

EXTRACTING LINEAR FACTORS IN FEYNMAN'S OPERATIONAL CALCULI : THE CASE OF TIME DEPENDENT NONCOMMUTING OPERATORS

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ABSTRACT. Disentangling is the essential operation of Feynman's operational calculus for noncommuting operators. Thus formulas which simplify this operation are central to the subject. In a recent paper the procedure for "extracting a linear factor" has been established in the setting of Feynman's operational calculus for time independent operators A_1, \dots, A_n and associated probability measures μ_1, \dots, μ_n . While the setting just described is natural in many circumstances, it is not natural for evolution problems. There the measures should not be restricted to probability measures and it is worthwhile to allow the operators to depend on time. The main purpose for this paper is to extend the procedure for extracting a linear factor to this latter setting. We should mention that Feynman's primary motivation for developing an operational calculus for noncommuting operators came from a desire to describe the evolution of certain quantum systems.

1. Introduction

This paper extends the basic result of Johnson and Kim [7] on "extracting a linear factor" to the time-dependent and not necessarily probability measure setting described in the abstract. Feynman's operational calculus for noncommuting operators began with the paper [1]. There have been various later developments among which we particularly note the work of Maslov [10] and of Nazaikinsii, Shatalov and Sternin [11]. It is in [10 and 11] where the related concepts of "autonomous brackets" and "extracting a linear factor" were introduced.

We are working here in the framework of the recent approach to Feynman's operational calculus initiated by Jefferies and Johnson [2, 3,

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4, 5] and by the same authors and Nielsen [6]. Further discussion and many further references can be found in [2, 3, 7, 8, 11].

Passing from probability measures to measures which are finite on any bounded interval of \mathbb{R} is not difficult as discussed in [3], but time-dependence of the operators as in [6] does yield a more complicated framework and so a somewhat more complicated proof than in [7]. However, the most essential ideas of the proof remain the same.

In [7], the basic results, Theorems 2.1 and Theorem 2.3, are followed up by various consequences. Such corollaries, remarks, etc. could be given in the present setting as well but we leave that for the reader.

Let X be a separable Banach space over the complex numbers and let $\mathcal{L}(X)$ denote the space of bounded linear operators on X . Fix $T > 0$. For $i = 1, \dots, n$ let $A_i : [0, T] \rightarrow \mathcal{L}(X)$ be maps that are measurable in the sense that $A_i^{-1}(E)$ is a Borel set in $[0, T]$ for any strong operator open set $E \subset \mathcal{L}(X)$. To each $A_i(\cdot)$ we associate a finite continuous Borel measure μ_i on $[0, T]$ and we require that, for each i ,

$$r_i = \int_{[0, T]} \|A_i(s)\|_{\mathcal{L}(X)} |\mu_i|(ds) < \infty.$$

For n positive numbers r_1, \dots, r_n , let $\mathbb{A}(r_1, \dots, r_n)$ be the space of complex-valued functions of n complex variables $f(z_1, \dots, z_n)$, which are analytic at $(0, \dots, 0)$, and are such that their power series expansion

$$f(z_1, \dots, z_n) = \sum_{m_1, \dots, m_n=0}^{\infty} c_{m_1, \dots, m_n} z_1^{m_1} \dots z_n^{m_n}$$

converges absolutely, at least on the closed polydisk $|z_1| \leq r_1, \dots, |z_n| \leq r_n$. Such functions are analytic at least in the open polydisk $|z_1| < r_1, \dots, |z_n| < r_n$.

To the algebra $\mathbb{A}(r_1, \dots, r_n)$ we associate as in [2] a disentangling algebra by replacing the z_i 's with formal commuting objects $(A_i(\cdot), \mu_i)\checkmark$, $i = 1, \dots, n$. Rather than using the notation $(A_i(\cdot), \mu_i)\checkmark$ below, we will often abbreviate to $A_i(\cdot)\checkmark$. Consider the collection $\mathbb{D}((A_1(\cdot), \mu_1)\checkmark, \dots, (A_n(\cdot), \mu_n)\checkmark)$ of all expressions of the form

$$f(A_1(\cdot)\checkmark, \dots, A_n(\cdot)\checkmark) = \sum_{m_1, \dots, m_n=0}^{\infty} c_{m_1, \dots, m_n} (A_1(\cdot)\checkmark)^{m_1} \dots (A_n(\cdot)\checkmark)^{m_n}$$

where $c_{m_1, \dots, m_n} \in \mathbb{C}$ for all $m_1, \dots, m_n = 0, 1, \dots$, and

$$\begin{aligned}
 (1) \quad & \|f(A_1(\cdot)\tilde{\gamma}, \dots, A_n(\cdot)\tilde{\gamma})\| \\
 &= \|f(A_1(\cdot)\tilde{\gamma}, \dots, A_n(\cdot)\tilde{\gamma})\|_{\mathbb{D}(A_1(\cdot)\tilde{\gamma}, \dots, A_n(\cdot)\tilde{\gamma})} \\
 &:= \sum_{m_1, \dots, m_n=0}^{\infty} |c_{m_1, \dots, m_n}| r_1^{m_1} \dots r_n^{m_n} < \infty.
 \end{aligned}$$

The function on $\mathbb{D}((A_1(\cdot), \mu_1)\tilde{\gamma}, \dots, (A_n(\cdot), \mu_n)\tilde{\gamma})$ defined by (1) makes $\mathbb{D}((A_1(\cdot), \mu_1)\tilde{\gamma}, \dots, (A_n(\cdot), \mu_n)\tilde{\gamma})$ into a commutative Banach algebra [6].

We refer to $\mathbb{D}((A_1(\cdot), \mu_1)\tilde{\gamma}, \dots, (A_n(\cdot), \mu_n)\tilde{\gamma})$ as the disentangling algebra associated with the n -tuple $((A_1(\cdot), \mu_1)\tilde{\gamma}, \dots, (A_n(\cdot), \mu_n)\tilde{\gamma})$.

We will often write \mathbb{D} in place of $\mathbb{D}(A_1(\cdot)\tilde{\gamma}, \dots, A_n(\cdot)\tilde{\gamma})$ or $\mathbb{D}((A_1(\cdot), \mu_1)\tilde{\gamma}, \dots, (A_n(\cdot), \mu_n)\tilde{\gamma})$.

For $m = 0, 1, \dots$, let S_m denote the set of all permutations of the integers $\{1, \dots, m\}$, and given $\pi \in S_m$, we let

$$\Delta_m(\pi) = \{(s_1, \dots, s_m) \in [0, T]^m : 0 < s_{\pi(1)} < \dots < s_{\pi(m)} < T\}.$$

Now for nonnegative integers m_1, \dots, m_n and $m = m_1 + \dots + m_n$, we define

$$(2) \quad C_i(s) = \begin{cases} A_1(s), & \text{if } i \in \{1, \dots, m_1\} \\ A_2(s), & \text{if } i \in \{m_1 + 1, \dots, m_1 + m_2\} \\ \dots & \\ A_n(s), & \text{if } i \in \{m_1 + \dots + m_{n-1} + 1, \dots, m\} \end{cases}$$

for $i = 1, \dots, m$ and for all $0 \leq s \leq T$.

DEFINITION 1. Let $P^{m_1, \dots, m_n}(z_1, \dots, z_n) = z_1^{m_1} \dots z_n^{m_n}$. We define the disentangling map on this monomial by

$$\begin{aligned}
 & \mathcal{T}_{\mu_1, \dots, \mu_n} P^{m_1, \dots, m_n}(A_1(\cdot)\tilde{\gamma}, \dots, A_n(\cdot)\tilde{\gamma}) \\
 &= \mathcal{T}_{\mu_1, \dots, \mu_n} ((A_1(\cdot)\tilde{\gamma})^{m_1} \dots (A_n(\cdot)\tilde{\gamma})^{m_n}) \\
 &= \sum_{\pi \in S_m} \int_{\Delta_m(\pi)} C_{\pi(m)}(s_{\pi(m)}) \dots C_{\pi(1)}(s_{\pi(1)}) \\
 & \quad (\mu_1^{m_1} \times \dots \times \mu_n^{m_n})(ds_1, \dots, ds_m).
 \end{aligned}$$

Finally for $f \in \mathbb{D}((A_1(\cdot), \mu_1)\tilde{\gamma}, \dots, (A_n(\cdot), \mu_n)\tilde{\gamma})$ given by

$$f(A_1(\cdot)\tilde{\gamma}, \dots, A_n(\cdot)\tilde{\gamma}) = \sum_{m_1, \dots, m_n=0}^{\infty} c_{m_1, \dots, m_n} (A_1(\cdot)\tilde{\gamma})^{m_1} \dots (A_n(\cdot)\tilde{\gamma})^{m_n}$$

we set

$$\begin{aligned} & \mathcal{T}_{\mu_1, \dots, \mu_n} f(A_1(\cdot), \dots, A_n(\cdot)) \\ := & \sum_{m_1, \dots, m_n=0}^{\infty} c_{m_1, \dots, m_n} \mathcal{T}_{\mu_1, \dots, \mu_n} P^{m_1, \dots, m_n}(A_1(\cdot), \dots, A_n(\cdot)). \end{aligned}$$

We will often use the alternate notation indicated in the next two equalities :

$$P_{\mu_1, \dots, \mu_n}^{m_1, \dots, m_n}(A_1(\cdot), \dots, A_n(\cdot)) = \mathcal{T}_{\mu_1, \dots, \mu_n} P^{m_1, \dots, m_n}(A_1(\cdot), \dots, A_n(\cdot))$$

and

$$f_{\mu_1, \dots, \mu_n}(A_1(\cdot), \dots, A_n(\cdot)) = \mathcal{T}_{\mu_1, \dots, \mu_n} f(A_1(\cdot), \dots, A_n(\cdot)).$$

2. Extraction of a linear factor

Let μ_1, \dots, μ_n be finite continuous measures on $\mathcal{B}[0, T]$ such that , for each i ,

$$r_i = \int_{[0, T]} \|A_i(s)\|_{\mathcal{L}(X)} |\mu_i|(ds) < \infty.$$

THEOREM 1. *Suppose that the measures μ_1, \dots, μ_k are supported by $[a, b] \subset [0, T]$ and the measures μ_{k+1}, \dots, μ_n are supported by $[0, a] \cup [b, T]$. Let*

$$K_{m_1, \dots, m_k} := P_{\mu_1, \dots, \mu_k}^{m_1, \dots, m_k}(A_1(\cdot), \dots, A_k(\cdot))$$

and let μ_0 be any continuous probability measure supported by $[a, b]$. Then we have

$$\begin{aligned} (3) \quad & P_{\mu_1, \dots, \mu_n}^{m_1, \dots, m_n}(A_1(\cdot), \dots, A_n(\cdot)) \\ & = P_{\mu_0, \mu_{k+1}, \dots, \mu_n}^{1, m_{k+1}, \dots, m_n}(K_{m_1, \dots, m_k}, A_{k+1}(\cdot), \dots, A_n(\cdot)). \end{aligned}$$

Proof. For any permutation $\pi \in S_m$ for which any of $s_{m_1+\dots+m_{k+1}}, \dots, s_m$ lies between any two of $s_1, \dots, s_{m_1+\dots+m_k}$ in the list $s_{\pi(1)}, \dots, s_{\pi(m)}$, we have

$$(\mu_1^{m_1} \times \dots \times \mu_n^{m_n})(\Delta_m(\pi)) = 0.$$

Such permutations can be omitted from the sum which defines the left hand side of (3). In the integrands corresponding to the remaining set, say S'_m , of permutations, each of $s_{m_1+\dots+m_k+1}, \dots, s_m$ comes before or after all of $s_1, \dots, s_{m_1+\dots+m_k}$ in the list $s_{\pi(1)}, \dots, s_{\pi(m)}$. But there is a unique correspondence between the permutations $\pi \in S'_m$ and triples (ρ, σ, p) where $p \in \{0, 1, \dots, m_{k+1} + \dots + m_n\}$, $\rho \in S_{m_1+\dots+m_k}$ and σ is a permutation of the $m_{k+1} + \dots + m_n$ integers $m_1 + \dots + m_k + 1, \dots, m$. We denote this last set of permutations by $S_{m_1+\dots+m_k+1, m}$. The triple (ρ, σ, p) corresponds to the following ordering of the variables s_i ;

$$0 < s_{\sigma(m_1+\dots+m_k+1)} < \dots < s_{\sigma(m_1+\dots+m_k+p)} < s_{\rho(1)} < \dots < s_{\rho(m_1+\dots+m_k)} < s_{\sigma(m_1+\dots+m_k+p+1)} < \dots < s_{\sigma(m)} < T.$$

Let

$$\begin{aligned} \Delta_m(\rho, \sigma, p) = \{ & (s_1, \dots, s_m) : 0 < s_{\sigma(m_1+\dots+m_k+1)} < \dots \\ & < s_{\sigma(m_1+\dots+m_k+p)} < s_{\rho(1)} < \dots \\ & < s_{\rho(m_1+\dots+m_k)} < s_{\sigma(m_1+\dots+m_k+p+1)} < \dots \\ & < s_{\sigma(m)} < T \}. \end{aligned}$$

But from the supports of the measures μ_1, \dots, μ_n , we know that

$$(\mu_1^{m_1} \times \dots \times \mu_n^{m_n})(\Delta_m(\rho, \sigma, p)) = (\mu_1^{m_1} \times \dots \times \mu_n^{m_n})(\Delta'_m(\rho, \sigma, p)),$$

where

$$\begin{aligned} \Delta'_m(\rho, \sigma, p) = \{ & (s_1, \dots, s_m) : 0 < s_{\sigma(m_1+\dots+m_k+1)} < \dots \\ & < s_{\sigma(m_1+\dots+m_k+p)} < a < s_{\rho(1)} < \dots \\ & < s_{\rho(m_1+\dots+m_k)} < b < s_{\sigma(m_1+\dots+m_k+p+1)} < \dots \\ & < s_{\sigma(m)} < T \}. \end{aligned}$$

Moreover it is easy to see that

$$\Delta'_m(\rho, \sigma, p) = \Delta'_{m_1+\dots+m_k}(\rho) \times \Delta'_{m_1+\dots+m_k+1, m}(\sigma, p)$$

where

$$\begin{aligned} \Delta'_{m_1+\dots+m_k}(\rho) = \{ & (s_1, \dots, s_{m_1+\dots+m_k}) : a < s_{\rho(1)} < \dots \\ & < s_{\rho(m_1+\dots+m_k)} < b \} \end{aligned}$$

and

$$\begin{aligned} \Delta'_{m_1+\dots+m_k+1,m}(\sigma,p) = & \{(s_{m_1+\dots+m_k+1}, \dots, s_m) : \\ & 0 < s_{\sigma(m_1+\dots+m_k+1)} < \dots \\ & < s_{\sigma(m_1+\dots+m_k+p)} < a < b \\ & < s_{\sigma(m_1+\dots+m_k+p+1)} < \dots < s_{\sigma(m)} < T\}. \end{aligned}$$

Hence we can write

$$\begin{aligned} & P_{\mu_1, \dots, \mu_n}^{m_1, \dots, m_n}(A_1(\cdot), \dots, A_n(\cdot)) \\ = & \sum_{\rho \in S_{m_1+\dots+m_k}} \sum_{\sigma \in S_{m_1+\dots+m_k+1,m}} \sum_{p=0}^{m_{k+1}+\dots+m_n} \\ & \int_{\Delta'_{m_1+\dots+m_k}(\rho) \times \Delta'_{m_1+\dots+m_k+1,m}(\sigma,p)} \\ & C_{\sigma(m)}(s_{\sigma(m)}) \cdots C_{\sigma(m_1+\dots+m_k+p+1)}(s_{\sigma(m_1+\dots+m_k+p+1)}) \\ & C_{\rho(m_1+\dots+m_k)}(s_{\rho(m_1+\dots+m_k)}) \cdots C_{\rho(1)}(s_{\rho(1)}) \\ & C_{\sigma(m_1+\dots+m_k+p)}(s_{\sigma(m_1+\dots+m_k+p)}) \cdots \\ & C_{\sigma(m_1+\dots+m_k+1)}(s_{\sigma(m_1+\dots+m_k+1)}) (\mu_1^{m_1} \times \dots \times \mu_n^{m_n})(ds_1, \dots, ds_m). \end{aligned}$$

Note that

$$(\mu_1^{m_1} \times \dots \times \mu_k^{m_k})(\Delta'_{m_1+\dots+m_k}(\rho)) = (\mu_1^{m_1} \times \dots \times \mu_k^{m_k})(\Delta_{m_1+\dots+m_k}(\rho))$$

and so

$$\begin{aligned} & \sum_{\rho \in S_{m_1+\dots+m_k}} \int_{\Delta'_{m_1+\dots+m_k}(\rho)} C_{\rho(m_1+\dots+m_k)}(s_{\rho(m_1+\dots+m_k)}) \cdots \\ & C_{\rho(1)}(s_{\rho(1)}) (\mu_1^{m_1} \times \dots \times \mu_k^{m_k})(ds_1, \dots, ds_{m_1+\dots+m_k}) \\ & = K_{m_1, \dots, m_k}. \end{aligned}$$

Hence we obtain

$$\begin{aligned} & P_{\mu_1, \dots, \mu_n}^{m_1, \dots, m_n}(A_1(\cdot), \dots, A_n(\cdot)) \\ = & \sum_{\sigma \in S_{m_1+\dots+m_k+1,m}} \sum_{p=0}^{m_{k+1}+\dots+m_n} \int_{\Delta'_{m_1+\dots+m_k+1,m}(\sigma,p)} \\ & C_{\sigma(m)}(s_{\sigma(m)}) \cdots C_{\sigma(m_1+\dots+m_k+p+1)}(s_{\sigma(m_1+\dots+m_k+p+1)}) \\ & K_{m_1, \dots, m_k} C_{\sigma(m_1+\dots+m_k+p)}(s_{\sigma(m_1+\dots+m_k+p)}) \cdots \\ & C_{\sigma(m_1+\dots+m_k+1)}(s_{\sigma(m_1+\dots+m_k+1)}) \\ & (\mu_{k+1}^{m_{k+1}} \times \dots \times \mu_n^{m_n})(ds_{m_1+\dots+m_k+1}, \dots, ds_m). \end{aligned}$$

But since μ_0 is a continuous probability measure whose support is contained within $[a, b]$ and $K_{m_1, \dots, m_k} = P_{\mu_1, \dots, \mu_k}^{m_1, \dots, m_k}(A_1(\cdot), \dots, A_k(\cdot))$ is independent of variable s , we can write

$$\begin{aligned}
 (4) \quad & P_{\mu_1, \dots, \mu_n}^{m_1, \dots, m_n}(A_1(\cdot), \dots, A_n(\cdot)) \\
 &= \sum_{\sigma \in S_{m_1 + \dots + m_{k+1}, m}} \sum_{p=0}^{m_{k+1} + \dots + m_n} \int_{[a, b] \times \Delta'_{m_1 + \dots + m_{k+1}, m}(\sigma, p)} \\
 & C_{\sigma(m)}(s_{\sigma(m)}) \cdots C_{\sigma(m_1 + \dots + m_{k+1})}(s_{\sigma(m_1 + \dots + m_{k+1})}) \\
 & K_{m_1, \dots, m_k}(s_0) C_{\sigma(m_1 + \dots + m_{k+p})}(s_{\sigma(m_1 + \dots + m_{k+p})}) \cdots \\
 & C_{\sigma(m_1 + \dots + m_{k+1})}(s_{\sigma(m_1 + \dots + m_{k+1})}) \\
 & (\mu_0 \times \mu_{k+1}^{m_{k+1}} \times \cdots \times \mu_n^{m_n})(ds_0, ds_{m_1 + \dots + m_{k+1}}, \dots, ds_m).
 \end{aligned}$$

On the other hand,

$$\begin{aligned}
 & P_{\mu_0, \mu_{k+1}, \dots, \mu_n}^{1, m_{k+1}, \dots, m_n}(K_{m_1, \dots, m_k}, A_{k+1}(\cdot), \dots, A_n(\cdot)) \\
 &= \sum_{\pi \in S_{1+m_{k+1} + \dots + m_n}} \int_{\Delta_{1+m_{k+1} + \dots + m_n}(\pi)} \\
 & C'_{\pi(1+m_{k+1} + \dots + m_n)}(s_{\pi(1+m_{k+1} + \dots + m_n)}) \cdots C'_{\pi(1)}(s_{\pi(1)}) \\
 & (\mu_0 \times \mu_{k+1}^{m_{k+1}} \times \cdots \times \mu_n^{m_n})(ds_1, \dots, ds_{1+m_{k+1} + \dots + m_n}),
 \end{aligned}$$

where

$$C'_i(s) := \begin{cases} K_{m_1, \dots, m_k}(s), & \text{if } i = 1 \\ A_{k+1}(s), & \text{if } i \in \{2, \dots, 1 + m_{k+1}\} \\ \dots & \\ A_n(s), & \text{if } i \in \{1 + m_{k+1} + \dots + m_{n-1} + 1, \dots, \\ & 1 + m_{k+1} + \dots + m_n\} \end{cases}$$

for $i = 1, \dots, 1 + m_{k+1} + \dots + m_n$ and for all $0 \leq s \leq T$. But there is a unique correspondence between the permutations $\pi \in S_{1+m_{k+1} + \dots + m_n}$ and pairs (σ', p) where $\sigma' \in S_{2, 1+m_{k+1} + \dots + m_n}$ and $p \in \{0, 1, \dots, m_{k+1} + \dots + m_n\}$. The pair (σ', p) corresponds to the following ordering of the variables s_i ;

$$\begin{aligned}
 0 &< s_{\sigma'(2)} < \cdots < s_{\sigma'(p+1)} < s_1 \\
 &< s_{\sigma'(p+2)} < \cdots < s_{\sigma'(1+m_{k+1} + \dots + m_n)} < T.
 \end{aligned}$$

Let

$$\begin{aligned} \Delta_{1+m_{k+1}+\dots+m_n}(\sigma', p) = & \{(s_1, \dots, s_{1+m_{k+1}+\dots+m_n}) : 0 < \\ & s_{\sigma'(2)} < \dots < s_{\sigma'(p+1)} < s_1 \\ & < s_{\sigma'(p+2)} < \dots < s_{\sigma'(1+m_{k+1}+\dots+m_n)} < T\}. \end{aligned}$$

But from the supports of the measures $\mu_0, \mu_{k+1}, \dots, \mu_n$, we know that

$$\begin{aligned} & (\mu_0 \times \mu_{k+1}^{m_{k+1}} \times \dots \times \mu_n^{m_n})(\Delta_{1+m_{k+1}+\dots+m_n}(\sigma', p)) \\ & = (\mu_0 \times \mu_{k+1}^{m_{k+1}} \times \dots \times \mu_n^{m_n})(\Delta'_{1+m_{k+1}+\dots+m_n}(\sigma', p)) \end{aligned}$$

where

$$\begin{aligned} \Delta'_{1+m_{k+1}+\dots+m_n}(\sigma', p) = & \{(s_1, \dots, s_{1+m_{k+1}+\dots+m_n}) : 0 < \\ & s_{\sigma'(2)} < \dots < s_{\sigma'(p+1)} < a < s_1 < b \\ & < s_{\sigma'(p+2)} < \dots < s_{\sigma'(1+m_{k+1}+\dots+m_n)} < T\} \end{aligned}$$

and note that

$$\Delta'_{1+m_{k+1}+\dots+m_n}(\sigma', p) = (a, b) \times \Delta'_{2,1+m_{k+1}+\dots+m_n}(\sigma', p)$$

where

$$\begin{aligned} \Delta'_{2,1+m_{k+1}+\dots+m_n}(\sigma', p) = & \{(s_2, \dots, s_{1+m_{k+1}+\dots+m_n}) : 0 < \\ & s_{\sigma'(2)} < \dots < s_{\sigma'(p+1)} < a < b \\ & < s_{\sigma'(p+2)} < \dots < s_{\sigma'(1+m_{k+1}+\dots+m_n)} < T\}. \end{aligned}$$

Hence we can write

$$\begin{aligned} & P_{\mu_0, \mu_{k+1}, \dots, \mu_n}^{1, m_{k+1}, \dots, m_n}(K_{m_1, \dots, m_k}, A_{k+1}(\cdot), \dots, A_n(\cdot)) \\ & = \sum_{\sigma' \in S_{2,1+m_{k+1}+\dots+m_n}} \sum_{p=0}^{m_{k+1}+\dots+m_n} \int_{[a,b] \times \Delta'_{2,1+m_{k+1}+\dots+m_n}(\sigma', p)} \\ & \quad C'_{\sigma'(1+m_{k+1}+\dots+m_n)}(s_{\sigma'(1+m_{k+1}+\dots+m_n)}) \cdots C'_{\sigma'(p+2)}(s_{\sigma'(p+2)}) \\ & \quad K_{m_1, \dots, m_k}(s_1) C'_{\sigma'(p+1)}(s_{\sigma'(p+1)}) \cdots C'_{\sigma'(2)}(s_{\sigma'(2)}) \\ & \quad (\mu_0 \times \mu_{k+1}^{m_{k+1}} \times \dots \times \mu_n^{m_n})(ds_1, \dots, ds_{1+m_{k+1}+\dots+m_n}). \end{aligned}$$

Define $\sigma(m_1 + \dots + m_k - 1 + j) = m_1 + \dots + m_k - 1 + \sigma'(j)$ for $j = 2, \dots, 1 + m_{k+1} + \dots + m_n$. Then $\sigma \in S_{m_1 + \dots + m_{k+1}, m}$ and

$$C_{\sigma(m_1 + \dots + m_k - 1 + j)}(s_{\sigma(m_1 + \dots + m_k - 1 + j)}) = C'_{\sigma'(j)}(s_{\sigma'(j)}),$$

for $j = 2, \dots, 1 + m_{k+1} + \dots + m_n$, where $C_i(s)$ is given by (2). Hence we obtain

$$\begin{aligned} (5) \quad & P_{\mu_0, \mu_{k+1}, \dots, \mu_n}^{1, m_{k+1}, \dots, m_n}(K_{m_1, \dots, m_k}, A_{k+1}(\cdot), \dots, A_n(\cdot)) \\ &= \sum_{\sigma \in S_{m_1 + \dots + m_{k+1}, m}} \sum_{p=0}^{m_{k+1} + \dots + m_n} \int_{[a, b] \times \Delta'_{m_1 + \dots + m_{k+1}, m}(\sigma, p)} \\ & C_{\sigma(m)}(s_{\sigma(m)}) \cdots C_{\sigma(m_1 + \dots + m_k + p + 1)}(s_{\sigma(m_1 + \dots + m_k + p + 1)}) \\ & K_{m_1, \dots, m_k}(s_1) C_{\sigma(m_1 + \dots + m_k + p)}(s_{\sigma(m_1 + \dots + m_k + p)}) \cdots \\ & C_{\sigma(m_1 + \dots + m_k + 1)}(s_{\sigma(m_1 + \dots + m_k + 1)}) \\ & (\mu_0 \times \mu_{k+1}^{m_{k+1}} \times \cdots \times \mu_n^{m_n})(ds_1, ds_{m_1 + \dots + m_k + 1}, \dots, ds_m). \end{aligned}$$

Comparing (4) and (5), we see that the proof is complete. □

LEMMA 2. For any nonnegative integers m_1, \dots, m_n , and for any continuous probability measure μ_0 , the map $\Phi : \mathcal{L}(X) \rightarrow \mathcal{L}(X)$ defined by

$$\Phi(A) = P_{\mu_0, \mu_1, \dots, \mu_n}^{1, m_1, \dots, m_n}(A, A_1(\cdot), \dots, A_n(\cdot))$$

is a bounded linear operator.

Proof. Since the power associated with the operator-measure pair (A, μ_0) is one, the linearity is trivial. For any $A \in \mathcal{L}(X)$, since A is independent of variable s and μ_0 is a probability measure, by the same method as in the proof of Proposition 2.2 of [6], we have

$$\begin{aligned} \|\Phi(A)\| &= \|P_{\mu_0, \mu_1, \dots, \mu_n}^{1, m_1, \dots, m_n}(A, A_1(\cdot), \dots, A_n(\cdot))\| \\ &\leq \int_{[0, T]} \|A(s)\| |\mu_0|(ds) \left[\int_{[0, T]} \|A_1(s)\| |\mu_1|(ds) \right]^{m_1} \cdots \\ &\quad \left[\int_{[0, T]} \|A_n(s)\| |\mu_n|(ds) \right]^{m_n} \\ &\leq \|A\| \left[\int_{[0, T]} \|A_1(s)\| |\mu_1|(ds) \right]^{m_1} \cdots \\ &\quad \left[\int_{[0, T]} \|A_n(s)\| |\mu_n|(ds) \right]^{m_n}. \end{aligned}$$

So Φ is bounded and

$$\|\Phi\| \leq \left[\int_{[0,T]} \|A_1(s)\| |\mu_1|(ds) \right]^{m_1} \cdots \left[\int_{[0,T]} \|A_n(s)\| |\mu_n|(ds) \right]^{m_n}$$

□

THEOREM 3. (Extraction of a Linear Factor). Let μ_1, \dots, μ_n be given as in the Theorem 1. Assume that $g(A_1(\cdot), \dots, A_k(\cdot)) \in \mathbb{D}(A_1(\cdot), \dots, A_k(\cdot))$ and $h(A_{k+1}(\cdot), \dots, A_n(\cdot)) \in \mathbb{D}(A_{k+1}(\cdot), \dots, A_n(\cdot))$. Let

$$f(z_1, \dots, z_n) = g(z_1, \dots, z_k)h(z_{k+1}, \dots, z_n).$$

Let $K := \mathcal{T}_{\mu_1, \dots, \mu_k} g(A_1(\cdot), \dots, A_k(\cdot))$ and μ_0 be any continuous probability measure supported by $[a, b]$. Then $f(A_1(\cdot), \dots, A_n(\cdot)) \in \mathbb{D}(A_1(\cdot), \dots, A_n(\cdot))$ and

$$\begin{aligned} & \mathcal{T}_{\mu_1, \dots, \mu_n} f(A_1(\cdot), \dots, A_n(\cdot)) \\ &= \mathcal{T}_{\mu_0, \mu_{k+1}, \dots, \mu_n} F(\tilde{K}, A_{k+1}(\cdot), \dots, A_n(\cdot)) \end{aligned}$$

where $F(z_0, z_{k+1}, \dots, z_n) = z_0 h(z_{k+1}, \dots, z_n)$.

Proof. Suppose that g and h are given by

$$g(z_1, \dots, z_k) = \sum_{m_1, \dots, m_k=0}^{\infty} d_{m_1, \dots, m_k} z_1^{m_1} \cdots z_k^{m_k}$$

and

$$h(z_{k+1}, \dots, z_n) = \sum_{m_{k+1}, \dots, m_n=0}^{\infty} e_{m_{k+1}, \dots, m_n} z_{k+1}^{m_{k+1}} \cdots z_n^{m_n}.$$

Then we have

$$f(z_1, \dots, z_n) = \sum_{m_1, \dots, m_n=0}^{\infty} c_{m_1, \dots, m_n} z_1^{m_1} \cdots z_n^{m_n}$$

where

$$c_{m_1, \dots, m_n} = d_{m_1, \dots, m_k} e_{m_{k+1}, \dots, m_n}$$

for all $m_1, \dots, m_n = 0, 1, 2 \dots$. Now by Definition 1 and Theorem 1,

$$\begin{aligned} & \mathcal{T}_{\mu_1, \dots, \mu_n} f(A_1(\cdot), \dots, A_n(\cdot)) \\ &= \sum_{m_1, \dots, m_n=0}^{\infty} c_{m_1, \dots, m_n} P_{\mu_1, \dots, \mu_n}^{m_1, \dots, m_n}(A_1(\cdot), \dots, A_n(\cdot)) \\ &= \sum_{m_1, \dots, m_n=0}^{\infty} d_{m_1, \dots, m_k} e_{m_{k+1}, \dots, m_n} \\ & \quad P_{\mu_0, \mu_{k+1}, \dots, \mu_n}^{1, m_{k+1}, \dots, m_n}(K_{m_1, \dots, m_k}, A_{k+1}(\cdot), \dots, A_n(\cdot)). \end{aligned}$$

We have

$$\begin{aligned} & \sum_{m_1, \dots, m_k=0}^{\infty} |d_{m_1, \dots, m_k}| \\ & \quad \|P_{\mu_0, \mu_{k+1}, \dots, \mu_n}^{1, m_{k+1}, \dots, m_n}(K_{m_1, \dots, m_k}, A_{k+1}(\cdot), \dots, A_n(\cdot))\| \\ & \leq \sum_{m_1, \dots, m_k=0}^{\infty} |d_{m_1, \dots, m_k}| \|K_{m_1, \dots, m_k}\| \\ & \quad \left[\int_{[0, T]} \|A_{k+1}(s)\| |\mu_{k+1}|(ds) \right]^{m_{k+1}} \dots \\ & \quad \left[\int_{[0, T]} \|A_n(s)\| |\mu_n|(ds) \right]^{m_n} \\ & \leq \sum_{m_1, \dots, m_k=0}^{\infty} |d_{m_1, \dots, m_k}| \left[\int_{[0, T]} \|A_1(s)\| |\mu_1|(ds) \right]^{m_1} \\ & \quad \dots \left[\int_{[0, T]} \|A_k(s)\| |\mu_k|(ds) \right]^{m_k} \\ & \quad \left[\int_{[0, T]} \|A_{k+1}(s)\| |\mu_{k+1}|(ds) \right]^{m_{k+1}} \\ & \quad \dots \left[\int_{[0, T]} \|A_n(s)\| |\mu_n|(ds) \right]^{m_n} \\ & < \infty, \end{aligned}$$

since $g(A_1(\cdot), \dots, A_k(\cdot)) \in \mathbb{D}(A_1(\cdot), \dots, A_k(\cdot))$. Hence

$$\sum_{m_1, \dots, m_k=0}^{\infty} d_{m_1, \dots, m_k} P_{\mu_0, \mu_{k+1}, \dots, \mu_n}^{1, m_{k+1}, \dots, m_n}(K_{m_1, \dots, m_k}, A_{k+1}(\cdot), \dots, A_n(\cdot))$$

$$= \lim_{N \rightarrow \infty} \sum_{m_1, \dots, m_k=0}^N d_{m_1, \dots, m_k} P_{\mu_0, \mu_{k+1}, \dots, \mu_n}^{1, m_{k+1}, \dots, m_n} (K_{m_1, \dots, m_k}, A_{k+1}(\cdot), \dots, A_n(\cdot)).$$

By Lemma 2, the map Φ defined by

$$\Phi(A) = P_{\mu_0, \mu_{k+1}, \dots, \mu_n}^{1, m_{k+1}, \dots, m_n} (A, A_{k+1}(\cdot), \dots, A_n(\cdot))$$

is a bounded linear operator. Moreover the series

$$\sum_{m_1, \dots, m_k=0}^{\infty} d_{m_1, \dots, m_k} K_{m_1, \dots, m_k}$$

also converges absolutely. Hence we obtain

$$\begin{aligned} & \sum_{m_1, \dots, m_k=0}^{\infty} d_{m_1, \dots, m_k} P_{\mu_0, \mu_{k+1}, \dots, \mu_n}^{1, m_{k+1}, \dots, m_n} (K_{m_1, \dots, m_k}, A_{k+1}(\cdot), \dots, A_n(\cdot)) \\ &= P_{\mu_0, \mu_{k+1}, \dots, \mu_n}^{1, m_{k+1}, \dots, m_n} \left(\sum_{m_1, \dots, m_k=0}^{\infty} d_{m_1, \dots, m_k} K_{m_1, \dots, m_k}, A_{k+1}(\cdot), \dots, A_n(\cdot) \right). \end{aligned}$$

But

$$\begin{aligned} & \sum_{m_1, \dots, m_k=0}^{\infty} d_{m_1, \dots, m_k} K_{m_1, \dots, m_k} \\ &= \mathcal{T}_{\mu_1, \dots, \mu_k} g(A_1(\cdot), \dots, A_k(\cdot)) = K. \end{aligned}$$

So we have

$$\begin{aligned} & \mathcal{T}_{\mu_1, \dots, \mu_n} f(A_1(\cdot), \dots, A_n(\cdot)) \\ &= \sum_{m_{k+1}, \dots, m_n=0}^{\infty} e_{m_{k+1}, \dots, m_n} P_{\mu_0, \mu_{k+1}, \dots, \mu_n}^{1, m_{k+1}, \dots, m_n} (K, A_{k+1}(\cdot), \dots, A_n(\cdot)) \\ &= \mathcal{T}_{\mu_0, \mu_{k+1}, \dots, \mu_n} F(\tilde{K}, A_{k+1}(\cdot), \dots, A_n(\cdot)) \end{aligned}$$

where

$$\begin{aligned} F(z_0, z_{k+1}, \dots, z_n) &= \sum_{m_{k+1}, \dots, m_n=0}^{\infty} e_{m_{k+1}, \dots, m_n} z_0^{m_{k+1}} \cdots z_n^{m_n} \\ &= z_0 h(z_{k+1}, \dots, z_n). \end{aligned}$$

□

COROLLARY 4. Suppose that the measures μ_1, \dots, μ_k are supported by $[0, a]$ and the measures μ_{k+1}, \dots, μ_n are supported by $[a, T]$. Then we have

$$\begin{aligned} & P_{\mu_1, \dots, \mu_n}^{m_1, \dots, m_n}(A_1(\cdot), \dots, A_n(\cdot)) \\ &= P_{\mu_{k+1}, \dots, \mu_n}^{m_{k+1}, \dots, m_n}(A_{k+1}(\cdot), \dots, A_n(\cdot)) P_{\mu_1, \dots, \mu_k}^{m_1, \dots, m_k}(A_1(\cdot), \dots, A_k(\cdot)) \end{aligned}$$

Proof. Applying Theorem 1, we obtain

$$\begin{aligned} & P_{\mu_1, \dots, \mu_n}^{m_1, \dots, m_n}(A_1(\cdot), \dots, A_n(\cdot)) \\ &= P_{\nu_0, \mu_{k+1}, \dots, \mu_n}^{1, m_{k+1}, \dots, m_n}(K_{m_1, \dots, m_k}, A_{k+1}(\cdot), \dots, A_n(\cdot)) \end{aligned}$$

where

$$K_{m_1, \dots, m_k} = P_{\mu_1, \dots, \mu_k}^{m_1, \dots, m_k}(A_1(\cdot), \dots, A_k(\cdot))$$

and ν_0 is a continuous probability measure supported by $[0, a]$. Applying again Theorem 1, we gain

$$\begin{aligned} & P_{\nu_0, \mu_{k+1}, \dots, \mu_n}^{1, m_{k+1}, \dots, m_n}(K_{m_1, \dots, m_k}, A_{k+1}(\cdot), \dots, A_n(\cdot)) \\ &= P_{\nu_0, \nu_1}^{1, 1}(K_{m_1, \dots, m_k}, K_{m_{k+1}, \dots, m_n}) \end{aligned}$$

where

$$K_{m_{k+1}, \dots, m_n} = P_{\mu_{k+1}, \dots, \mu_n}^{m_{k+1}, \dots, m_n}(A_{k+1}(\cdot), \dots, A_n(\cdot))$$

and ν_1 is a continuous probability measure supported by $[a, T]$. Proposition 4.5 from [3] implies that

$$P_{\mu_1, \dots, \mu_n}^{m_1, \dots, m_n}(A_1(\cdot), \dots, A_n(\cdot)) = K_{m_{k+1}, \dots, m_n} K_{m_1, \dots, m_k}$$

as desired. □

COROLLARY 5. Let μ_1, \dots, μ_n be as in Corollary 4. Let f, g and h be given as in Theorem 3. Then

$$\begin{aligned} & \mathcal{T}_{\mu_1, \dots, \mu_n} f(A_1(\cdot), \dots, A_n(\cdot)) \\ &= \mathcal{T}_{\mu_{k+1}, \dots, \mu_n} h(A_{k+1}(\cdot), \dots, A_n(\cdot)) \\ & \quad \mathcal{T}_{\mu_1, \dots, \mu_k} g(A_1(\cdot), \dots, A_k(\cdot)) \end{aligned}$$

Proof. By Corollary 4, we have

$$\begin{aligned}
& \mathcal{T}_{\mu_1, \dots, \mu_n} f(A_1(\cdot), \dots, A_n(\cdot)) \\
&= \sum_{m_1, \dots, m_n=0}^{\infty} c_{m_1, \dots, m_n} P_{\mu_1, \dots, \mu_n}^{m_1, \dots, m_n}(A_1(\cdot), \dots, A_n(\cdot)) \\
&= \sum_{m_1, \dots, m_n=0}^{\infty} d_{m_1, \dots, m_k} e_{m_{k+1}, \dots, m_n} \\
&\quad P_{\mu_{k+1}, \dots, \mu_n}^{m_{k+1}, \dots, m_n}(A_{k+1}(\cdot), \dots, A_n(\cdot)) P_{\mu_1, \dots, \mu_k}^{m_1, \dots, m_k}(A_1(\cdot), \dots, A_k(\cdot)) \\
&= \sum_{m_{k+1}, \dots, m_n=0}^{\infty} e_{m_{k+1}, \dots, m_n} P_{\mu_{k+1}, \dots, \mu_n}^{m_{k+1}, \dots, m_n}(A_{k+1}(\cdot), \dots, A_n(\cdot)) \\
&\quad \sum_{m_1, \dots, m_k=0}^{\infty} d_{m_1, \dots, m_k} P_{\mu_1, \dots, \mu_k}^{m_1, \dots, m_k}(A_1(\cdot), \dots, A_k(\cdot)) \\
&= \mathcal{T}_{\mu_{k+1}, \dots, \mu_n} h(A_{k+1}(\cdot), \dots, A_n(\cdot)) \\
&\quad \mathcal{T}_{\mu_1, \dots, \mu_k} g(A_1(\cdot), \dots, A_k(\cdot))
\end{aligned}$$

as desired. \square

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