta}{bz_0^3}+1}\sqrt{1-\frac{b\eta^2}{4R_0^2}} \quad \text{and}$$

$$z = z_0\sqrt[3]{\frac{3R_0^2\zeta}{bz_0^3}+1}\left(1-\frac{b\eta^2}{2R_0^2}\right)-z_0 \tag{13}$$

It follows from these equations that the only possibility to satisfy equation (4) is to assume that

$$\lim_{b\to 0}\left(\frac{bz_0^3}{R_0^2}\right) = \infty \tag{14}$$

Using equation (14) and expanding equation (13) in a series lead to

$$r = \frac{\eta z_0\sqrt{b}}{R_0}\left(1+\frac{R_0^2\zeta}{bz_0^3}\right)\left(1-\frac{b\eta^2}{8R_0^2}\right) \quad \text{and}$$

$$z = \frac{\zeta R_0^2}{bz_0^2}-\frac{bz_0\eta^2}{2R_0^2} \tag{15}$$

as $b \to 0$. Comparing equations (4) and (15) one can conclude that the behavior of the function $z_0(b)$ in a vicinity of the point $b = 0$ must satisfy the following rule

$$\lim_{b\to 0}\left(\frac{bz_0^2}{R_0^2}\right) = 1 \tag{16}$$

Since $z_0 \to \infty$ as $b \to 0$, the assumption equation (14) is verified and $bz_0 \to 0$ as $b \to 0$. The latter has been used in comparing equations (4) and (15). Applying l'Hospital's rule to equation (16) gives

$$\lim_{b\to 0}\left(\frac{2b\sqrt{b}}{R_0}\frac{dz_0}{db}\right) = -1 \tag{17}$$

It follows from equations (7) and (12) that the orientation of the edge, $\theta = \theta_0$, is determined by

$$\cos\theta_0 = 1-b/2 \tag{18}$$

and the radius of the upper and lower surfaces by (Figure 1)

$$r_u = \sqrt[3]{z_0^3+\frac{3R_0^2\zeta_0}{b}} \quad \text{and} \quad r_l = \sqrt[3]{z_0^3+\frac{3R_0^2(\zeta_0-H_0)}{b}} \tag{19}$$

Then, the current thickness is given by

$$h = r_u - r_l = \sqrt[3]{z_0^3+\frac{3R_0^2\zeta_0}{b}} - \sqrt[3]{z_0^3+\frac{3R_0^2(\zeta_0-H_0)}{b}} \tag{20}$$

Since the maximum magnitude of θ_0 is π, it follows from equation (18) that

$$0 \le b \le 4 \tag{21}$$

In the Lagrangian coordinates, the non-zero physical components of the strain rate tensor can be found from equation (9) in the form

$$\xi_{\eta\eta} = \frac{\dot{g}_{\eta\eta}}{2g_{\eta\eta}} = \frac{\dot{P}}{P}+\frac{\dot{\Theta}_{,\eta}}{\Theta_{,\eta}}, \quad \xi_{\varphi\varphi} = \frac{\dot{g}_{\varphi\varphi}}{2g_{\varphi\varphi}} = \frac{\dot{P}}{P}+\frac{\cos\Theta}{\sin\Theta}\dot{\Theta},$$

$$\xi_{\zeta\zeta} = \frac{\dot{g}_{\zeta\zeta}}{2g_{\zeta\zeta}} = \frac{\dot{P}_{,\zeta}}{P_{,\zeta}} \tag{22}$$

Here, the superimposed dot denotes the partial time derivative in the Lagrangian coordinates. Equation (22) shows that the shear components of the strain rate tensor in the Lagrangian coordinates vanish and, therefore, the trajectories of the principal strain rates are fixed in the material. Thus the ideal flow conditions are satisfied. Using equation (12) it is possible to get

$$\frac{\dot{\Theta}_{,\eta}}{\Theta_{,\eta}} = \frac{2\dot{b}}{b(4-b\eta^2/R_0^2)}, \quad \frac{\dot{P}_{,\zeta}}{P_{,\zeta}} = -\frac{\dot{b}}{b}\left[1+2\frac{(b^2z_0^2dz_0/db-R_0^2\zeta)}{(3R_0^2\zeta+bz_0^3)}\right]$$

$$\frac{\dot{P}}{P} = \frac{\dot{b}}{b}\frac{(b^2z_0^2dz_0/db-R_0^2\zeta)}{(3R_0^2\zeta+bz_0^3)}, \quad \frac{\cos\Theta}{\sin\Theta}\dot{\Theta} = \frac{\dot{b}}{b}\frac{(2-b\eta^2/R_0^2)}{(4-b\eta^2/R_0^2)}$$

Then, equation (22) becomes

$$\xi_{\zeta\zeta} = -\frac{\dot{b}}{b}[1+2f(\bar{\zeta})]$$

$$\xi_{\eta\eta} = \frac{\dot{b}}{b}[f(\bar{\zeta})+g(\bar{\eta})]$$

$$\xi_{\varphi\varphi} = \frac{\dot{b}}{b}[1-g(\bar{\eta})+f(\bar{\zeta})] \tag{23}$$

where,

$$f(\bar{\zeta}) = \frac{b^2z_0^2(dz_0/db)-R_0^2H_0\bar{\zeta}}{bz_0^3+3R_0^2H_0\bar{\zeta}}$$

$$g(\bar{\eta}) = \frac{2}{4-b\bar{\eta}^2} \tag{24}$$

and $\bar{\zeta} = \zeta/H_0$ and $\bar{\eta} = \eta/R_0$.

Assume that $\xi_3 \equiv \xi_{\zeta\zeta}$, $\xi_2 \equiv \xi_{\varphi\varphi}$, and $\xi_1 \equiv \xi_{\eta\eta}$. Then, in regime (*i*) introduced in Section Statement of the problem, $\xi_{\varphi\varphi} > 0$ and $\xi_{\eta\eta} > 0$. Note that for any choice of function $z_0(b)$ satisfying equations (14), (16) and (17), these conditions are valid as confirmed by equations (23) and (24). Also, in regime (*ii*) $\xi_{\varphi\varphi} < 0$ and $\xi_{\eta\eta} < 0$. These conditions can be also verified in a similar manner. Thus the domain of existence of the solution is determined by the conditions

$$\xi_{\varphi\varphi} = 0 \quad \text{and} \quad \xi_{\eta\eta} = 0 \tag{25}$$

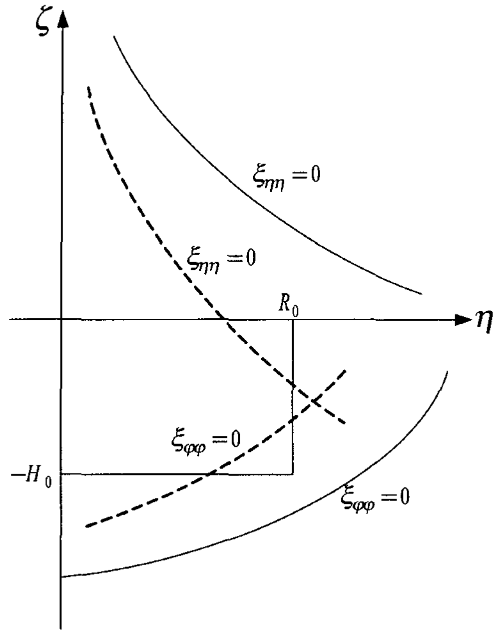

Figure 3. Illustration of the condition for the existence of the solution.

These existence conditions can be illustrated geometrically. In the Lagrangian coordinates, at any instant the workpiece is represented by the rectangular, as follows from equation (5). This rectangular is shown in Figure 3. Substituting equation (25) into equation (23) determines two curves moving in the $\zeta\eta$ space. Two couples of such curves are schematically shown in Figure 3. The ideal flow exists if no one of these curves passes through the rectangular at any stage of the process. The solid curves represent a possible ideal flow. The ideal flow conditions are violated if one or both of the curves pass through the rectangular at any instant (dashed curves in Figure 3). Note that the solution is also not unique because it involves one undetermined function $z_0(b)$.

Conclusions

The ideal flow theory has been applied to a nonsteady axi-symmetric process. The existence of the ideal flow solution has been demonstrated and kinematic quantities have been determined. The solution is however not unique and involves one undetermined function $z_0(b)$. This function can only be found if an optimality condition is superimposed. One possible condition is to search for a minimum of the plastic work with respect to the initial thickness of the disk while keeping its volume constant. This criterion has been successfully applied to a similar plane strain problem (Alexandrov *et al.* [11]). The application of this criterion to the axi-symmetric problem will be the subject of a subsequent investigation.

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